Bi-rewriting, a Term Rewriting Technique for Monotonic Order Relations*

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Abstract. We propose an extension of rewriting techniques to derive inclusion relations $a \subseteq b$ between terms built from monotonic operators. Instead of using only a rewriting relation $\stackrel{\subseteq}{\longrightarrow}$ and rewriting a to b, we use another rewriting relation $\stackrel{\supseteq}{\longrightarrow}$ as well and seek a common expression c such that $a \stackrel{\subseteq}{\longrightarrow}^* c$ and $b \stackrel{\supseteq}{\longrightarrow}^* c$. Each component of the bi-rewriting system $\langle \stackrel{\subseteq}{\longrightarrow}, \stackrel{\supseteq}{\longrightarrow} \rangle$ is allowed to be a subset of the corresponding inclusion $\subseteq$ or $\supseteq$. In order to assure the decidability and completeness of the proof procedure we study the commutativity of $\stackrel{\subseteq}{\longrightarrow}$ and $\stackrel{\supseteq}{\longrightarrow}$. We also extend the existing techniques of rewriting modulo equalities to bi-rewriting modulo a set of inclusions. We present the canonical bi-rewriting system corresponding to the theory of non-distributive lattices.

1 Introduction

Rewriting systems are usually associated with rewriting on equivalence classes of terms, defined by a set of equations. However term rewriting techniques may be used to compute other relations than congruences. Particularly interesting are non-symmetric relations like pre-orders. For instance, logics of inequalities [7], rewriting logic [21], ordered algebras [8], subset logic [12, 24], unified algebras [2, 22], taxonomies [1, 23, 26], subtypes [5], refinement calculus [20], all them use some kind of pre-order on expressions. In this paper we will show the applicability of rewriting techniques to monotonic pre-order relations on first order terms (inequality logics), that is the deduction of inequalities —here we call them inclusions— from a given set of them, the axioms.

The idea of applying rewriting techniques to the deduction of inclusions between terms, like $a \subseteq b$, is very simple. We compute by repeatedly replacing both 1) subterms of a by "bigger" terms using the axioms and 2) subterms of b by "smaller" terms using the same axioms until a connection is found between a and b. Evidently there are many paths starting from a in the direction $\stackrel{\subseteq}{\longrightarrow}$ and from b in the direction $\stackrel{\supseteq}{\longrightarrow}$ (see figure 2). Many of them are blind alleys and others are

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not terminating. Thus, it is essential that the search avoids blind alleys for efficiency reasons and, specially, avoids infinite sequences of rewritings with infinite different terms (infinite paths due to cycles are avoided easily). Evidently infinite different rewritings would prevent the decidability of the procedure. The solution to non-termination is to orient the axioms using a well founded ordering(s) on terms. Because the relation is non-symmetric, the orientation results in a pair of rewriting systems $\langle \xrightarrow{\subseteq}_{R_1}, \xrightarrow{\supseteq}_{R_2} \rangle$, that is, we get a bi-rewriting system. We introduce the definitions of a Church-Rosser and quasi-terminating bi-rewriting system in order to assure the decidability and the completeness of the search procedure. That is, given a set of axioms, if we can orient and complete them obtaining a confluent bi-rewriting system, then we will have a semi-decidable procedure to test $a \subseteq b$. The procedure is decidable if the bi-rewriting system is quasi-terminating.

Most of the notions of rewriting can be extended to bi-rewriting and the development of the subject follows the same pattern as rewriting: from Church-Rosser property to critical pairs lemma and then the completion process. However there are also some differences. Equational rewriting is in essence a theory of normal forms, while bi-rewriting disregards this notion since is based on quasi-termination and Church-Rosser properties. Bi-rewriting can also be seen as a generalization of equational rewriting: equations can be translated to pairs of inclusions and then we can reproduce the equational case. The price of this generalization is that bi-rewriting is based on a search procedure —which is avoided in canonical rewriting systems— and as we will see in section 2, the set of critical pairs of a non-left-linear bi-rewriting system may be infinite and then the study of confluence is case dependent.[2]

This paper proceeds as follows. In section 2 we present a version of the critical pairs theorem [10, 16, 17] for bi-rewriting systems using an extended definition of critical pairs. We also give a counter-example that invalidates this theorem stated in terms of standard critical pairs and a counter-example for the Toyama theorem [27].

In section 3 we generalize the results of section 2 to bi-rewriting systems modulo a set of (non-orientable) inclusions. We will see that the characterization of Huet for left-linear rules (in terms of α and γ properties [10, lemma 2.8]), the generalization of Peterson and Stickel [25] for non-left-linear rules (in terms of E-compatibility), and the result of Jouannaud & Kirchner [13, 14, 15] (in terms of E-coherence or confluence of cliffs), all of them are not valid for inclusions. We present a new characterization of bi-rewriting modulo a set of inclusions where stronger properties are required. We have divided section 3 in two subsections, the first devoted to abstract bi-rewriting properties and the second to term dependent properties.

In section 4 we present two examples of canonical bi-rewriting systems. We

[2] The possibility of a infinite set of critical pairs does not apply to the translation of a set of equations into a bi-rewriting system. In fact, the set of critical pairs obtained in the translation is a subset of those obtained in the equational case, and the only disadvantage is the loss of efficiency due to the use of a search algorithm.

sketch a method able to handle schemes of critical pairs, which are needed in non-left-linear bi-rewriting systems. We also show some of the disadvantages of modeling inclusions with equations containing unions or intersections.

2 Inclusions and Bi-rewriting Systems

If nothing is said, we follow the notation used in [6, 10, 16]. We shall be concerned with first-order terms over a nonempty signature. We will denote the p occurrence or position in t by $t|_p$, and the substitution of the occurrence p by s in t by $t[s]_p$. We use the relational logic notation to present the abstract bi-rewriting properties. The inverse of the relation $\longrightarrow_R$ will be denoted by $\longleftarrow_R$, its reflexive-transitive closure by $\longrightarrow^*_R$, the transitive composition by $\longrightarrow_{R_1} \circ \longrightarrow_{R_2}$, and the union by $\longrightarrow_{R_1} \cup \longrightarrow_{R_2}$.

An inclusion is an ordered pair of terms $\langle s, t\rangle$ written $s \subseteq t$. Given a finite set of inclusions I, $\subseteq_I$ will denote the monotonic (stable and compatible) closure of I. That is, $u \subseteq_I v$ iff u is $w[\sigma(s)]_p$ and v is $w[\sigma(t)]_p$ for some term w, occurrence p of w, substitution σ and inclusion $s \subseteq t$ in I. The reflexive-transitive closure $\subseteq^*_I$ defines the inclusion theory presented by I.

The orientation of a finite set of inclusions I, for rewriting purposes, may result in two sets of rewriting rules, R_1 with rules like $s \xrightarrow{\subseteq} t$ and R_2 with rules like $s \xrightarrow{\supseteq} t$. The pair $\langle R_1, R_2\rangle$ is called a bi-rewriting system. For example, inclusions defining the union may be oriented as it is shown in figure 1.

$$I = \begin{cases} X \cup X \subseteq X \\ X \subseteq X \cup Y \\ Y \subseteq X \cup Y \end{cases} \qquad R_1 = \left\{ r_1 : X \cup X \xrightarrow{\subseteq} X \right. \qquad R_2 = \begin{cases} r_2 : X \cup Y \xrightarrow{\supseteq} X \\ r_3 : X \cup Y \xrightarrow{\supseteq} Y \end{cases}$$

Fig. 1. Orientation of the inclusion theory of the union.

In this section we suppose that each inclusion may be oriented putting it in R_1 or in R_2, or may be in both sets. In the next section we will consider the case of inclusions which can not be oriented.

Given a bi-rewriting system $\langle R_1, R_2\rangle$ its monotonic closure results in a pair of rewriting relations $\langle \longrightarrow_{R_1}, \longrightarrow_{R_2}\rangle$. Then the relation $(\longrightarrow_{R_1} \cup \longleftarrow_{R_2})^*$ is equal to $\subseteq^*_I$.

Based on the pair of rewriting relations $\langle \longrightarrow_{R_1}, \longrightarrow_{R_2}\rangle$ a sound breadth-first search proof procedure for the inclusion theory $\subseteq^*_I$ can be easily defined (see figure 2). The procedure is complete and semi-decidable iff the bi-rewriting system is Church-Rosser –the branches being enumerable– and it is decidable iff the bi-rewriting system is also quasi-terminating. These two notions are studied

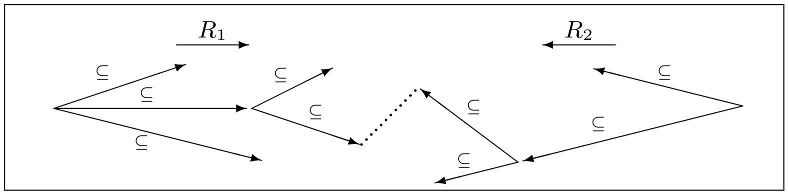

Fig. 2. An image of the bi-rewriting algorithm

in the following paragraph, and defined as extensions of the standard definitions for term rewriting systems.

A bi-rewriting system $\langle R_1, R_2\rangle$ is said to terminate iff $\longrightarrow^*_{R_1}$ and $\longrightarrow^*_{R_2}$ are well founded orderings. It is said to quasi-terminate (globally finite) iff the sets $\{x \mid a \longrightarrow^*_{R_1} x\}$ and $\{x \mid a \longrightarrow^*_{R_2} x\}$ are finite for any term a. It is Church-Rosser iff $(\longrightarrow_{R_1} \cup \longleftarrow_{R_2})^* \subseteq \longrightarrow^*_{R_1} \circ \longleftarrow^*_{R_2}$.

In order to test automatically the Church-Rosser property we extend the standard procedure of rewriting to bi-rewriting. So we reduce the Church-Rosser property to three simpler properties, namely bi-confluence (or commutativity), local bi-confluence and critical pairs bi-confluence.

A bi-rewriting system $\langle R_1, R_2\rangle$ is bi-confluent or commutative iff $\longleftarrow^*_{R_2} \circ \longrightarrow^*_{R_1} \subseteq \longrightarrow^*_{R_1} \circ \longleftarrow^*_{R_2}$. It is locally bi-confluent iff $\longleftarrow_{R_2} \circ \longrightarrow_{R_1} \subseteq \longrightarrow^*_{R_1} \circ \longleftarrow^*_{R_2}$. A pair of terms s, t is bi-confluent $s \downarrow t$ iff there exists u such that $s \longrightarrow^*_{R_1} u$ and $t \longrightarrow^*_{R_2} u$. The Newman's lemma is also true in bi-rewriting systems: a terminating bi-rewriting system is Church-Rosser iff it is locally bi-confluent.

A simple extension of the standard critical pairs definition can be given for bi-rewriting systems. However, as we will see, it is not sufficient to prove the critical pairs lemma [17]. The simple definition of critical pair arises from the most general non-variable overlap between the left hand side of a rule in R_1 and the left hand side of a rule in R_2. Given $l \longrightarrow_{R_1} r$ and $s \longrightarrow_{R_2} t$, a position p of a non-variable subterm of s, and the most general unifier σ of l and $s|_p$, the pair $\sigma(t) \subseteq \sigma(s[r]_p)$ is a critical pair; and the same for critical pairs between R_2 and R_1.

Unfortunately, in the presence of non-left-linear rules, the critical pair lemma can not be proved because the confluence of variable overlaps is no longer possible. Here is a simple counter-example to the validity of this lemma. The bi-rewriting system $\langle\{f(X,X) \stackrel{\subseteq}{\longrightarrow} X\}, \{a \stackrel{\supseteq}{\longrightarrow} b\}\rangle$ has no critical pairs, and $f(a,b) \stackrel{\subseteq}{\longleftarrow} f(a,a) \stackrel{\subseteq}{\longrightarrow} a$ does not satisfy the Church-Rosser property. This problem would be avoided if $\langle a \stackrel{\subseteq}{\longrightarrow} b\rangle \in R_1$.

Non-left-linear rules also unvalidate the Toyama theorem [27] for bi-rewriting systems as the following counter-example shows. The following two bi-rewriting systems

$$R_1 = \begin{cases} X \cup X \xrightarrow{\subseteq} X \\ X \cup Y \xrightarrow{\subseteq} Y \cup X \\ X \cup (Y \cup Z) \xrightarrow{\subseteq} (X \cup Y) \cup Z \end{cases} \qquad R_2 = \begin{cases} X \cup Y \xrightarrow{\supseteq} X \\ X \cup Y \xrightarrow{\supseteq} Y \end{cases}$$

and

$$R'_1 = \begin{cases} X \cap Y \xrightarrow{\subseteq} X \\ X \cap Y \xrightarrow{\subseteq} Y \end{cases} \quad R'_2 = \begin{cases} X \cap X \xrightarrow{\supseteq} X \\ X \cap Y \xrightarrow{\supseteq} Y \cap X \\ X \cap (Y \cap Z) \xrightarrow{\supseteq} (X \cap Y) \cap Z \end{cases}$$

are both Church-Rosser and have disjoint alphabets, but their union $\langle R_1 \cup R'_1, R_2 \cup R'_2 \rangle$ is not Church-Rosser as the following rewriting sequence shows.[3]

$$(A{\cap}B){\cup}(A{\cap}C) \xleftarrow[R_2]{\subseteq} \big(A{\cap}(B{\cup}C)\big){\cup}(A{\cap}C) \xleftarrow[R_2]{\subseteq} \big(A{\cap}(B{\cup}C)\big){\cup}\big(A{\cap}(B{\cup}C)\big) \xrightarrow[R_1]{\subseteq} A{\cap}(B{\cup}C)$$

Using the previous definition of critical pairs, the critical pairs lemma is only true for left-linear systems: a terminating and left-linear bi-rewriting system is Church-Rosser iff all critical pairs are bi-confluent. In order to keep this lemma for non-left-linear bi-rewriting systems, we have to enlarge the set of critical pairs as follows.

Definition 1. If $\langle \alpha_1 \xrightarrow{\subseteq} \beta_1 \rangle \in R_1$ and $\langle \alpha_2 \xrightarrow{\supseteq} \beta_2 \rangle \in R_2$ are two rewriting rules (with variables distinct) and p a position in α_1, then

1. if $\alpha_1|_p$ is non-variable subterm and σ is the most general unifier of $\alpha_1|_p$ and α_2 then $\langle \sigma(\alpha_1[\beta_2]_p), \sigma(\beta_1) \rangle$ is a (standard) critical pair,
2. if $\alpha_1|_p = x$ is a repeated variable in α_1, F a term $x \notin \mathcal{V}(F)$, q an occurrence in F, and $\alpha_2 \longrightarrow^*_{R_1} \beta_2$ is not satisfied,[4] then $\langle \sigma(\alpha_1[F[\beta_2]_q]_p), \sigma(\beta_1) \rangle$ is an (extended) critical pair where σ only substitutes x by $F[\alpha_2]_q$.

The same for critical pairs between R_2 and R_1.

The set of (extended) critical pairs of the previous definition is in general infinite —$\langle \sigma(\alpha_1[F[\beta_2]_q]_p), \sigma(\beta_1) \rangle$ is a critical pair scheme— (in section 4 we will see two examples using these schemes). So the critical pairs lemma even if true with this definition of critical pairs, will be of little practical help to test bi-confluence. Then the conditions of confluence have to be studied in each case taking into account the particular shape of the non-left-linear rules.

Nevertheless, if all rules come from the translation of an equational theory then we can always have $\langle \alpha \xrightarrow{\subseteq} \beta \rangle \in R_1$ iff $\langle \alpha \xrightarrow{\supseteq} \beta \rangle \in R_2$ and the extended critical pairs schemes will not appear.[5] Notice also that an inclusion $a \subseteq b$ could be used by both R_1 and R_2 systems —as rules $a \xrightarrow{\subseteq} b$ and $b \xrightarrow{\supseteq} a$— without losing necessarily the termination property of the bi-rewriting system $\langle R_1, R_2 \rangle$.[6]

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[3] The non-confluence of this inclusion sequence is due to the addition of new symbols in the signature, not to the addition of new rules.

[4] If this condition is satisfied then we can make the pair confluent like in the equational case.

[5] Any equation $a = b$ is translated into $a \subseteq b$ and $b \subseteq a$ and these are oriented as $a \xrightarrow{\subseteq} b$ and $a \xrightarrow{\supseteq} b$, in the case we have the same orientation ordering for R_1 and R_2.

[6] Both rewriting systems can have different orientation orderings.

3 Bi-rewriting Modulo a Set of Inclusions

Like in equational rewriting, in bi-rewriting it is not always possible to orient all inclusions of a theory presentation in two terminating rewrite relations, as shown in the previous section. Frequently enough, we must handle three rewrite relations, the terminating relations $\longrightarrow_{R_1}$ and $\longrightarrow_{R_2}$ resulting from the inclusions oriented to the right and to the left respectively, and the non-terminating relation $\longrightarrow_{I}$ resulting from the non-oriented inclusions. We name these three relations a $\langle R_1, R_2\rangle$ bi-rewriting system modulo I.[7] Figure 3 shows an example of them.

3.1 From Church-Rosser to Local Confluence

The simplest way to have a complete and decidable proof procedure for $\langle R_1, R_2\rangle$ modulo I is reducing it to the bi-rewriting system $\langle R_1 \cup I, R_2 \cup I\rangle$ and, like in the previous section, to require of it the following properties

$$\xrightarrow[R_1]{} \cup \xrightarrow[I]{} \text{ and } \xrightarrow[R_2]{} \cup \xleftarrow[I]{} \text{are quasi-terminating, and} \tag{1}$$

$$(\xrightarrow[R_1]{} \cup \xrightarrow[I]{} \cup \xleftarrow[R_2]{})^* \subseteq (\xrightarrow[R_1]{} \cup \xrightarrow[I]{})^* \circ (\xleftarrow[R_2]{} \cup \xrightarrow[I]{})^* \tag{2}$$

However, the quasi-termination of $\longrightarrow_{R_1} \cup \longrightarrow_{I}$ and $\longrightarrow_{R_2} \cup \longleftarrow_{I}$ is not enough to reduce the property (2) —called $\langle R_1, R_2\rangle$ weak Church-Rosser modulo I— to the corresponding local bi-confluence (4). To do this we would need the (strong) termination of $\longrightarrow_{R_1} \cup \longrightarrow_{I}$ and of $\longrightarrow_{R_2} \cup \longleftarrow_{I}$, which are not true. The solution to this problem comes from requiring the following property stronger than (1)

$$\xrightarrow[I]{*} \circ \xrightarrow[R_1]{} \text{ and } \xleftarrow[I]{*} \circ \xrightarrow[R_2]{} \text{ are terminating, and } \xrightarrow[I]{} \text{ is quasi-terminating} \tag{3}$$

Notice that from the fact $(\longrightarrow_{I} \cup \longrightarrow_{R})^* = (\longrightarrow_{I}^{*} \circ \longrightarrow_{R})^* \circ \longrightarrow_{I}^{*}$ one can see that (3) implies (1). Using the stronger termination property (3), the weak Church-Rosser property (2) can be reduced to the following local confluence property:

$$\xleftarrow[R_2]{} \circ \xrightarrow[I]{*} \circ \xrightarrow[R_1]{} \subseteq (\xrightarrow[I]{*} \circ \xrightarrow[R_1]{})^* \circ \xrightarrow[I]{*} \circ (\xleftarrow[R_2]{} \circ \xrightarrow[I]{*})^* \tag{4}$$

The equivalence of (2) and (4) can be proved using noetherian induction on $\longrightarrow_{I}^{*} \circ \longrightarrow_{R_1}$ and $\longleftarrow_{I}^{*} \circ \longrightarrow_{R_2}$. In fact, to prove this equivalence, it is not necessary for $\longrightarrow_{I}$ to be quasi-terminating. If $\longrightarrow_{I}$ is symmetric the above termination property (3) becomes similar to the termination property required in rewriting modulo a set of equations [3]. That is, I symmetric means we can define equivalence classes ($[s]_I \longrightarrow_{R} [t]_I$ iff $s \longrightarrow_{I}^{*} \circ \longrightarrow_{R} \circ \longrightarrow_{I}^{*} t$) and, the termination of $\longrightarrow_{I}^{*} \circ \longrightarrow_{R_1}$ and $\longleftarrow_{I}^{*} \circ \longrightarrow_{R_2}$ is equivalent to the existence of two well founded I-compatible order relations $\succ_1$ and $\succ_2$ satisfying $\longrightarrow_{R_1} \subseteq \succ_1$ and $\longrightarrow_{R_2} \subseteq \succ_2$;

[7] Although we use the word "modulo", it does not mean that $\longrightarrow_{I}^{*}$ is a congruence, be aware it is a non-symmetric relation (monotonic pre-order).

and the quasi-termination of $\longrightarrow_I$ is equivalent to the finiteness of the equivalence classes.

However, we know by analogy with rewriting modulo a set of equations, that the proof procedure based on these properties is not a practical one. Like in the equational case, rewriting by $\longrightarrow^*_I \circ \longrightarrow_R$ is inefficient, if decidable at all. Therefore we will approximate it by a weaker, but more practical notion of bi-rewriting named $\langle I\backslash R_1, I^{-1}\backslash R_2\rangle$ by similarity to the corresponding equational definitions. As we will see later, this new rewriting relation will have to satisfy what is called a $\langle I\backslash R_1, I^{-1}\backslash R_2\rangle$ strong Church-Rosser modulo I property, defined as follows:

$$(\xrightarrow[I\backslash R_1]{} \cup \xrightarrow[I]{} \cup \xleftarrow[I^{-1}\backslash R_2]{})^* \subseteq \xrightarrow[I\backslash R_1]{*} \circ \xrightarrow[I]{*} \circ \xleftarrow[I^{-1}\backslash R_2]{*} \qquad (5)$$

This property plus the quasi-termination of $I\backslash R_1$ and $I^{-1}\backslash R_2$ and the decidability of the I-unification are sufficient to have a more efficient complete and decidable proof procedure. The solution we will propose comes mainly from the two solutions known for the equational case [13, 25]. In the following we consider how they can be adapted to bi-rewriting.

Huet [10] and Jouannaud & Kirchner [14, 13] have proved that given a set of rules R and equations E such that $\longleftrightarrow^*_E \circ \longrightarrow_R$ is terminating, R is strong Church-Rosser modulo E iff all peaks and cliffs are confluent: $\longleftarrow_R \circ \longrightarrow_R \subseteq \longrightarrow^*_R \circ \longleftrightarrow^*_E \circ \longleftarrow^*_R$ and $\longleftrightarrow_E \circ \longrightarrow_R \subseteq \longrightarrow^*_R \circ \longleftrightarrow^*_E \circ \longleftarrow^*_R$. Notice these are sufficient and, what is also important, necessary conditions. Besides, the finiteness of the E-equivalence classes is not required. These confluence properties, stated by Huet, are too strong and can not be reduced to the confluence of critical pairs unless the rules are left-linear. To overcome this limitation of non-left-linear systems Jouannaud & Kirchner [3, 6, 13] propose a new rewriting relation $E\backslash R$ satisfying $\longrightarrow_R \subseteq \longrightarrow_{E\backslash R} \subseteq \longleftrightarrow^*_E \circ \longrightarrow_R$. This relation is proved to be strong Church-Rosser modulo E iff all critical peaks $\longleftarrow_R \circ \longrightarrow_{E\backslash R}$ and critical cliffs $\longleftrightarrow_E \circ \longrightarrow_R$ are confluent. Then this confluence can be reduced to critical pairs confluence and to extended rules. We are interested in extending the same kind of result to bi-rewriting systems because on it are based the proof and completion procedures.

Then the direct translation of the previous result to the bi-rewriting case may be stated as follows. $\langle R_1, R_2\rangle$ is strong Church-Rosser modulo I iff $\longleftarrow_{R_2} \circ \longrightarrow_{R_1} \subseteq \longrightarrow^*_{R_1} \circ \longrightarrow^*_I \circ \longleftarrow^*_{R_2}$ and $\longrightarrow_I \circ \longrightarrow_{R_1} \subseteq \longrightarrow^*_{R_1} \circ \longrightarrow^*_I \circ \longleftarrow^*_{R_2}$ and $\longleftarrow_{R_2} \circ \longrightarrow_I \subseteq \longrightarrow^*_{R_1} \circ \longrightarrow^*_I \circ \longleftarrow^*_{R_2}$ where $\longrightarrow^*_I \circ \longrightarrow_{R_1}$ and $\longleftarrow^*_I \circ \longrightarrow_{R_2}$ are terminating. Unfortunately this result is not true unless $\longrightarrow_{R_1}$ and $\longrightarrow_{R_2}$ have the same set of normal forms, which is semantically meaningless. Here is a counter-example of its validity. Let $I = \{a \xleftrightarrow{\subseteq} b, c \xleftrightarrow{\subseteq} d\}$, $R_1 = \{b \xrightarrow{\subseteq} c\}$ and $R_2 = \{c \xrightarrow{\supseteq} b\}$, then $\longrightarrow^*_I \circ \longrightarrow_{R_1}$ and $\longleftarrow^*_I \circ \longrightarrow_{R_2}$ are terminating and all peaks and cliffs are confluent, nevertheless $a \longrightarrow_I b \longrightarrow_{R_1} c \longrightarrow_I d$ is not confluent.[8]

Another way of having the strong Church-Rosser property is by means of the stronger requirement on R rewriting modulo E given by Peterson &

[8] Note that if we translate the counter-example to the classical case defining $\longrightarrow_R \stackrel{def}{=} \longrightarrow_{R_1} \cup \longrightarrow_{R_2}$, then R becomes non-terminating, and the hypothesis of the Jouannaud theorem is not satisfied.

Stickel in [25]. They define a rewriting relation between E-equivalence classes which can be modeled by $(\longleftrightarrow_E^* \circ \longrightarrow_R)^* \circ \longleftrightarrow_E^*$. They also formulate what they call an E-completeness property, equivalent to what we have called weak Church-Rosser property. They were the first to propose the mentioned relation $E\backslash R$. When this relation is E-compatible, that is, when $\longleftrightarrow_E^* \circ \longrightarrow_R \subseteq \longrightarrow_{E\backslash R} \circ \longleftrightarrow_E^* \circ (\longleftarrow_R \circ \longleftrightarrow_E^*)^*$, then the corresponding weak and strong Church-Rosser properties both become equivalent to the peaks confluence property $\longleftarrow_{E\backslash R} \circ \longrightarrow_{E\backslash R} \subseteq \longrightarrow_{E\backslash R}^* \circ \longleftrightarrow_E^* \circ \longleftarrow_{E\backslash R}^*$. The E-compatibility is not a necessary condition although it is a sufficient one. To adapt this same result to the bi-rewriting case we will need a requirement even stronger than E-compatibility, as shown below.

Given a $\langle R_1, R_2 \rangle$ bi-rewriting system modulo I, the problem is to find which requirements two new relations $I\backslash R_1$ and $I^{-1}\backslash R_2$ have to satisfy in order to prove (5), the $\langle I\backslash R_1, I^{-1}\backslash R_2 \rangle$ strong Church-Rosser modulo I property. Since $I\backslash R_1$ and $I^{-1}\backslash R_2$ are required to satisfy at least $\longrightarrow_{R_1} \subseteq \longrightarrow_{I\backslash R_1} \subseteq \longrightarrow_I^* \circ \longrightarrow_{R_1}$ and $\longrightarrow_{R_2} \subseteq \longrightarrow_{I^{-1}\backslash R_2} \subseteq \longleftarrow_I^* \circ \longrightarrow_{R_2}$, the termination of $\longrightarrow_I^* \circ \longrightarrow_{R_1}$ and $\longleftarrow_I^* \circ \longrightarrow_{R_2}$ ensures the termination of $\longrightarrow_I^* \circ \longrightarrow_{I\backslash R_1}$ and $\longleftarrow_I^* \circ \longrightarrow_{I^{-1}\backslash R_2}$. From a computational point of view, this relations are to be based on the suppression of those applications of $\longrightarrow_I$ in $\longrightarrow_I^* \circ \longrightarrow_R$ not conducting to a new way of applying $\longrightarrow_R$ later. That is, with the new relations, all this unnecessary I-rewritings before R-rewritings could be suppressed or moved to the final I-unification. This requirement is captured by the following local commutativity property of I and $I\backslash R_1$, and of I^{-1} and $I^{-1}\backslash R_2$:

$$\begin{array}{l} \xrightarrow[I]{} \circ \xrightarrow[I\backslash R_1]{} \subseteq \xrightarrow[I\backslash R_1]{*} \circ \xrightarrow[I]{*} \\ \xleftarrow[I]{} \circ \xrightarrow[I^{-1}\backslash R_2]{} \subseteq \xrightarrow[I^{-1}\backslash R_2]{*} \circ \xleftarrow[I]{*} \end{array} \tag{6}$$

These requirements are stronger than the E-compatibility in [25] and the confluence of cliffs in [13]. Furthermore, if $\longrightarrow_I^* \circ \longrightarrow_{R_1}$ and $\longleftarrow_I^* \circ \longrightarrow_{R_2}$ are terminating then $\longrightarrow_I^* \circ \longrightarrow_{I\backslash R_1}$ and $\longleftarrow_I^* \circ \longrightarrow_{I^{-1}\backslash R_2}$ are also terminating and the local commutativities are equivalent to the global commutativities: $\longrightarrow_I^* \circ \longrightarrow_{I\backslash R_1}^* \subseteq \longrightarrow_{I\backslash R_1}^* \circ \longrightarrow_I^*$ and $\longleftarrow_I^* \circ \longrightarrow_{I^{-1}\backslash R_2}^* \subseteq \longrightarrow_{I^{-1}\backslash R_2}^* \circ \longleftarrow_I^*$. These global commutativity properties lead to the equivalence of the $\langle I\backslash R_1, I^{-1}\backslash R_2 \rangle$ weak Church-Rosser and the $\langle I\backslash R_1, I^{-1}\backslash R_2 \rangle$ strong Church-Rosser modulo I properties. On the other hand, using the previously proved equivalence between weak Church-Rosser (2) and local bi-confluence (4) modulo I, the $\langle I\backslash R_1, I^{-1}\backslash R_2 \rangle$ weak Church-Rosser property becomes equivalent to the following local bi-confluence property:

$$\xleftarrow[I^{-1}\backslash R_2]{} \circ \xrightarrow[I]{*} \circ \xrightarrow[I\backslash R_1]{} \subseteq \Big(\xrightarrow[I]{*} \circ \xrightarrow[I\backslash R_1]{}\Big)^* \circ \xrightarrow[I]{*} \circ \Big(\xleftarrow[I^{-1}\backslash R_2]{} \circ \xrightarrow[I]{*}\Big)^*$$

And again, the commutativity properties and the inclusions $\longrightarrow_{R_1} \subseteq \longrightarrow_{I\backslash R_1}$ and $\longrightarrow_{R_2} \subseteq \longrightarrow_{I^{-1}\backslash R_2}$ allows us to reduce this condition to the following one

$$\xleftarrow[R_2]{} \circ \xrightarrow[I]{*} \circ \xrightarrow[R_1]{} \subseteq \xrightarrow[I\backslash R_1]{*} \circ \xrightarrow[I]{*} \circ \xleftarrow[I^{-1}\backslash R_2]{*} \tag{7}$$

and from this to whatever of the following ones

$$\xleftarrow[I^{-1}\backslash R_2]{} \circ \xrightarrow[R_1]{} \subseteq \xrightarrow[I\backslash R_1]{*} \circ \xrightarrow[I]{*} \circ \xleftarrow[I^{-1}\backslash R_2]{*}$$

$$\text{or } \xleftarrow[R_2]{} \circ \xrightarrow[I\backslash R_1]{} \subseteq \xrightarrow[I\backslash R_1]{*} \circ \xrightarrow[I]{*} \circ \xleftarrow[I^{-1}\backslash R_2]{*}$$

This results can be summarized in the following lemma:

Lemma 2. *If* $\xrightarrow[I]{*} \circ \xrightarrow[R_1]{}$ *and* $\xleftarrow[I]{*} \circ \xrightarrow[R_2]{}$ *are terminating, and*

$$\begin{aligned}
\xrightarrow[R_1]{} &\subseteq \xrightarrow[I\backslash R_1]{} \subseteq \xrightarrow[I]{*} \circ \xrightarrow[R_1]{} \\
\xrightarrow[R_2]{} &\subseteq \xrightarrow[I^{-1}\backslash R_2]{} \subseteq \xleftarrow[I]{*} \circ \xrightarrow[R_2]{} \\
\xrightarrow[I]{} \circ \xrightarrow[I\backslash R_1]{} &\subseteq \xrightarrow[I\backslash R_1]{*} \circ \xrightarrow[I]{*} \\
\xleftarrow[I^{-1}\backslash R_2]{} \circ \xrightarrow[I]{} &\subseteq \xrightarrow[I]{*} \circ \xleftarrow[I^{-1}\backslash R_2]{*} \\
\xleftarrow[R_2]{} \circ \xrightarrow[I]{*} \circ \xrightarrow[R_1]{} &\subseteq \xrightarrow[I\backslash R_1]{*} \circ \xrightarrow[I]{*} \circ \xleftarrow[I^{-1}\backslash R_2]{*}
\end{aligned}$$

then $\langle I\backslash R_1, I^{-1}\backslash R_2\rangle$ *is strongly Church-Rosser modulo* I*.*

This lemma reproduces adapted to bi-rewriting the results of Huet, Peterson, Stickel, Jouannaud and Kirchner but is based on stronger properties.

A generalization of lemma 2 was given in [18], and we summarize it bellow. The set I of non-oriented inclusions of a theory presentation is divided into two subsets I_1 and I_2 ($I = I_1 \cup I_2$). From them two non-terminating rewrite relations $\longrightarrow_{I_1}$ and $\longrightarrow_{I_2}$ can be defined such that $(\longrightarrow_{R_1} \cup \longrightarrow_{I_1} \cup \longleftarrow_{I_2} \cup \longleftarrow_{R_2})^*$ corresponds to the inclusion theory. These four relations constitute a $\langle R_1, R_2\rangle$ bi-rewriting system modulo $\langle I_1, I_2\rangle$. We say that such a system is strong Church-Rosser iff

$$(\xrightarrow[R_1]{} \cup \xrightarrow[I_1]{} \cup \xleftarrow[I_2]{} \cup \xleftarrow[R_2]{})^* \subseteq \xrightarrow[R_1]{*} \circ \xrightarrow[I_1]{*} \circ \xleftarrow[I_2]{*} \circ \xleftarrow[R_2]{*}$$

Then the generalization of lemma 2 can be stated as follows:

Lemma 3. *If* $\xrightarrow[I_1]{*} \circ \xrightarrow[R_1]{}$ *and* $\xrightarrow[I_2]{*} \circ \xrightarrow[R_2]{}$ *are terminating, and*

$$\begin{aligned}
\xrightarrow[I_1]{} \circ \xrightarrow[I_1\backslash R_1]{} &\subseteq \xrightarrow[I_1\backslash R_1]{*} \circ \xrightarrow[I_1]{*} \\
\xleftarrow[I_2\backslash R_2]{} \circ \xleftarrow[I_2]{} &\subseteq \xleftarrow[I_2]{*} \circ \xleftarrow[I_2\backslash R_2]{*} \\
\xleftarrow[I_2]{*} \circ \xrightarrow[I_1]{*} &\subseteq \xrightarrow[I_1\backslash R_1]{*} \circ \xrightarrow[I_1]{*} \circ \xleftarrow[I_2]{*} \circ \xleftarrow[I_2\backslash R_2]{*} \\
\xleftarrow[I_2\backslash R_2]{} \circ \xrightarrow[I_1]{*} &\subseteq \xrightarrow[I_1\backslash R_1]{*} \circ \xrightarrow[I_1]{*} \circ \xleftarrow[I_2]{*} \circ \xleftarrow[I_2\backslash R_2]{*} \\
\xleftarrow[I_2\backslash R_2]{} \circ \xrightarrow[R_1]{} &\subseteq \xrightarrow[I_1\backslash R_1]{*} \circ \xrightarrow[I_1]{*} \circ \xleftarrow[I_2]{*} \circ \xleftarrow[I_2\backslash R_2]{*} \\
\xleftarrow[I_2]{*} \circ \xrightarrow[I_1\backslash R_1]{} &\subseteq \xrightarrow[I_1\backslash R_1]{*} \circ \xrightarrow[I_1]{*} \circ \xleftarrow[I_2]{*} \circ \xleftarrow[I_2\backslash R_2]{*} \\
\xleftarrow[R_2]{} \circ \xrightarrow[I_1\backslash R_1]{} &\subseteq \xrightarrow[I_1\backslash R_1]{*} \circ \xrightarrow[I_1]{*} \circ \xleftarrow[I_2]{*} \circ \xleftarrow[I_2\backslash R_2]{*}
\end{aligned}$$

then $\langle R_1, R_2\rangle$ *is (strongly) Church-Rosser modulo* $\langle I_1, I_2\rangle$*.*

The generalization comes from the fact that now the commutativity property is only required between $I_1\backslash R_1$ and I_1 and between $I_2\backslash R_2$ and I_2, and is not needed between $I_1\backslash R_1$ and I_2 or $I_2\backslash R_2$ and I_1.

Till now, we have studied Church-Rosser, termination, confluence and local confluence properties in the framework of relational algebra [4]. All proofs can be done without references to the structure of terms. In the following subsection we will consider the term structure in order to reduce the local confluence properties to the confluence of (extended) critical pairs.

3.2 From Local Confluence to (Extended) Critical Pairs

We begin defining the rewrite relations $I\backslash R_1$ and $I^{-1}\backslash R_2$ that were only axiomatically characterized by the commutativity and local confluence properties in the previous subsection.

Definition 4. We say that s rewrites to t modulo I at $[p, \sigma, \alpha \longrightarrow \beta]$, written $s \longrightarrow_{I\backslash R} t$, iff there exists a rule $\langle \alpha \longrightarrow \beta \rangle \in R$, an occurrence p in s, and a substitution σ such that $s|_p \longrightarrow_I^* \sigma(\alpha)$ and $t = s[\sigma(\beta)]_p$.

With this definition $I\backslash R$ verifies $\longrightarrow_R \subseteq \longrightarrow_{I\backslash R} \subseteq \longrightarrow_I^* \circ \longrightarrow_R$ (although in general $\longrightarrow_I^* \circ \longrightarrow_R \not\subseteq \longrightarrow_{I\backslash R}$). The relations $\longrightarrow_{I\backslash R_1}$ and $\longrightarrow_{I^{-1}\backslash R_2}$ are defined in this way.

We are using the notions of E-matching and E-unification from [25] but adapted to bi-rewriting. Given two terms s and t, we say that s I-matches t iff there exists a substitution σ such that $s \longrightarrow_I^* \sigma(t)$, and we say that s I-unify with t iff there exists a substitution σ such that $\sigma(s) \longrightarrow_I^* \sigma(t)$. Notice that, since $\longrightarrow_I$ is not necessarily symmetric s I-unify t is equivalent to t I^{-1}-unify s, but not to t I-unify s. We will suppose in the following that I-unification and I and I^{-1}-matching are decidable.

As in the equational case (to prove confluence of cliffs or E-compatibility), we will prove the commutativity properties by means of the extensionally closed property defined as follows.

Definition 5. Given a set of rules R and inclusions I, R is said to be right (left) I-extensionally closed iff whenever $\langle \alpha_1 \subseteq \beta_1 \rangle \in I$, $\langle \alpha_2 \longrightarrow \beta_2 \rangle \in R$, $\beta_1|_p$ ($\alpha_1|_p$) and α_2 I-unify (I^{-1}-unify) with minimum unifier σ and $\beta_1|_p$ ($\alpha_1|_p$) is not a variable, then $\sigma(\alpha_1) \longrightarrow_{I\backslash R} \sigma(\beta_1[\beta_2]_p)$ (then $\sigma(\beta_1) \longrightarrow_{I^{-1}\backslash R} \sigma(\alpha_1[\beta_2]_p)$).

Since $\longrightarrow_I$ is non-symmetric, we have had to distinguish between right and left extensionally closed in the previous definition. We will suppose in the following that R_1 is right I-extensionally closed, and that R_2 is right I^{-1}-extensionally closed, or what is the same left I-extensionally closed.

Let's study now the conditions for the satisfiability of lemma 2. This conditions will be the premises of theorem 7. The rest of the section is an sketch of the proof of this theorem.

We start with the commutativity properties (6). Both properties may be generalized to $\longrightarrow_I \circ \longrightarrow_{I\backslash R} \subseteq \longrightarrow_{I\backslash R}^* \circ \longrightarrow_I^*$ where I and $I\backslash R$ stands for I and $I\backslash R_1$ in one case, and for I^{-1} and $I^{-1}\backslash R_2$ in the other. Suppose $a \longrightarrow_I b$ at $[p_1, \sigma_1, \alpha_1 \longrightarrow_I \beta_1]$ and $b \longrightarrow_{I\backslash R} c$ at $[p_2, \sigma_2, \alpha_2 \longrightarrow_R \beta_2]$, where p_i are positions, σ_i are substitutions, $\alpha_1 \longrightarrow_I \beta_1$ is an inclusion and $\alpha_2 \longrightarrow_R \beta_2$ is a rule. We have to consider the following three cases in its commutativity.

case $p_1|p_2$ It can be easily proved that $a \longrightarrow_{I\backslash R} d \longrightarrow_I c$ where $d = a[\sigma_2(\beta_2)]_{p_2} = b[\sigma_1(\alpha_1)]_{p_1}[\sigma_2(\beta_2)]_{p_2} = c[\sigma_1(\alpha_1)]_{p_1}$.

case $p_1 \prec p_2$ Let v satisfy $p_2 = p_1 \cdot v$. We have $\beta_1|_v$ I-unify α_2. If $\beta_1|_v$ is not a variable, we are in the conditions of definition 5, and if R is right I-extensionally closed, then $a \longrightarrow_{I \backslash R} c$ at $[p_1, \sigma, \alpha_2 \longrightarrow_R \beta_2]$ for some σ.
Otherwise, there exist two occurrences v_1 and v_2 satisfying $p_1 \cdot v_1 \cdot v_2 = p_2$ and $\beta_1|_{v_1} = x$, x being a variable. If all inclusions in I are left linear (and non-erasing) then x occurs once in α_1. Let v'_1 be this occurrence. It can be proved that $a \longrightarrow_{I \backslash R} d$ at $[p'_2, \sigma_2, \alpha_2 \longrightarrow_R \beta_2]$ and $d \longrightarrow_I c$ at $[p_1, \sigma'_1, \alpha_1 \longrightarrow_I \beta_1]$ where $p'_2 = p_1 \cdot v'_1 \cdot v_2$, $\sigma'_1(y) = \sigma_1(y)$ for $y \neq x$ and $\sigma'_1(x) = \sigma_1(x)[\sigma_2(\beta_2)]_{v_2}$ and $d = c[\sigma'_1(\alpha_1)]_{p_1} = a[\sigma_2(\beta_2)]_{p'_2}$.

case $p_1 \succeq p_2$ Let v be the occurrence such that $p_2 \cdot v = p_1$. We have $a|_{p_2} \longrightarrow_I b|_{p_2}$ at $[v, \sigma_1, \alpha_1 \longrightarrow_I \beta_1]$ and therefore $a \longrightarrow_{I \backslash R} c$ at $[p_2, \sigma_2, \alpha_2 \longrightarrow_R \beta_2]$.

It must be noticed that like in [25], and differently from [13], the inclusions in I are required to be right-linear in order to prove commutativity of $\longrightarrow_I$ and $\longrightarrow_{I \backslash R_1}$, and left-linear in order to prove commutativity of $\longleftarrow_I$ and $\longrightarrow_{I^{-1} \backslash R_2}$; so, all inclusions in I have to be linear. If all inclusions are left- or right-linear, but they are not all linear, then we can oversee this problem using lemma 3 by putting right-linear inclusions in I_1 and left-linear inclusions in I_2.

Let's study now the condition (7) for the confluence of peaks. Suppose we have $a \longleftarrow_{R_2} b \longrightarrow^*_I c \longrightarrow_{R_1} d$ where reduction $c \longrightarrow_{R_1} d$ takes place at p_1 and $b \longrightarrow_{R_2} a$ at p_2. Three cases must be considered:

case $p_1 | p_2$ We can reduce the problem to the confluence of $a \longleftarrow_{R_2} b \longrightarrow_{I \backslash R_1} d'$ where reductions also take place at $[p_1, \sigma_1, \alpha_1 \longrightarrow_{R_1} \beta_1]$ and $[p_2, \sigma_2, \alpha_2 \longrightarrow_{R_2} \beta_2]$, and as in the commutativity case, both reductions can be permuted.

case $p_1 \prec p_2$ The middle I rules commute with R_2 in (7) and the problem is reduced to the confluence of $a' \longleftarrow_{I^{-1} \backslash R_2} c \longrightarrow_{R_1} d$. This case is equal to the next one if we exchange the indexes 1 by 2 and and we reverse the order of the relations in both sides of the inclusion.

case $p_1 \succeq p_2$ We commute I with R_1 in (7) and we test the confluence of $\longleftarrow_{R_2} \circ \longrightarrow_{I \backslash R_1}$. The previous case, as well as this one are generalized by the confluence of $a \longleftarrow_{R_2} b \longrightarrow_{I \backslash R_1} c$ where reductions take place at p_1 and p_2 respectively, and $p_1 \succeq p_2$. It corresponds to the equational case in the study of the confluence of $\longleftarrow_R \circ \longrightarrow_{E \backslash R}$ where we can always suppose that the $E \backslash R$ reduction takes place below the R reduction. As we have seen in the previous section, if there is a variable overlap, and the rule used in $b \longrightarrow_{R_2} a$ is left-linear or $\alpha_1 \longrightarrow^*_{R_2} \beta_1$ is satisfied, the pair is always confluent. Otherwise we have to include this kind of overlap in the critical pairs definition given below.

Definition 6. If $\langle \alpha_1 \xrightarrow{\subseteq} \beta_1 \rangle \in R_1$ and $\langle \alpha_2 \xrightarrow{\supseteq} \beta_2 \rangle \in R_2$ are two rewriting rules normalized apart, and p is a position in α_1, then

1. if $\alpha_2|_p$ is not a variable and σ is a minimum I-unifier of $\alpha_2|_p$ and α_1, then $\langle \sigma(\beta_2), \sigma(\alpha_2[\beta_1]_p) \rangle$ is a (standard) critical pair,

2. if $\alpha_2|_p = x$ is a repeated variable in α_2, F is a term $x \notin \mathcal{V}(F)$, q a position in F, and $\alpha_1 \longrightarrow^*_{R_2} \beta_1$ is not satisfied, then $\langle \sigma(\beta_2), \sigma(\alpha_2[F[\beta_1]_q]_p) \rangle$ is an (extended) critical pair where the domain of σ is $\{x\}$ and $\sigma(x) = F[\alpha_1]_q$.

$ECP(I\backslash R_1, R_2)$ denotes this set of standard and extended critical pairs. The set $ECP(R_1, I^{-1}\backslash R_2)$ can be defined similarly.

Again we have had to introduce critical pair schemes which may generate infinite critical pairs. Using this extended definition of critical pairs we can prove the following theorem which characterizes the strong Church-Rosser property of a $\langle R_1, R_2 \rangle$ bi-rewriting system modulo I.

Theorem 7. *Given two sets of rules R_1 and R_2 and a set of inclusions I, if $I^* R_1$ and $I^{-1^*} R_2$ are terminating, R_1 is right I-extensionally closed, R_2 left I-extensionally closed, all inclusions in I are linear, and all standard and extended critical pairs $ECP(I\backslash R_1, R_2)$ and $ECP(R_1, I^{-1}\backslash R_2)$ are confluent, then $\langle I\backslash R_1, I^{-1}\backslash R_2 \rangle$ is (strongly) Church-Rosser modulo I.*

4 Two Examples: Towards a Completion Procedure

As we said in the previous sections, bi-rewriting compared with equational rewriting, faces the extra difficulty of a possible infinite set of critical pairs. Non-left-linear rules may generate what we called critical pair schemes (see definitions 1 and 6). The process of completion with these schemes is an open problem. In this section instead of giving the completion procedure we sketch out the possibilities of completion of two examples of bi-rewriting by means of rule schemes.

4.1 Inclusion Theory of the Union Operator

Figure 1 shows the first bi-rewriting system that we want to complete corresponding to the union operator. Its termination can be proved using the interpretation $|X \cup Y| = |X| + |Y|$. Although the standard critical pairs (scp) of this system are confluent, the presence of the non-left-linear rule $X \cup X \xrightarrow{\subseteq} X$ also makes necessary the consideration of the extended critical pairs (ecp). We will do this in two steps dividing the set of ecp in two subsets. First, we consider scp and the finite subset of ecp of the particular form $\langle \sigma(\alpha_1[\beta_2]_p), \sigma(\beta_1) \rangle$ where $\alpha_1|_p = x$ is a repeated variable in the non-left-linear rule $\langle \alpha_1 \xrightarrow{\subseteq} \beta_1 \rangle \in R_1$, $\langle \alpha_2 \xrightarrow{\supseteq} \beta_2 \rangle \in R_2$ being the other rule, and σ substitutes x by α_2. Between all these critical pairs we may focus into the following two sequences of oriented rules and non-oriented inclusions:

r_4 $Y \cup (X \cup Y) \xrightarrow{\subseteq} X \cup Y$ ecp from r_1 and r_3
r_5 $Y \cup X \xleftrightarrow{\subseteq} X \cup Y$ scp from r_2 and r_4

r_6 $(X \cup Y) \cup Y \xrightarrow{\subseteq} X \cup Y$ ecp from r_1 and r_3
r_7 $(X \cup Y) \cup (Y \cup Z) \xrightarrow{\subseteq} X \cup (Y \cup Z)$ ecp from r_2 and r_6
r_8 $(X \cup Y) \cup Z \xleftrightarrow{\subseteq} X \cup (Y \cup Z)$ scp from r_3 and r_7

Using the commutativity r_5 and the associativity r_8 all the other rules generated by the subset of ecp become redundant. The fact that these inclusions can not be oriented makes necessary the use of $\langle \{r_1\}, \{r_3\} \rangle$ bi-rewriting modulo $I = \{r_5, r_8\}$. Notice that in this case $\longrightarrow_I^* = \longleftarrow_I^*$, and so we can use the standard equational I-matching and I-unification, and also the flattered notation for $\cup$.

Let's consider now the scp and the rest of ecp $\langle \sigma(\alpha_1[F[\beta_2]_q]_p), \sigma(\beta_1) \rangle$ where F is an expression, q is an occurrence in F, and σ substitutes $\alpha_1|_p = x$ by $F[\beta_2]_q$. Using them we can obtain the sequence:

r_9 $F[X] \cup F[X \cup Y] \xrightarrow{\subseteq} F[X \cup Y]$ ecp from r_1 and r_2
r_{10} $F[X \cup Y] \xrightarrow{\supseteq} F[X] \cup F[Y]$ scp from r_2 and r_9
r_{11} $F[X \cup Y \cup Z] \xrightarrow{\supseteq} F[X \cup Y] \cup F[Y \cup Z]$ ecp from r_2 and r_9

Where the orientation in the last two rules depends on the orientation ordering used for the other symbols in the signature. Another possible orientation of r_{10} could be:

r'_{10} $F[X] \cup F[Y] \xrightarrow{\subseteq} F[X \cup Y]$ from r_2 and r_9

and, then r_9 would be subsumed by r_1 and r'_{10}, and r_{11} would become confluent.

Notice that we are dealing with rule schemes instead of ordinary rules, and that the use of rule schemes in completion is an open problem. However, in this case, the rule scheme r'_{10} may be subsumed by the following (finite) set of rules:

For any $f \in Sig^n$
$r_{12}^{(f)}$ $f(X_1, \ldots, X_n) \cup f(X'_1, \ldots, X'_n) \xrightarrow{\subseteq} f(X_1 \cup X'_1, \ldots, X_n \cup X'_n)$

where f is any n-ary symbol in the signature with $n > 0$. This results from the following compositional property:

$$F[G[X]] \cup F[G[Y]] \xrightarrow[r'_{10}]{\subseteq} F[G[X] \cup G[Y]] \xrightarrow[r'_{10}]{\subseteq} F[G[X \cup Y]]$$

Similarly, rules r_{10} and r_{11} are subsumed by

For any $f \in Sig^n$
$r_{13}^{(f)}$ $f(\ldots X_i \cup X'_i \ldots) \xrightarrow{\supseteq} f(\ldots X_i \ldots) \cup f(\ldots X'_i \ldots)$
$r_{14}^{(f)}$ $f(\ldots X_i \cup X'_i \cup X''_i \ldots) \xrightarrow{\supseteq} f(\ldots X_i \cup X'_i \ldots) \cup f(\ldots X'_i \cup X''_i \ldots)$

but the same does not apply to r_9. Because of this we choose r'_{10} instead of r_{10}. Finally, using this transformation we obtain the confluent $\langle R_1, R_2 \rangle$ bi-rewriting modulo I system shown in figure 3 where r_1^{ext} and r_{12}^{ext} are the I-extensions of r_1 and r_{12}, and $r_{12}^{(\cup)}$ is not necessary because is subsumed by r_1^{ext}.

$$R_1 = \begin{cases} r_1 & X \cup X \xrightarrow{\subseteq} X \\ r_1^{ext} & X \cup X \cup Y \xrightarrow{\subseteq} X \cup Y \\ r_{12}^{(f)} & f(\ldots X \ldots) \cup f(\ldots Y \ldots) \xrightarrow{\subseteq} f(\ldots X \cup Y \ldots) \\ r_{12}^{(f)^{ext}} & f(\ldots X \ldots) \cup f(\ldots Y \ldots) \cup Z \xrightarrow{\subseteq} f(\ldots X \cup Y \ldots) \cup Z \end{cases}$$

$$R_2 = \begin{cases} r_2 \; X \cup Y \xrightarrow{\supseteq} X \end{cases}$$

$$I = \begin{cases} r_5 \; Y \cup X \xleftrightarrow{\subseteq} X \cup Y \\ r_8 \; (X \cup Y) \cup Z \xleftrightarrow{\subseteq} X \cup (Y \cup Z) \end{cases}$$

Fig. 3. A canonical bi-rewriting system for the inclusion theory of the union.

4.2 The Inclusion Theory of Non-Distributive Lattices

The presentation of non-distributive lattices may be given by the following set of inclusions:

$$\begin{array}{ll} X \cup X \subseteq X & X \subseteq X \cap X \\ X \subseteq X \cup Y & X \cap Y \subseteq X \\ Y \subseteq X \cup Y & X \cap Y \subseteq Y \end{array}$$

Applying to them the completion process of the previous subsection we get the confluent $\langle R_1, R_2 \rangle$ bi-rewriting modulo I system of figure 4. Notice that rule $r_4^{(\cap)}$ is subsumed by r_7 , and $r_8^{(\cup)}$ is subsumed by r_3.

$$R_1 = \begin{cases} r_1 & X \cup X \xrightarrow{\subseteq} X \\ r_1^{ext} & X \cup X \cup Y \xrightarrow{\subseteq} X \cup Y \\ r_2 & X \cap Y \xrightarrow{\subseteq} X \\ r_3 & X \cup (Y \cap Z) \xrightarrow{\subseteq} (X \cup Y) \cap (X \cup Z) \\ r_3^{ext} & X \cup (Y \cap Z) \cup T \xrightarrow{\subseteq} \Big((X \cup Y) \cap (X \cup Z) \Big) \cup T \\ r_4^{(f)} & f(\ldots X \ldots) \cup f(\ldots Y \ldots) \xrightarrow{\subseteq} f(\ldots X \cup Y \ldots) \\ r_4^{(f)^{ext}} & f(\ldots X \ldots) \cup f(\ldots Y \ldots) \cup Z \xrightarrow{\subseteq} f(\ldots X \cup Y \ldots) \cup Z \end{cases}$$

$$R_2 = r_5, r_5^{ext}, r_6, r_7, r_7^{ext}, r_8^{(f)}, r_8^{(f)^{ext}} \text{(Dual of } R_1)$$

$$I = \begin{cases} r_9 \;\; Y \cup X \xleftrightarrow{\subseteq} X \cup Y & r_{11} \; Y \cap X \xleftrightarrow{\subseteq} X \cap Y \\ r_{10} \; (X \cup Y) \cup Z \xleftrightarrow{\subseteq} X \cup (Y \cup Z) & r_{12} \; (X \cap Y) \cap Z \xleftrightarrow{\subseteq} X \cap (Y \cap Z) \end{cases}$$

Fig. 4. A canonical bi-rewriting system for the inclusion theory of non-distributive lattices.

We don't know of any canonical rewriting system for non-distributive lattices, although they are known for distributive lattices [11] and for boolean rings [9]. So its modelization by a bi-rewriting system represents a contribution to rewriting techniques (see also [19]). The lack of disjunctive and conjunctive normal forms is the cause of non-existence of a canonical rewriting system. On the contrary, the proposed bi-rewriting system has two normalizing rules. Rules r_3 and r_7 acting in opposite directions allow to get a disjunctive normal form the first, and the other a conjunctive normal form. In a non-distributive lattice these rules are strict inclusions and they can not be used as equational rewrite rules. Furthermore, if they are put together in a unique rewriting system then we lose termination.

4.3 Why Inclusions and not Equations

In the previous subsection we discussed briefly the advantage of modeling the deduction in a non-distributive lattice by a bi-rewriting system: there is no canonical rewrite system for it. In general inclusions express weaker constraints between terms than equations, allowing to use rules like r_3 and r_7 in the previous example. Even in the case of lattices where inclusions may be modeled by equations —like $a \subseteq b$ by $a \cup b = b$ or $a \cap b = a$— inclusions are more natural and have some advantages. The transitivity and monotonicity of inclusions which are captured implicitly by bi-rewriting systems, must be "implemented" explicitly by equational rewrite rules. Let's consider a little further the case of transitivity. The inclusions $a \subseteq b$ and $b \subseteq c$ can be oriented like $a \xrightarrow{\subseteq} b$ and $b \xrightarrow{\subseteq} c$ and we can prove $a \subseteq c$ rewriting a into b and b into c. However, their translation to equations results in two rules $a \cup b \longrightarrow b$ and $b \cup c \longrightarrow c$. These rules generate non-confluent critical pairs with the other rules defining the union and intersection, and the completion process leads to add the following rules $a \cap b \longrightarrow a$, $b \cap c \longrightarrow b$, $a \cup c \longrightarrow c$ and $a \cap c \longrightarrow a$. In general, the completion of a sequence $a_1 \subseteq \ldots \subseteq a_n$ lead to add rules $a_i \cup a_j \longrightarrow a_j$ and $a_i \cap a_j \longrightarrow a_i$ for any $i < j$. This means that the transitivity of inclusions is not captured by the transitivity of the equality relation or by the transitivity of the rewriting relation $\longrightarrow^*$, loosing so one of the main powers of rewriting systems.

References

1. D. M. Allester, B. Givan, and T. Fatima. Taxonomic syntax for first order inference. In *Proc. of the First Int. Conf. on Princ. of Knowledge Representation and Reasoning*, pages 289–300, 1989.
2. V. Antimirov. Term rewriting in unified algebras: an order-sorted approach. In *9th WADT - 4th Compass Workshop*, Caldes de Malavella, Spain, 1992.
3. L. Bachmair and N. Dershowitz. Completion for rewriting module a congruence. *J. of Theoretical Computer Science*, 67:173–201, 1989.
4. H. Bäumer. On the use of relation algebra in the theory of reduction systems. Technical report, Dept. Informatica, Univ. of Twente, Enschede, The Netherlands, 1992.
5. L. Cardelli. A semantics of multiple inheritance. *Information and Computation*, 76:138–164, 1988.

6. N. Dershowitz and J.-P. Jouannaud. Rewrite systems. In J. V. Leeuwen, editor, *Handbook of Theoretical Computer Science*. Elsevier Science Publishers, 1990.
7. J. Gallier. The semantics of recursive programs with function parameters of finite types: n-rational algebras and logic of inequalities. In N. Nivat and J. Reynolds, editors, *Algebraic Methods in Semantics*. Cambridge University Press, 1985.
8. I. Guesarian. *Algebraic Semantics*, volume 99 of *Lecture Notes in Computer Science*. Springer-Verlag, 1981.
9. J. Hsiang and N. Dershowitz. Rewrite methods for clausal and non-clausal theorem proving. In *10th Int. Colloquium on Automata, Languages and Programming*, Barcelona, Spain, 1983. Springer-Verlag.
10. G. Huet. Confluent reductions: Abstract properties and applications to term rewriting systems. *Journal of the ACM*, 27(4):797–821, 1980.
11. J.-M. Hullot. A catalogue of canonical term rewriting systems. Technical Report CSL-113, Computer Science Laboratory, Menlo Park, California, 1980.
12. B. Jayaraman. Impplementation of subset-equational programs. *J. of Logic Programming*, 12:229–324, 1992.
13. J.-P. Jouannaud and H. Kirchner. Completion on a set of rules modulo a set of equations. *SIAM J. computing*, 15(1):1155–1194, 1986.
14. C. Kirchner. *Methodes et Outils de Conception Systematique d'Algorithmes d'Unification dans les Theories Equationnelles*. PhD thesis, Universite de Nancy I, 1985.
15. H. Kirchner. *Preuves par Completion dans les Varietes d'Algebres*. PhD thesis, Universite de Nancy I, 1985.
16. J. W. Klop. Term rewriting systems: A tutorial. *Bulletin of the EATCS*, 32:143–183, 1987.
17. D. E. Knuth and P. B. Bendix. Simple word problems in universal algebras. In J. Leech, editor, *Computational Problems in Abstract Algebra*, pages 263–297. Pergamon Press, Elmsford, N. Y., 1970.
18. J. Levy and J. Agustí. Bi-rewriting, a rewriting technique for monotonic order relation. Technical Report IIIA 92/26, Institut d'Investigació en Intel·ligència Artificial, Blanes, Spain, 1992.
19. J. Levy and J. Agustí. Implementing inequality specifications with bi-rewriting systems. In *4th Compass Workshop*, Lecture Notes in Computer Science, Caldes de Malavella, Spain, 1992. Springer-Verlag.
20. J. Levy, J. Agustí, F. Esteva, and P. García. An ideal model for an extended λ-calculus with refinements. Technical Report ECS-LFCS-91-188, Laboratory for Foundations of Computer Science, Edinburgh, Great Britain, 1991.
21. J. Meseguer. Conditional rewriting logic as a unified model of concurrency. *J. of Theoretical Computer Science*, 96:73–155, 1992.
22. P. D. Mosses. Unified algebras and institutions. In *Principles of Programming Languages Conference*, pages 304–312. ACM Press, 1989.
23. B. Nebel. *Reasoning and Revision in Hybrid Representation Systems*. Lecture Notes in Artificial Intelligence. Springer-Verlag, 1990.
24. M. J. O'Donnell. Term-rewriting implementation of equational logic programming. In P. Lescanne, editor, *Proc. of Rewriting Techniques and Applications*, pages 1–12, Bordeaux, France, 1987. Springer-Verlag.
25. G. E. Peterson and M. E. Stickel. Complete sets of reductions for some equational theories. *Journal of the ACM*, 28(2):233–264, 1981.
26. G. Smolka and H. Aït-Kaci. Inheritance hierarchies: Semantics and unification. *Journal of Symbolic Computation*, 7:343–370, 1989.

27. Y. Toyama. On the Church-Roser property for the direct sum of term rewriting systems. *J. of the ACM*, 34(1):128–143, 1987.