Prime numbers and implication free reducts of MV_n-chains

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Abstract

Let $\mathbf{Ł}_{n+1}$ be the MV-chain on the $n+1$ elements set $Ł_{n+1} = \{0, 1/n, 2/n, \dots, (n-1)/n, 1\}$ in the algebraic language $\{\to, \neg\}$ [3]. As usual, further operations on $Ł_{n+1}$ are definable by the following stipulations: $1 = x \to x$, $0 = \neg 1$, $x \oplus y = \neg x \to y$, $x \odot y = \neg(\neg x \oplus \neg y)$, $x \wedge y = x \odot (x \to y)$, $x \vee y = \neg(\neg x \wedge \neg y)$. Moreover, we will pay special attention to the also definable unary operator ${}^*x = x \odot x$.

In fact, the aim of this paper is to continue the study initiated in [4] of the $\{^*, \neg, \vee\}$-reducts of the MV-chains $\mathbf{Ł}_{n+1}$, denoted $\mathbf{Ł}^*_{n+1}$. In fact $\mathbf{Ł}^*_{n+1}$ is the algebra on $Ł_{n+1}$ obtained by replacing the implication operator $\to$ by the unary operation * which represents the square operator ${}^*x = x \odot x$ and which has been recently used in [5] to provide, among other things, an alternative axiomatization for the four-valued matrix logic $J_4 = \langle \mathbf{Ł}_4, \{1/3, 2/3, 1\}\rangle$. In this contribution we make a step further in studying the expressive power of the * operation, in particular our main result provides a full characterization of those prime numbers n for which the structures $\mathbf{Ł}_{n+1}$ and $\mathbf{Ł}^*_{n+1}$ are term-equivalent. In other words, we characterize for which n the Łukasiewicz implication $\to$ is definable in $\mathbf{Ł}^*_{n+1}$, or equivalenty, for which n $\mathbf{Ł}^*_{n+1}$ is in fact an MV-algebra. We also recall that, in any case, the matrix logics $\langle \mathbf{Ł}^*_{n+1}, F\rangle$, where F is an order filter, are algebraizable.

Term-equivalence between $\mathbf{Ł}_{n+1}$ and $\mathbf{Ł}^*_{n+1}$

Let X be a subset of $Ł_{n+1}$. We denote by $\langle X\rangle^*$ the subalgebra of $\mathbf{Ł}^*_{n+1}$ generated by X (in the reduced language $\{^*, \neg, \vee\}$). For $n \geq 1$ define recursively $(^*)^n x$ as follows: $(^*)^1 x = {}^*x$, and $(^*)^{i+1} x = {}^*((^*)^i x)$, for $i \geq 1$.

A nice feature of the $\mathbf{Ł}^*_{n+1}$ algebras is that we can always define terms characterising the principal order filters $F_a = \{b \in Ł_{n+1} \mid a \leq b\}$, for every $a \in Ł_{n+1}$. A proof of the following result can be found in [4].

Proposition 1. *For each $a \in Ł_{n+1}$, the unary operation Δ_a defined as*

$$\Delta_a(x) = \begin{cases} 1 & \text{if } x \in F_a \\ 0 & \text{otherwise.} \end{cases}$$

is definable in $\mathbf{Ł}^*_{n+1}$. Therefore, for every $a \in Ł_{n+1}$, the operation χ_a, i.e., the characteristic function of a (i.e. $\chi_a(x) = 1$ if $x = a$ and $\chi_a(x) = 0$ otherwise) is definable as well.

It is now almost immediate to check that the following implication-like operation is definable in every $\mathbf{Ł}^*_{n+1}$: $x \Rightarrow y = 1$ if $x \leq y$ and 0 otherwise. Indeed, $\Rightarrow$ can be defined as

$$x \Rightarrow y = \bigvee_{0 \leq i \leq j \leq n} (\chi_{i/n}(x) \wedge \chi_{j/n}(y)).$$

Actually, one can also define Gödel implication on Ł^*_{n+1} by putting $x \Rightarrow_G y = (x \Rightarrow y) \vee y$.

It readily follows from Proposition 1 that all the Ł^*_{n+1} algebras are simple as, if $a > b \in \text{Ł}_{n+1}$ would be congruent, then $\Delta_a(a) = 1$ and $\Delta_a(b) = 0$ should be so. Recall that an algebra is called *strictly simple* if it is simple and does not contain proper subalgebras. It is clear that if Ł_{n+1} and Ł^*_{n+1} are strictly simple, then $\{0, 1\}$ is their only proper subalgebra.

Remark 2. It is well-known that Ł_{n+1} is strictly simple iff n is prime. Note that, for every n, if $\mathbf{B} = (B, \neg, \rightarrow)$ is an MV-subalgebra of Ł_{n+1}, then $\mathbf{B}^* = (B, \vee, \neg, *)$ is a subalgebra of Ł^*_{n+1} as well. Thus, if Ł_{n+1} is not strictly simple, then Ł^*_{n+1} is not strictly simple as well. Therefore, if n is not prime, Ł^*_{n+1} is not strictly simple. However, in contrast with the case of Ł_{n+1}, n being prime is not a sufficient condition for Ł^*_{n+1} being strictly simple.

We now introduce the following procedure $\mathtt{P}$: given n and an element $a \in \text{Ł}^*_{n+1} \setminus \{0, 1\}$, it iteratively computes a sequence $[a_1, \ldots, a_k, \ldots]$ where $a_1 = a$ and for every $k \geq 1$,

$$a_{k+1} = \begin{cases} {}^*(a_k), & \text{if } a_k > 1/2 \\ \neg(a_k), & \text{otherwise (i.e, if } a_k < 1/2) \end{cases}$$

until it finds an element a_i such that $a_i = a_j$ for some $j < i$, and then it stops. Since Ł^*_{n+1} is finite, this procedure always stops and produces a finite sequence $[a_1, a_2, \ldots, a_m]$, where $a_1 = a$ and a_m is such that $\mathtt{P}$ stops at a_{m+1}. In the following, we will denote this sequence by $\mathtt{P}(n, a)$.

Lemma 3. *For each odd number n, let $a_1 = (n-1)/n$. Then the procedure $\mathtt{P}$ stops after reaching $1/n$, that is, if $\mathtt{P}(n, a_1) = [a_1, a_2, \ldots, a_m]$ then $a_m = 1/n$.*

Furthermore, for any $a \in \text{Ł}^*_{n+1} \setminus \{0, 1\}$, the set A_1 of elements reached by $\mathtt{P}(n, a)$, i.e. $A_1 = \{b \in \text{Ł}^*_{n+1} \mid b \text{ appears in } \mathtt{P}(n, a)\}$, together with the set A_2 of their negations, 0 and 1, define the domain of a subalgebra of Ł^*_{n+1}.

Lemma 4. *Ł^*_{n+1} is strictly simple iff $\langle (n-1)/n \rangle^* = \text{Ł}^*_{n+1}$.*

Proof. (Sketch) The 'if' direction is trivial. As for the other direction, call $a_1 = (n-1)/n$ and assume that $\langle a_1 \rangle^* = \text{Ł}^*_{n+1}$. Launch the procedure $\mathtt{P}(n, a_1)$ and let $\mathbf{A}$ be the subalgebra of Ł^*_{n+1} whose universe is $A_1 \cup A_2 \cup \{0, 1\}$ defined as above. Clearly $a_1 \in A$, hence $\langle a_1 \rangle^* \subseteq \mathbf{A}$. But $\mathbf{A} \subseteq \langle a_1 \rangle^*$, by construction. Therefore $\mathbf{A} = \langle a_1 \rangle^* = \text{Ł}^*_{n+1}$.

Fact: Under the current hypothesis (namely, $\langle a_1 \rangle^* = \text{Ł}^*_{n+1}$) if n is even, then $n = 2$ or $n = 4$.

Thus, assume n is odd, and hence Lemma 3 shows that $1/n \in A_1$. Now, let $c \in \text{Ł}^*_{n+1} \setminus \{0, 1\}$ such that $c \neq a_1$. If $c \in A_1$ then the process of generation of A from c will produce the same set A_1 and so $\mathbf{A} = \text{Ł}^*_{n+1}$, showing that $\langle c \rangle^* = \text{Ł}^*_{n+1}$. Otherwise, if $c \in A_2$ then $\neg c \in A_1$ and, by the same argument as above, it follows that $\langle c \rangle^* = \text{Ł}^*_{n+1}$. This shows that Ł^*_{n+1} is strictly simple. □

Lemma 5 ([4])**.** *If Ł_{n+1} is term-equivalent to Ł^*_{n+1} then:*
(i) Ł^*_{n+1} is strictly simple.
(ii) n is prime

Theorem 6. *Ł_{n+1} is term-equivalent to Ł^*_{n+1} iff Ł^*_{n+1} is strictly simple.*

Proof. The 'only if' part is (i) of Lemma 5. For the 'if' part, since Ł^*_{n+1} is strictly simple then, for each $a, b \in \text{Ł}_{n+1}$ where $a \notin \{0, 1\}$ there is a definable term $\mathbf{t}_{a,b}(x)$ such that $\mathbf{t}_{a,b}(a) = b$. Otherwise, if for some $a \notin \{0, 1\}$ and $b \in \text{Ł}_{n+1}$ there is no such term then $\mathbf{A} = \langle a \rangle^*$ would be a proper subalgebra of $\mathbf{Ł}^*_{n+1}$ (since $b \notin \mathbf{A}$) different from $\{0,1\}$, a contradiction. By Proposition 1 the operations $\chi_a(x)$ are definable for each $a \in Ł_{n+1}$, then in $\mathbf{Ł}^*_{n+1}$ we can define Łukasiewicz implication $\to$ as follows:

$$x \to y = (x \Rightarrow y) \vee \left(\bigvee_{n>i>j\geq 0} \chi_{i/n}(x) \wedge \chi_{j/n}(y) \wedge \mathbf{t}_{i/n,a_{ij}}(x) \right) \vee \left(\bigvee_{n>j\geq 0} \chi_1(x) \wedge \chi_{j/n}(y) \wedge y \right)$$

where $a_{ij} = 1 - i/n + j/n$. □

We have seen that n being prime is a necessary condition for $\mathbf{Ł}_{n+1}$ and $\mathbf{Ł}^*_{n+1}$ being term-equivalent. But this is not a sufficient condition: in fact, there are prime numbers n for which $\mathbf{Ł}_{n+1}$ and $\mathbf{Ł}^*_{n+1}$ are not term-equivalent and this is the case, for instance, of $n = 17$.

Definition 7. Let Π be the set of odd primes n such that 2^m is not congruent with ± 1 mod n for all m such that $0 < m < (n-1)/2$.

Since, for every odd prime n, 2^m is congruent with ± 1 mod n for $m = (n-1)/2$ then n is in Π iff n is an odd prime such that $(n-1)/2$ is the least $0 < m$ such that 2^m is congruent with ± 1 mod n.

The following is our main result and it characterizes the class of prime numbers for which the Łukasiewicz implication is definable in $\mathbf{Ł}^*_{n+1}$.

Theorem 8. *For every prime number $n > 5$, $n \in \Pi$ iff $\mathbf{Ł}_{n+1}$ and $\mathbf{Ł}^*_{n+1}$ are term-equivalent.*

The proof of theorem above makes use of the procedure $\mathtt{P}$ defined above. Let $a_1 = (n-1)/n$ and let $\mathtt{P}(n, a_1) = [a_1, \ldots, a_l]$. By the definition of the procedure $\mathtt{P}$, the sequence $[a_1, \ldots, a_l]$ is the concatenation of a number r of subsequences $[a^1_1, \ldots, a^1_{l_1}]$, $[a^2_1, \ldots, a^2_{l_2}]$, $\ldots$, $[a^r_1, \ldots, a^r_{l_r}]$, with $a^1_1 = a_1$ and $a^r_{l_r} = a_l$, where for each subsequence $1 \leq j \leq r$, only the last element $a^i_{l_i}$ is below $1/2$, while the rest of elements are above $1/2$.

Now, by the very definition of *, it follows that the last elements $a^j_{l_j}$ of every subsequence are of the form

$$a^j_{l_j} = \begin{cases} \frac{kn-2^m}{n}, & \text{if } j \text{ is odd} \\ \frac{2^m-kn}{n}, & \text{otherwise, i.e. if } j \text{ is even} \end{cases}$$

for some $m, k > 0$, where in particular m is the number of strictly positive elements of $\mathbf{Ł}_{n+1}$ which are obtained by the procedure before getting $a^j_{l_j}$.

Now, Lemma 3 shows that if n is odd then $1/n$ is reached by $\mathtt{P}$, i.e. $a_l = a^r_{l_r} = 1/n$. Thus,

$$\begin{cases} kn - 2^m = 1, & \text{if } r \text{ is odd (i.e., } 2^m \equiv -1 \pmod{n} \text{ if } r \text{ is odd)} \\ 2^m - kn = 1, & \text{otherwise (i.e., } 2^m \equiv 1 \pmod{n} \text{ if } r \text{ is even)} \end{cases}.$$

where m is now the number of strictly positive elements in the list $\mathtt{P}(n, a_1)$, i.e. that are reached by the procedure.

Therefore 2^m is congruent with ± 1 mod n. If n is a prime such that $\mathbf{Ł}^*_{n+1}$ is strictly simple, the integer m must be exactly $(n-1)/2$, for otherwise $\langle a_1 \rangle^*$ would be a proper subalgebra of $Ł^*_{n+1}$ which is absurd. Moreover, for no $m' < m$ one has that $2^{m'}$ is congruent with ± 1 mod n because, in this case, the algorithm would stop producing a proper subalgebra of $\mathbf{Ł}^*_{n+1}$. This result, together with Theorem 6, shows the right-to-left direction of Theorem 8.

In order to show the other direction assume, by Theorem 6, that $\mathbf{Ł}^*_{n+1}$ is not strictly simple. Thus, by Lemma 4, $\langle a_1\rangle^*$ is a proper subalgebra of $\mathbf{Ł}^*_{n+1}$ and hence the algorithm above stops, in $1/n$, after reaching $m < (n-1)/2$ strictly positive elements of $\mathbf{Ł}^*_{n+1}$. Thus, 2^m is congruent with ± 1 (depending on whether r is even or odd, where r is the number of subsequences in the list $\mathtt{P}(n, a_1)$ as described above) mod n, showing that $n \notin \Pi$.

Algebraizability of $\langle \mathbf{Ł}^*_{n+1}, F_{i/n}\rangle$

Given the algebra $\mathbf{Ł}^*_{n+1}$, it is possible to consider, for every $1 \le i \le n$, the matrix logic $Ł^*_{i,n+1} = \langle \mathbf{Ł}^*_{n+1}, F_{i/n}\rangle$. In this section we recall from [4] that all the $Ł^*_{i,n+1}$ logics are algebraizable in the sense of Blok-Pigozzi [1], and that, for every i, j, the quasivarieties associated to $Ł^*_{i,n+1}$ and $Ł^*_{j,n+1}$ are the same.

Observe that the operation $x \approx y = 1$ if $x = y$ and $x \approx y = 0$ otherwise is definable in $\mathbf{Ł}^*_{n+1}$. Indeed, it can be defined as $x \approx y = (x \Rightarrow y) \wedge (y \Rightarrow x)$. Also observe that $x \approx y = \Delta_1((x \Rightarrow_G y) \wedge (y \Rightarrow_G x))$ as well.

Lemma 9. *For every n, the logic $Ł^*_{n+1} := Ł^*_{n,n+1} = \langle \mathbf{Ł}^*_{n+1}, \{1\}\rangle$ is algebraizable.*

Proof. It is immediate to see that the set of formulas $\Delta(p, q) = \{p \approx q\}$ and the set of pairs of formulas $E(p, q) = \{\langle p, \Delta_0(p)\rangle\}$ satisfy the requirements of algebraizability. □

Blok and Pigozzi [2] introduce the following notion of equivalent deductive systems. Two propositional deductive systems S_1 and S_2 in the same language are *equivalent* if there are translations $\tau_i : S_i \to S_j$ for $i \ne j$ such that: $\Gamma \vdash_{S_i} \varphi$ iff $\tau_i(\Gamma) \vdash_{S_j} \tau_i(\varphi)$, and $\varphi \dashv\vdash_{S_i} \tau_j(\tau_i(\varphi))$. From very general results in [2] it follows that two equivalent logic systems are indistinguishable from the algebraic point of view, namely: if one of the systems is algebraizable then the other will be also algebraizable w.r.t. the same quasivariety. This can be applied to $Ł^*_{i,n+1}$.

Lemma 10. *For every n and every $1 \le i \le n-1$, the logics $Ł^*_{n+1}$ and $Ł^*_{i,n+1}$ are equivalent.*

Indeed, it is enough to consider the translation mappings $\tau_1 : Ł^*_{n+1} \to Ł^*_{i,n+1}$, $\tau_1(\varphi) = \Delta_1(\varphi)$, and $\tau_{i,2} : Ł^*_{i,n+1} \to Ł^*_{n+1}$, $\tau_{i,2}(\varphi) = \Delta_{i/n}(\varphi)$. Therefore, as a direct consequence of Lemma 9, Lemma 10 and the observations above, it follows the algebraizability of $Ł^*_{i,n+1}$.

Theorem 11. *For every n and for every $1 \le i \le n$, the logic $Ł^*_{i,n+1}$ is algebraizable.*

Therefore, for each logic $Ł^*_{i,n+1}$ there is a quasivariety $\mathcal{Q}(i, n)$ which is its equivalent algebraic semantics. Moreover, by Lemma 10 and by Blok and Pigozzi's results, $\mathcal{Q}(i, n)$ and $\mathcal{Q}(j, n)$ coincide, for every i, j. The question of axiomatizing $\mathcal{Q}(i, n)$ is left for future work.

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