On Fuzzy Description Logics

Àngel García-Cerdaña and Francesc Esteva

Institut d'Investigació en Intel·ligència Artificial-CSIC

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Description Logics

Knowledge representation languages suited to specify formal ontologies

Elements of a DL language

- Concepts: denote sets of individuals
- Roles: denote binary relations among individuals
- Constructors: from atomic concepts and rules they allow to build complex descriptions of both concepts and roles

DL knowledge base (KB)

- TBox: definitions and hierarchies of the relevant domain concepts
- ABox: specifications of properties of the domain instances

Issues

- The statements in the KB can be identified with formulas in FOL
- Obtain implicit knowledge from the explicit knowledge in the KB

Description Logics

Reference (classical DLs)

Baader, F.; Calvanese, D.; McGuinness, D. L.; Nardi, D. and Patel-Schneider, P. F. (ed.) *The description logic handbook: theory, implementation, and applications.* Cambridge University Press, 2003

Patient with high fever

Person living near Paris

First papers on Fuzzy Description Logic

- **Generalizing term subsumption languages to fuzzy logic**, John Yen. Proceedings of the IJCAI'91.
- **A Description Logic for Vague Knowledge**, Christopher B. Tresp and Ralf Molitor. Techreport of the Aachen University of Technology (1998). Proceedings of the ECAI'98.
- **Reasoning within Fuzzy Description Logics**, Umberto Straccia. Journal of Artificial Intelligence Research (2001).

Languages of description: Syntax

- Atomic concepts: $A_1, \ldots, A_n$
- Atomic roles: $R_1, \ldots, R_m$
- Construction of complex concepts and roles with connectives and quantifiers.

Semantics: Interpretation for a DL language

$\mathcal{I} = \langle M, (.)^{\mathcal{I}} \rangle$, where M is non-empty set, $(.)^{\mathcal{I}}$ is a mapping that assigns

- $A_i \quad \rightsquigarrow \quad A_i^{\mathcal{I}} \subseteq M$
- $R_j \quad \rightsquigarrow \quad R_j^{\mathcal{I}} \subseteq M \times M$

For complex concepts and roles:
$\mathcal{I}$ is recursively extended to any formula in the language.

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Semantics: Interpretation for a DL language (using characteristic functions)

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- $A_i \quad \rightsquigarrow \quad A_i^{\mathcal{I}} : M \rightarrow \{0, 1\}$
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- $A_i \quad \rightsquigarrow \quad A_i^{\mathcal{I}} : M \longrightarrow [0, 1]$
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$\mathcal{I}$ is recursively extended to any formula in the language.

The language $\mathcal{ALC}$: syntax

Metavariables and symbols

- A (atomic concepts); C, D (complex concepts); R (atomic roles)
- Connectives: $\&, \rightarrow, \bar{0}$ Quantifiers: $\forall, \exists$ Auxiliary symbol: .

Syntactic rules

$$
\begin{array}{lll}
C, D & \rightsquigarrow & A \mid \quad \text{(atomic concept)} \\
 & & \bar{0} \mid \quad \text{(false)} \\
 & & C \& D \mid \quad \text{(conjunction or fusion)} \\
 & & C \rightarrow D \mid \quad \text{(implication)} \\
 & & \forall R.C \mid \quad \text{(universal quantification)} \\
 & & \exists R.C \mid \quad \text{(existential quantification)}
\end{array}
$$

Defined connectives

$$\neg C := C \rightarrow \bar{0}, \quad C \veebar D := \neg C \& \neg D, \quad \bar{1} := \neg \bar{0}$$

Interpretation of complex $\mathcal{ALC}$-descriptions

$$
\begin{array}{rcl}
\bar{0}^{\mathcal{I}} & = & \emptyset \\
\bar{1}^{\mathcal{I}} & = & M \\
(\neg C)^{\mathcal{I}} & = & M \setminus C^{\mathcal{I}} \\
(C \& D)^{\mathcal{I}} & = & C^{\mathcal{I}} \cap D^{\mathcal{I}} \\
(C \veebar D)^{\mathcal{I}} & = & C^{\mathcal{I}} \cup D^{\mathcal{I}} \\
(C \rightarrow D)^{\mathcal{I}} & = & (M \setminus C^{\mathcal{I}}) \cup D^{\mathcal{I}} \\
(\forall R.C)^{\mathcal{I}} & = & \{x \in M : \{y \in M : \langle x, y \rangle \in R^{\mathcal{I}}\} \subseteq C^{\mathcal{I}}\} \\
(\exists R.C)^{\mathcal{I}} & = & \{x \in M : \{y \in M : \langle x, y \rangle \in R^{\mathcal{I}}\} \cap C^{\mathcal{I}} \neq \emptyset\}
\end{array}
$$

Using characteristic functions

$$
\begin{array}{rcl}
\bar{0}^{\mathcal{I}}(x) & = & 0 \\
\bar{1}^{\mathcal{I}}(x) & = & 1 \\
(\neg C)^{\mathcal{I}}(x) & = & 1 - C^{\mathcal{I}}(x) \\
(C \& D)^{\mathcal{I}}(x) & = & C^{\mathcal{I}}(x) \wedge D^{\mathcal{I}}(x) \\
(C \veebar D)^{\mathcal{I}}(x) & = & C^{\mathcal{I}}(x) \vee D^{\mathcal{I}}(x) \\
(C \rightarrow D)^{\mathcal{I}}(x) & = & (1 - C^{\mathcal{I}}(x)) \vee D^{\mathcal{I}}(x) \\
(\forall R.C)^{\mathcal{I}}(x) & = & \bigwedge_{y \in M}\{(1 - R^{\mathcal{I}}(x, y)) \vee C^{\mathcal{I}}(x)\} \\
(\exists R.C)^{\mathcal{I}}(x) & = & \bigvee_{y \in M}\{R^{\mathcal{I}}(x, y) \wedge C^{\mathcal{I}}(x)\}
\end{array}
$$

Interpretation of complex $\mathcal{ALC}$-descriptions

$$
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(C \veebar D)^{\mathcal{I}} &= C^{\mathcal{I}} \cup D^{\mathcal{I}} \\
(C \to D)^{\mathcal{I}} &= (M \setminus C^{\mathcal{I}}) \cup D^{\mathcal{I}} \\
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\end{aligned}
$$

Using fuzzy sets — Tresp & Molitor (98), Straccia (01)

$$
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$$

t-norms and their residuals

t-norm (t-conorm)

Binary operation defined in $[0,1]_{\mathbb{R}}$

1. commutative and associative,
2. non decreasing in both arguments,
3. having 1 (0) as unity element.

Residual of a continuous t-norm

Given a continuous t-norm $*$, there exists an unique operation $\rightarrow_*$ satisfying,

$$\forall a,b,c \in [0,1],\ a * c \leq b \Leftrightarrow c \leq a \rightarrow_* b$$

Associated negation

$$n_*(x) := x \rightarrow_* 0$$

Interpretation of complex $\mathcal{ALC}$-descriptions

- *Making fuzzy logic description more general*, Hájek 2005
- To deal with FDL taking as basis t-norm based fuzzy logics

$$
\begin{aligned}
\bar{0}^{\mathcal{I}}(x) &= 0\\
\bar{1}^{\mathcal{I}}(x) &= 1\\
(\neg C)^{\mathcal{I}}(x) &= n_* C^{\mathcal{I}}(x)\\
(C \& D)^{\mathcal{I}}(x) &= C^{\mathcal{I}}(x) * D^{\mathcal{I}}(x)\\
(C \veebar D)^{\mathcal{I}}(x) &= C^{\mathcal{I}}(x) \oplus D^{\mathcal{I}}(x)\\
(C \to D)^{\mathcal{I}}(x) &= C^{\mathcal{I}}(x) \to_* D^{\mathcal{I}}(x)\\
(\forall R.C)^{\mathcal{I}}(x) &= \bigwedge_{y \in M}\{R^{\mathcal{I}}(x,y) \to_* C^{\mathcal{I}}(x)\}\\
(\exists R.C)^{\mathcal{I}}(x) &= \bigvee_{y \in M}\{R^{\mathcal{I}}(x,y) * C^{\mathcal{I}}(x)\}
\end{aligned}
$$

$*$	:	Continuous t-norm
$\to_*$	:	Residual of this t-norm
n_*	:	Negation of this t-norm
$\oplus$	:	Continuous t-conorm

Mathematical Fuzzy Logic: the formal system *BL*

- Book: *Metamathematics of Fuzzy Logics* (Petr Hájek, 1998)
- He proposes the formal system *BL* and its expansion *BL*∀
- There are Hilbert-style axiomatizations (schemas of formulas and rules)
- Connectives: $\{\&, \rightarrow, \bar{0}\}$ (primitive); $\{\vee, \wedge, \leftrightarrow, \neg, \bar{1}\}$ (defined)
- Quantifiers: $\{\forall, \exists\}$
- To interpret the connectives he proposes as truth functions:

$\&$	(conjunction)	$\rightsquigarrow$	a continuous t-norm $*$
$\rightarrow$	(implication)	$\rightsquigarrow$	the residual $\rightarrow_*$ of $*$
$\bar{0}$	(false)	$\rightsquigarrow$	0

- The truth value for quantified formulas is defined analogously that in classical logic. Example (in a model **M** of universe *M*):

$$\|(\forall x)\varphi(x,\dots)\|^*_{\mathbf{M}} = \bigwedge_{a\in M}\{\|\varphi(a,\dots)\|^*_{\mathbf{M}}\}$$

$$\|(\exists x)\varphi(x,\dots)\|^*_{\mathbf{M}} = \bigvee_{a\in M}\{\|\varphi(a,\dots)\|^*_{\mathbf{M}}\}$$

Mathematical Fuzzy Logic: some notions

Standard algebras

$[0,1]_* = \langle [0,1]_{\mathbb{R}}, \leq, *, \rightarrow_*, n_*, 0, 1\rangle$

$\leq$	:	order of the real numbers
$*$	:	continuous t-norm
$\rightarrow_*$	:	residual of the t-norm
n_*	:	negation associated

Logic of a set *T* of continuous *t*-norms

A logic *L* such that,

- for every *t*-norm $*$ in *T*,
- every evaluation *e* of the formulas in $[0,1]_*$, and
- every formula φ,

$$\vdash_L \varphi \quad \text{iff} \quad e(\varphi) = 1$$

BL is the logic of all the continuous *t*-norms and their residua

Theorem (Esteva, Godo, Montagna, 2003)

The logic L^* of each continuous t-norm $*$ is a finite axiomatic extension of *BL*

Towards a general logical counterpart to FDL

- Choices oriented to search for the syntactical counterpart of the semantic calculi used in most of works dealing with FDLs:
  - the introduction of a involutive negation
  - the use of truth constants in the languages of description
- We present the languages $\mathcal{ALC}_{L^*(\mathbf{S})}$ and $\mathcal{ALC}_{L^*_{\sim}(\mathbf{S})}$

Adding an involutive negation to L^*

When the negation $\neg\varphi := \varphi \rightarrow \bar{0}$ defined in L^* is not involutive, we can do the following:

- to expand the language of L^* with a new unary connective $\sim$, and
- to add to L^* the axioms:
  - $\sim\sim\varphi \leftrightarrow \varphi$
  - $\sim(\varphi \vee \psi) \leftrightarrow (\sim\varphi \wedge \sim\psi)$
  - $\neg\varphi \rightarrow \sim\varphi$

This logic is denoted by $L^*_\sim$

$L^*(S)$ and $L^*_\sim(S)$

Adding truth constants to L^* (or $L^*_\sim$)

Let $\mathbf{S} = \langle S, *, \rightarrow_*, \max, \min, 0, 1\rangle$ be a countable subalgebra of the standard algebra $[0,1]_*$. Then,

- we expand the language of L^* (or $L^*_\sim$) with a constant $\bar{r}$ for each $r \in S$,
- for each $r, s \in S \setminus \{0, 1\}$ we add to L^* (or $L^*_\sim$) the *book-keeping* axioms:
  - $\bar{r} \& \bar{s} \leftrightarrow \overline{r * s}$
  - $(\bar{r} \rightarrow \bar{s}) \leftrightarrow \overline{r \rightarrow_* s}$

These logics are denoted by $L^*(\mathbf{S})$ and $L^*_\sim(\mathbf{S})$.

Predicate versions of these logics: $L^*(\mathbf{S})\forall$ and $L^*_\sim(\mathbf{S})\forall$

$$\frac{L^*(\mathbf{S})\forall}{L^*(\mathbf{S})} = \frac{\text{Classical Predicate Logic}}{\text{Classical Propositional Logic}}$$

Evaluated formulas

$$\bar{r} \rightarrow \varphi,\ \varphi \rightarrow \bar{r} \qquad \text{Interpretations: } e(\varphi) \geq r,\ r \leq e(\varphi)$$

(φ is a formula with no occurrences of truth constants)

The description languages $\mathcal{ALC}_{L^*(\mathbf{S})}$ and $\mathcal{ALC}_{L^*_\sim(\mathbf{S})}$

Syntactic rules for $\mathcal{ALC}_{L^*(\mathbf{S})}$ and $\mathcal{ALC}_{L^*_\sim(\mathbf{S})}$

C, D	$\rightsquigarrow$	$A \mid$	(atomic concept)
		$\bar{0} \mid$	(false)
		$\bar{r} \mid$, for every $r \in S$	(truth constants)
		$C\&D \mid$	(conjunction or fusion)
		$C \to D \mid$	(implication)
		$\forall R.C \mid$	(universal quantification)
		$\exists R.C \mid$	(existential quantification)

For $\mathcal{ALC}_{L^*_\sim(\mathbf{S})}$ we add:

$C \rightsquigarrow\ \sim C \mid$ (involutive negation)

Defined connectives: $\vee, \wedge, \neg, \bar{1}$

$$C \vee D := ((C \to D) \to D) \wedge ((D \to C) \to C), \quad C \wedge D := C\&(C \to D),$$
$$\neg C := C \to \bar{0}, \quad \bar{1} := \neg\bar{0}.$$

Instance of a concept

This notion allows us to read the concepts as formulas of the corresponding predicate fuzzy logic: for each term t (variable or constant), *the instance* $C(t)$ of a concept is defined as follows:

- $A(t)$ is the atomic formula in which A is interpreted as a unary predicate,
- $\bar{0}(t)$ is $\bar{0}$; and $\bar{r}(t)$ is $\bar{r}$,
- $(C\&D)(t)$ and $(C \to D)(t)$ are $C(t)\&D(t)$ and $C(t) \to D(t)$, respectively,

and, for the case of $\mathcal{ALC}_{L^*_\sim(\mathbf{s})}$,

- $(\sim C)(t)$ is $\sim (C(t))$,

and, if y is a variable not occurring in $C(t)$,

- $(\forall R.C)(t)$ is $(\forall y)(R(t, y) \to C(y))$,
- $(\exists R.C)(t)$ is $(\exists y)(R(t, y)\&C(y))$.

TBox and ABox for $\mathcal{ALC}_{L^*}(\mathbf{S})$ and $\mathcal{ALC}_{L^*_\sim}(\mathbf{S})$

Components of a *fuzzy KB* for our languages:

- *fuzzy TBox*: finite set of *fuzzy concept inclusion axioms*:
  $\bar{r} \to (\forall x)(C(x) \to D(x))$, $(\forall x)(C(x) \to D(x)) \to \bar{r}$.
- *fuzzy ABox*: finite set of *fuzzy assertion axioms*:
  $\bar{r} \to C(a)$, $C(a) \to \bar{r}$, $\bar{r} \to R(a,b)$, $R(a,b) \to \bar{r}$.

Sentence	Notation
$\bar{r} \to (\forall x)(C(x) \to D(x))$	$\langle C \sqsubseteq D, \bar{r} \rangle$
$\bar{r} \to C(a)$	$\langle a : C, \bar{r} \rangle$
$\bar{r} \to R(a,b)$	$\langle (a,b) : R, \bar{r} \rangle$

Reasoning with KBs in our languages involves evaluated formulas

Satisfaction of a fuzzy axiom in a model **M**

$$\begin{array}{lll} \mathbf{M} \models \langle C \sqsubseteq D, \bar{r}\rangle & \text{iff} & \inf\{C^{\mathbf{M}}(x) \to_* D^{\mathbf{M}}(x) : x \in M\} \geq r \\ \mathbf{M} \models \langle a : C, \bar{r}\rangle & \text{iff} & C^{\mathbf{M}}(a^{\mathbf{M}}) \geq r \\ \mathbf{M} \models \langle (a,b) : R, \bar{r}\rangle & \text{iff} & R^{\mathbf{M}}(a^{\mathbf{M}}, b^{\mathbf{M}}) \geq r \end{array}$$

The goal

- A KB is a set of evaluated formulas, say Γ.
- Our languages are defined as semantical calculus, i.e., φ is deduced from Γ iff for any semantic interpretation evaluating all formulas of Γ as 1, φ is also evaluated as 1.
- It is in this framework that the algorithms defined in some papers of FDL make sense.
- Which is the logic that underlies this semantical calculus?

Example: Straccia (2001)

1) He defines a $\mathcal{ALC}$-like language based on the minimum t-norm,

- (the framework is the Gödel Logic G),

2) taking the negation $n(x) = 1 - x$

- (the setting is $G_\sim$),

3) and adding truth-constants in the language

- (now the setting is $G_\sim(\mathbf{S})$).

So, the language is $\mathcal{ALC}_{G_\sim(\mathbf{S})}$ and the first order setting is $G_\sim(\mathbf{S})\forall$.

A completeness theorem

$$\Gamma \vdash_{G_\sim(\mathbf{S})\forall} \varphi \quad \text{iff} \quad \Gamma \models_{[0,1]_{G_\sim(\mathbf{S})}} \varphi$$

$\Gamma \cup \{\varphi\}$ is a set of evaluated formulas of type $\bar{r} \to \sigma$
$[0,1]_{G_\sim(\mathbf{S})} = \langle [0,1], \min, \max, \to_G, \sim, \{\bar{r} \mid r \in S\}, 0, 1\rangle$.

Some remarks

- In classical DLs, the logical system is always Classical Logic
- Each classical DL depends only on the constructors considered
- Main theoretical problem: relationships between expressiveness and complexity
- In our approach to FDLs the situation is more complex
- We need to choose both a *t*-norm $*$ and a countable subalgebra of $[0, 1]_*$ for each language
- The corresponding theoretical problems could take benefit from the recent advances on Mathematical Fuzzy Logic
- We work with fragments of *fuzzy first order logics*
- Thus, results coming from mathematical fuzzy logics could be applied to particular fragments.