On Fuzzy Modal Logics

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Many-Valued Modal Logics

- We want to combine modal logics and fuzzy logics (in the sense of P. HÁJEK). We are looking for the minimal logic.
- The language is going to be

$$\varphi ::= p \mid \bot \mid \top \mid \varphi_0 \wedge \varphi_1 \mid \varphi_0 \vee \varphi_1 \mid \varphi_0 \odot \varphi_1 \mid \varphi_0 \to \varphi_1 \mid \Box\varphi \mid \Diamond\varphi$$

- What about the semantics? Let us assume we have fixed a residuated lattice $\mathbf{A} = \langle A, 0, 1, \wedge, \vee, \odot, \to \rangle$ (i.e., $\mathbf{A} \in \mathrm{FL}_{ew}$).
- An **A-valued Kripke model** is a structure $\langle W, R, e \rangle$ such that
  - W is a set,
  - $R : W \times W \longrightarrow A$,
  - $e : Var \times W \longrightarrow A$.

Many-Valued Modal Logics

How to extend the valuation *e*?

It is extended to $\bar{e} : Fm \times W \longrightarrow A$ under the following conditions:

- $\bar{e}$ is an algebraic homomorphism, in its first component, for the connectives in the algebraic signature of **A**,
- $\bar{e}(\Box\varphi, w) = \bigwedge\{R(w, w') \rightarrow \bar{e}(\varphi, w') : w' \in W\}$,
- $\bar{e}(\Diamond\varphi, w) = \bigvee\{R(w, w') \odot \bar{e}(\varphi, w') : w' \in W\}$.

Definition

A formula φ is valid in this Kripke model in case that $e(\varphi, w) = 1$ for every world $w \in W$.

Definition

$\Gamma \models \varphi$ iff for every Kripke model $\langle W, R, e\rangle$ and every world $w \in W$ if $e(\Gamma, w) = 1$ then $e(\varphi, w) = 1$.

Axiom K : $\Box(p \to q) \to (\Box p \to \Box q)$

Example

Let us a consider **A** as the standard Łukasiewicz algebra and the Kripke model with only one point $\bullet$. This point satisfies $R(\bullet, \bullet) = \frac{1}{2}$, $e(p, \bullet) = \frac{1}{2}$ and $e(q, \bullet) = 0$. Then,

- $e(\Box(p \to q), \bullet) = R(\bullet, \bullet) \to (e(p, \bullet) \to e(q, \bullet)) = 1$,
- $e(\Box p, \bullet) = R(\bullet, \bullet) \to e(p, \bullet) = 1$,
- $e(\Box q, \bullet) = R(\bullet, \bullet) \to e(q, \bullet) = \frac{1}{2}$.

Several Remarks

- In general axiom K fails.
- In case that the accesibility relation only takes values in idempotent elements of **A**, then K holds.
- In general it is false that $\Box\varphi$ and $\Diamond\varphi$ are interdefinable (it holds for involutive negations).
- It is well known that modal formulas can be seen as first-order classical formulas with two variables,
  - $\Box p$ corresponds to $\forall v_1 (R v_0 v_1 \rightarrow P v_1)$,
  - $q \vee \Box p$ corresponds to $Q v_0 \ \vee \ \forall v_1 (R v_0 v_1 \rightarrow P v_1)$,
  - $\Box\Diamond p$ corresponds to $\forall v_1 (R v_0 v_1 \rightarrow \exists v_0 (R v_1 v_0 \wedge P v_0))$.

  The same translation embeds our many-valued modal language into the many-valued first-order logic given by (truth values in) **A**.

What do we know about the $\Box$ logic?

1. We have managed to axiomatize this modal logic when we have a constant for every element in the finite algebra, i.e., the modal logic of the canonical algebra $\mathbf{A^c} = \langle A, 0, 1, \wedge, \vee, \odot, \rightarrow, a_1, \ldots, a_n \rangle$.
2. Adding essentially the axiom (K) $\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$ to the previous minimum logic we can characterize the modal logic associated with idempotent Kripke frames.
3. Adding essentially the axioms $\Box 0 \vee (\Box a \leftrightarrow a)$ together with $\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$ to the previous minimum logic we can characterize the modal logic associated with Boolean Kripke frames.
4. Adding essentially the Prefixing metarule (i.e., if $\Gamma \vdash \varphi$ then $\Box\Gamma \vdash \Box\varphi$) to the previous minimum logic we can characterize the modal logic associated with crisp Kripke frames.
5. In the case of finite MV-algebras we can apply the method without using constants in the language, so directly to $\mathbf{A}$ $(= \mathbf{L_n})$.

Characterization of the quasivariety generated by $\mathbf{A^c}$

Remark

Our modal proof is based on the fact of having a strong complete axiomatization of the logic given by $\mathbf{A^c}$.

Theorem

If **A** is a finite residuated lattice with a coatom r (i.e., subdirectly irreducible) then a strong complete axiomatization of the logic given by $\mathbf{A^c}$ is given by:

- the axioms are the tautologies of **A**, plus book-keeping axioms and the axiom $\bigvee\{\varphi \leftrightarrow k : k \in A\}$.
- the rules of modus ponens and the rule $r \vee p \vdash p$.

Open Question

How can we axiomatize the arbitrary case?

Axiomatization of the modal $\Box$ logic

Theorem

Let us assume that we have a strong complete axiomatization of the logic given by $\mathbf{A^c}$. Then, a strong complete axiomatization of the modal logic over $\mathbf{A^c}$ is given by:

- the set of axioms is the smallest set containing
    - the tautologies of $\mathbf{A^c}$,
    - $\Box(\varphi \wedge \psi) \leftrightarrow (\Box\varphi \wedge \Box\psi)$,
    - $\Box 1$,
    - $\Box(k \rightarrow \varphi) \leftrightarrow (k \rightarrow \Box\varphi)$ (for every constant *k*)

  and that it is closed under the rules of inference
    - the rules of *Log*($\mathbf{A^c}$).
    - $\varphi \rightarrow \psi \vdash \Box\varphi \rightarrow \Box\psi$.
- the rules are the same ones than in *Log*($\mathbf{A^c}$).

Canonical Model

Definition

The canonical model is the Kripke model defined by

1. The universe is the set of non-modal homomorphisms $v : \mathbf{Fm} \longrightarrow \mathbf{A^c}$ such that all axioms are send to 1.
2. The accessibility relation is defined by $R(v_1, v_2) := \inf\{v_1(\Box\varphi) \rightarrow v_2(\varphi) : \varphi \in Fm\}$,
3. The evaluation function is defined by $e(p, v) = v(p)$ for every $p \in Var$.

Truth Lemma

In the canonical model it holds that $e(\varphi, v) = v(\varphi)$ for every formula φ.

Sketch of the Proof

- By induction it is enough to prove that $e(v, \Box\varphi) = v(\Box\varphi)$. The non trivial part is to see that $\bigwedge\{R(v, v') \rightarrow v'(\varphi) : v' \in W^{can}\} \leq v(\Box\varphi)$. Let us assume that k is the value of this infimum. We have to check that $k \leq v(\Box\varphi)$.
- for every constant s it holds that, $ModAx \cup \{s \rightarrow (j_\psi \rightarrow \psi) : \psi \in Fm \text{ and } j_\psi = v(\Box\psi)\} \models_{\mathbf{A^c}} s \rightarrow (k \rightarrow \varphi)$.
- $ModAx \models_{\mathbf{A^c}} (\bigwedge_{i \leq n}(j_{\psi_i} \rightarrow \psi_i)) \rightarrow (k \rightarrow \varphi)$.
- $ModAx \vdash_{\mathbf{A^c}} (\bigwedge_{i \leq n}(j_{\psi_i} \rightarrow \psi_i)) \rightarrow (k \rightarrow \varphi)$.
- $\vdash (\bigwedge_{i \leq n}(j_{\psi_i} \rightarrow \psi_i)) \rightarrow (k \rightarrow \varphi)$.
- Using our calculus from the previous fact we get that $\vdash (\bigwedge_{i \leq n}(j_{\psi_i} \rightarrow \Box\psi_i)) \rightarrow (k \rightarrow \Box\varphi)$.
- Since $v(\bigwedge_{i \leq n}(j_{\psi_i} \rightarrow \Box\psi_i)) = 1$ we conclude that $k \leq v(\Box\varphi)$.

Theorem (Completeness)

$\Gamma \vdash \varphi$ *iff* $\Gamma \models \varphi$.

Proof.

Let us assume that $\Gamma \nvdash \varphi$. Then, $ModAx \cup \Gamma \nvdash_{\mathbf{A^c}} \varphi$. Therefore, by the completeness theorem of $\models_{\mathbf{A^c}}$ we know that there is $v \in Hom(Fm, \mathbf{A^c})$ such that $v[ModAx] = \{1\}$, $v[\Gamma] \subseteq \{1\}$ and $v(\varphi) < 1$. By Truth Lemma there is a Kripke model $\langle W, R, e\rangle$ such that $v \in W$ and for every modal formula ψ, it holds that $e(\psi, v) = v(\psi)$. Thus, $e(\varphi, v) = v(\varphi) < 1$ and for every $\gamma \in \Gamma$, $e(\gamma, v) = v(\gamma) = 1$. ☐

Theorem

Closing the previous calculus under the Prefixing metarule

$$\frac{\Gamma \vdash \varphi}{\Box\Gamma \vdash \Box\varphi}$$

we get a strong complete axiomatization of the the modal logic associated with crisp (classical) Kripke frames.

Crisp Canonical Model

Definition

The crisp canonical model is the Kripke model defined by

1. The universe is the set of non-modal homomorphisms $v : \mathbf{Fm} \longrightarrow \mathbf{A^c}$ such that if $\Gamma \vdash \varphi$ (this is the smallest consequence relation closed under the Prefixing metarule) and $v[\Gamma] \subseteq \{1\}$ then $v(\varphi) = 1$.
2. The accessibility relation is defined by

$$R(v_1, v_2) := \begin{cases} 1 & \text{if } v_1(\Box\varphi) \leq v_2(\varphi) \text{ for every } \varphi \\ 0 & \text{if not} \end{cases}$$

3. The evaluation function is defined by $e(p, v) = v(p)$ for every $p \in Var$.

Truth Lemma

In the crisp canonical model, $e(\varphi, v) = v(\varphi)$ for every formula φ.

Sketch of the Proof

- By induction it is enough to prove that $e(v, \Box\varphi) = v(\Box\varphi)$. The non trivial part is to see that $\bigwedge\{v'(\varphi) : v' \in W^{can}, R(v, v') = 1\} \leq v(\Box\varphi)$. Let us assume that k is the value of this infimum. We have to check that $k \leq v(\Box\varphi)$.
- for every constant $s \in \{0, 1\}$ it holds that, $ModAx \cup \{s \to (j_\psi \to \psi) : \psi \in Fm \text{ and } j_\psi = v(\Box\psi)\} \models_{\mathbf{A^c}} s \to (k \to \varphi)$.
- $ModAx, j_{\psi_1} \to \psi_n, \ldots, j_{\psi_n} \to \psi_n \models_{\mathbf{A^c}} k \to \varphi$.
- $ModAx, j_{\psi_1} \to \psi_n, \ldots, j_{\psi_n} \to \psi_n \vdash_{\mathbf{A^c}} k \to \varphi$.
- $j_{\psi_1} \to \psi_n, \ldots, j_{\psi_n} \to \psi_n \vdash k \to \varphi$.
- $\Box(j_{\psi_1} \to \psi_n), \ldots, \Box(j_{\psi_n} \to \psi_n) \vdash \Box(k \to \varphi)$.
- $j_{\psi_1} \to \Box\psi_n, \ldots, j_{\psi_n} \to \Box\psi_n \vdash k \to \Box\varphi$.
- Since $v(j_{\psi_i} \to \Box\psi_i) = 1$ for every i, we conclude that $k \leq v(\Box\varphi)$.

Theorem

If **A** is a finite chain, then the multimodal logic $\Box_k$ (one modality for every constant k, which corresponds semantically to the k-cut) over $Log(\mathbf{A^c})$ is axiomatized using

- axioms and rules saying $\Box_k$ is a crisp Kripke frame (for every k),
- if $k \leq k'$ then $\Box_k\varphi \to \Box_{k'}\varphi$ is a theorem.

Theorem

In the multimodal logic $\Box_k$ it holds that $\Box\varphi$ is definable as $\bigwedge\{k \to \Box_k\varphi : k \in A\}$. In general $\Box_k\varphi$ is not definable from $\Box\varphi$ and the constants; an exception is involutive simple residuated lattices because there it holds that

$$\Box_k\varphi \leftrightarrow \bigwedge\left\{\left(\Delta\left(\bar{k} \to \Diamond\Delta(\varphi \leftrightarrow \bar{s})\right)\right) \to \bar{s} : s \in A\right\}.$$

Theorem

1. If **A** is a simple involutive residuated lattice then a strong complete axiomatization of the logic given by $\mathbf{A^c}$ is given by:
    - the axioms are the tautologies of **A**, plus book-keeping axioms and the axiom $\bigvee\{\varphi \leftrightarrow k : k \in A\}$.
    - the rule of modus ponens.
2. If **A** is a simple involutive residuated lattice then a strong complete axiomatization of the modal logic over $\mathbf{A^c}$ is given by:
    - the set of axioms is the smallest set containing
        - the tautologies of **A**, plus book-keeping axioms and the axiom $\bigvee\{\varphi \leftrightarrow k : k \in A\}$,
        - $\Box(\varphi \wedge \psi) \leftrightarrow (\Box\varphi \wedge \Box\psi)$,
        - $\Box 1$,
        - $\Box(k \rightarrow \varphi) \leftrightarrow (k \rightarrow \Box\varphi)$ (for every constant k)

      and that it is closed under the rules of inference
        - $\varphi \rightarrow \psi \vdash \Box\varphi \rightarrow \Box\psi$.
    - the rule of Modus Ponens.

Remark

In the previous proof of Truth Lemma the only use of constants is to have in our calculus a rule saying that

$$\frac{\vdash \big((k_1 \rightarrow \varphi_1) \wedge \ldots \wedge (k_n \rightarrow \varphi_n)\big) \rightarrow (k \rightarrow \varphi)}{\vdash \big((k_1 \rightarrow \Box\varphi_1) \wedge \ldots \wedge (k_n \rightarrow \Box\varphi_n)\big) \rightarrow (k \rightarrow \Box\varphi)}$$

Can we avoid constants? In the case we have non-modal formulas $\alpha_k(p)$ with the property that for every homomorphism v,

$$v(\alpha_k(p)) = \begin{cases} 1 & \text{if } k \leq v(p) \\ 0 & \text{if } v(p) < k \end{cases}$$

we could avoid the use of constants using rules like

$$\frac{\vdash (\alpha_{k_1}(\varphi_1) \wedge \ldots \wedge \alpha_{k_n}(\varphi_n)) \rightarrow \alpha_k(\varphi)}{\vdash (\alpha_{k_1}(\Box\varphi_1) \wedge \ldots \wedge \alpha_{k_n}(\Box\varphi_n)) \rightarrow \alpha_k(\Box\varphi)}$$

The problem with the previous rule is that it is not valid, but using a more complicated rule we can axiomatize the logic.

Case **A** is a finite MV chain (assume L_n)

Theorem

- the set of axioms is the smallest set closed under substitutions containing
  - the tautologies of L_n,
  - $\Box 1$ and $(\Box\varphi \wedge \Box\psi) \rightarrow \Box(\varphi \wedge \psi)$,
- the Modus Ponens rule, the Monotonicity rule and for every $a \in L_n \setminus \{0\}$ the rule (R_a)

$$\frac{(\eta_{a_2 \odot b}(\varphi_2) \wedge \eta_{a_3 \odot b}(\varphi_3) \wedge \ldots \wedge \eta_{a_n \odot b}(\varphi_n)) \rightarrow \eta_{a \odot b}(\varphi) \ \forall b \in L_n, b > \neg a}{(\eta_{a_2}(\Box\varphi_2) \wedge \eta_{a_3}(\Box\varphi_3) \ldots \wedge \eta_{a_n}(\Box\varphi_n)) \rightarrow \eta_a(\Box\varphi)}$$

where $a_2 = \frac{1}{n-1}$, $a_3 = \frac{2}{n-1}$, ..., $a_{n-1} = \frac{n-2}{n-1}$ and $a_n = 1$.

Case **A** is a finite MV chain (assume L_{n+1})

Remark

The particular case of crisp accessibility relations is much easier because in this particular case it holds that $\vdash \alpha_k(\Box\varphi) \leftrightarrow \Box\alpha_k(\varphi)$ (this commutativity holds for every monotone non-modal formula).

Theorem (Hansoul & Teheux)

A strong complete axiomatization of the modal logic over a finite MV chain (without constants) with crisp accessibility relations is given by:

- the set of axioms is the smallest set containing
  - the tautologies of $\mathbf{L_{n+1}}$, $\Box(\varphi \wedge \psi) \leftrightarrow (\Box\varphi \wedge \Box\psi)$, $\Box 1$ and $\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$,
  - $\alpha_k(\Box\varphi) \leftrightarrow \Box\alpha_k(\varphi)$, [It is enough to write $\Box(\varphi \oplus \varphi) \leftrightarrow \Box\varphi \oplus \Box\varphi$ and $\Box(\varphi \odot \varphi) \leftrightarrow \Box\varphi \odot \Box\varphi$]

  and that it is closed under the rules of inference
  - $\varphi \rightarrow \psi \vdash \Box\varphi \rightarrow \Box\psi$,
- the rule of Modus Ponens.

Open Questions?

1. Can we improve the presentation of the axiomatization for finite MV chains?
2. How can we deal with both modalities $\Box$ and $\Diamond$?
3. Is the modal logic defined over the standard Lukasiewicz algebra decidable? [It is known that is is the intersection of all modal logics over finite MV chains]
4. How can we axiomatize the modal logic over standard Lukasiewicz algebra allowing infinitary rules (and perhaps constants)?
5. Standard Lukasiewicz algebra with crisp accessibility relation is axiomatizable with the infinitary rule (Hansoul & Teheux)

$$\frac{\varphi \oplus \varphi, \varphi \oplus \varphi^2, \ldots, \varphi \oplus \varphi^m, \ldots}{\varphi}$$

Is this rule admissible? That is, if we delete it do we get the same theorems?