Introducing Grades in Deontic Logics

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Abstract. In this paper we define a framework to introduce gradedness in Deontic logics through the use of fuzzy modalities. By way of example, we instantiate the framework to Standard Deontic logic (SDL) formulas. Given a deontic formula $\Phi \in SDL$, our language contains formulas of the form $\overline{r} \rightarrow N\Phi$ or $\overline{r} \rightarrow P\Phi$, where $r \in [0,1]$, expressing that the preference or probability degree respectively of a norm Φ is at least r. We present sound and complete axiomatisations for these logics.

Keywords: Deontic Logic, Fuzzy Logic, Norms, Institutions.

1 Introduction

In their article [4], Tom R. Burns and Marcus Carson describe how agents adhere to and implement rule and normative systems to varying degrees. Agents conform to rule and normative systems to varying degrees, depending on their identity or status, their knowledge of the rules, the interpretations they attribute to them, the sanctions a group or organization imposes for noncompliance, the structure of situational incentives, and the degree competing or contradictory rules are activated in the situation, among other factors. Actually, the claim that obligations come in degrees goes back to W. D. Ross in his system of ethics, when dealing with the possibility of conflicting moral obligations (for a reference see [23]).

In hierarchical normative systems not every norm may have the same importance. In such a case, it seems interesting that agents can attach a level or degree of importance to each of these norms. These importance or preference degrees may be in turn useful for resolving conflicts among norms that may arise due to different reasons. Within a Multi-Agent System (MAS), normative conflicts may arise due to the dynamic nature of the MAS and simultaneous agents' actions. In a normative structure, one action can be simultaneously forbidden and obliged. Ensuring conflict freedom of normative structures at design time is computationally intractable as shown in [9], and thus real-time conflict resolution is required. In multi-institutional contexts, different institutions could have contradictory norms and therefore agents that participate in these institutions should decide which norm they follow. Attaching a preference degree to norms could help agents in order to take this kind of decisions.

Moreover, in hierarchical normative multi-agent systems, even if a set of norms may have a same rank, there might be different expectations about their compliance or violation by agents. In such situations, it may also be useful to represent and reason about the probability of compliance of norms.

In this paper we would like to define a logical framework able to capture different graded aspects of norms. Taking Standard Deontic Logic (SDL) as the basic formalism to model normative systems as way of exemple, we present in this paper preliminary steps towards defining Graded Deontic logics. We are aware that SDL suffers from a number of paradoxes, mostly inherent in the normal modal Kripke semantics of its operators. Thus, our proposal is not to represent graded normative reasoning in MAS over the logic SDL. But we believe that begining the study in this basic logic, graded SDL, could led us to a better understanding of the main characteristics of graded normative systems in general.

Our fuzzy modal approach has been already used to define a number of uncertainty logics (probability, possibility, belief functions [13, 10] or even graded BDI agent architectures [7]). More specifically, we define fuzzy modal languages over SDL to reason about preference (understood as necessity, in the possibilistic sense) and probability of deontic propositions. To this end we introduce two fuzzy modal-like operators N and P that apply over SDL , in such a way that e.g. the truth-degree of a formula $NO\varphi$ o $PO\varphi$ is respectively interpreted as the necessity degree or probability degree of φ being obliged. Then we use suitable fuzzy logics to reason about these intermediate truth-degrees, truth-degrees which are of neither of propositions φ nor $O\varphi$ (which remain two-valued) but of *fuzzy* propositions $NO\varphi$ and $PO\varphi$. Namely, the language of *Necessity-valued Standard Deontic Logic* NSDL will result from the union of the language of the logic $G_{\Delta}(C)$ (Gödel Logic expanded with the Δ operator and a finite set $C \subset [0,1]$ of truth-constants) and the language of SDL extended with the fuzzy unary operator N. On the other hand, the language of *Probability-valued Standard Deontic Logic* PSDL will result from the union of the language of Rational Pavelka Logic RPL (Łukasiewicz Logic expanded with rational truth-constants) and the language of SDL extended with a fuzzy unary operator P.

The main features of the Graded Deontic Logics we introduce in this paper are:

1. they are conservative extensions of SDL
2. they have a finite and recursive set of axioms
3. they keep classical semantics for formulas of SDL, in particular their truth-values remain always 0 or 1
4. they have as semantics extensions of the standard Kripke frames for SDL with necessity and probability measures over worlds respectively.
5. they contain formulas of the form $\overline{r} \to N\Phi$ or $\overline{r} \to P\Phi$, where $r \in [0,1]$, expressing that the necessity or probability degree respectively of a norm Φ is at least r, where Φ is any closed formula of SDL (not only a propositional one).

The main objective of this article is to present the above mentioned four variants of Graded Deontic Logics and prove soundness and completeness results. This constitutes a purely logical study of these formalisms. Note that our purpose is not to *fuzzify* Deontic Logic by providing a different interpretation to its modalities in the sense of having fuzzy deontic modalities, see Section 6 for a discussion. Instead we have fuzzy, many-valued modalities (of necessity and probability) applying over classical deontic formulas.

This paper is structured as follows. In Section 2 we present some rather long preliminaries on the $G_\Delta(C)$ and RPL fuzzy logics that will be needed later. In Section 3, Necessity-valued and Probability-valued Deontic logics are defined over Standard Deontic logic and in Section 4 we present two small examples of application of the two graded logics. Finally Section 5 is devoted to related and future work.

2 Preliminaries on the $G_\Delta(C)$ and RPL Fuzzy Logics

Probably the most studied and developed many-valued systems related to fuzzy logic are those corresponding to logical calculi with the real interval $[0, 1]$ as set of truth-values and defined by a conjunction & and an implication $\rightarrow$ interpreted respectively by a (left-continuous) t-norm $*$ and its residuum $\Rightarrow$[1], and where negation is defined as $\neg\varphi = \varphi \rightarrow \overline{0}$, with $\overline{0}$ being the truth-constant for falsity. In the framework of these logics, called *t-norm based fuzzy logics*, each (left continuous) t-norm $*$ uniquely determines a semantical (propositional) calculus $PC(*)$ over formulas defined in the usual way from a countable set of propositional variables, connectives $\wedge$, & and $\rightarrow$ and truth-constant $\overline{0}$ [13]. Evaluations of propositional variables are mappings e assigning each propositional variable p a truth-value $e(p) \in [0, 1]$, which extend univocally to compound formulas as follows:

$$
\begin{aligned}
e(\overline{0}) &= 0 \\
e(\varphi \wedge \psi) &= \min(e(\varphi), e(\psi)) \\
e(\varphi \& \psi) &= e(\varphi) * e(\psi) \\
e(\varphi \rightarrow \psi) &= e(\varphi) \Rightarrow e(\psi)
\end{aligned}
$$

Note that, by definition of residuum, $e(\varphi \rightarrow \psi) = 1$ iff $e(\varphi) \le e(\psi)$, in other words, the implication $\rightarrow$ captures the ordering. Further connectives are defined as follows:

$\varphi \vee \psi$ is $((\varphi \rightarrow \psi) \rightarrow \psi) \wedge ((\psi \rightarrow \varphi) \rightarrow \varphi)$,
$\neg\varphi$ is $\varphi \rightarrow \bar{0}$,
$\varphi \equiv \psi$ is $(\varphi \rightarrow \psi)\&(\psi \rightarrow \varphi)$.

Note that, from the above defintions, $e(\varphi \vee \psi) = \max(e(\varphi), e(\psi))$, $\neg\varphi = e(\varphi) \Rightarrow 0$

[1] Defined as $x \Rightarrow y = \max\{z \in [0, 1] \mid x * z \le y\}$, which always exists provided $*$ is left-continuous.

and $e(\varphi \equiv \psi) = e(\varphi \to \psi) * e(\psi \to \varphi)$. A formula φ is a said to be a 1-tautology of $PC(*)$ if $e(\varphi) = 1$ for each evaluation e, and will be denoted as $\models_* \varphi$. The associated consequence relation is defined as usual: if T is a theory (set of formulas), then $T \models_* \varphi$ whenever $e(\varphi) = 1$ for all evaluations e such that $e(\psi) = 1$ for all $\psi \in T$. Two outstanding examples of *continuous* t-norm based fuzzy logic calculi are:

Gödel logic calculus: defined by the operations

$$x *_G y = \min(x, y)$$
$$x \Rightarrow_G y = \begin{cases} 1, & \text{if } x \le y \\ y, & \text{otherwise.} \end{cases}$$

Łukasiewicz logic calculus: defined by the operations

$$x *_Ł y = \max(x + y - 1, 0)$$
$$x \Rightarrow_Ł y = \begin{cases} 1, & \text{if } x \le y \\ 1 - x + y, & \text{otherwise.} \end{cases}$$

Actually, in these two calculi (and in general when $*$ is continuous) the min operation is also definable from $*$ and $\Rightarrow$ as :

$$\min(x, y) = x * (x \Rightarrow y)$$

and hence the connective $\wedge$ can be also considered as definable. These two fuzzy logic calculi turn out to correspond to the well-known infinitely-valued Łukasiewicz and Gödel logics[2], already studied much before fuzzy logic was born (see e.g. [13] for references there). If we denote by $\vdash_Ł$ and $\vdash_G$ the provability relations in Lukasiewicz and Gödel logics respectively, the following *standard* completeness hold:

$\vdash_Ł \varphi$ iff $\models_Ł \varphi$
$\vdash_G \varphi$ iff $\models_G \varphi$

where, for the sake of simpler notation, we have written $\models_Ł$ and $\models_G$ instead of $\models_{*_Ł}$ and $\models_{*_G}$ respectively. Interestingly enough, both Łukasiewicz and Gödel logics have been shown to be axiomatic extensions of Hájek's Basic fuzzy logic BL [13] which axiomatizes the set of all common tautologies to every calculus PC(*) with $*$ being a continuous t-nom. As a matter of fact, Łukasiewicz logic is the extension of BL by the axiom

(Ł) $\neg\neg\varphi \to \varphi$,

forcing the negation to be involutive, and Gödel logic is the extension of BL by the axiom

[2] Gödel logic is also known as Dummett logic and is the axiomatic extension of Intuitionistic logic with the pre-linearity axiom $(\varphi \to \psi) \vee (\psi \to \varphi)$.

(G) $\varphi \to (\varphi \& \varphi)$.

forcing the conjunction to be idempotent. The above mentioned completeness for theorems extend to deductions from arbitrary theories in case of Gödel logic and only to deductions from finite theories in case of Łukasiewicz logic:

$T \vdash_{Ł} \varphi$ iff $T \models_{Ł} \varphi$, if T is finite
$T \vdash_G \varphi$ iff $T \models_G \varphi$

In a sense, due to the residuation property of implications, a t-norm based fuzzy logic L as defined above can be considered as a logic of *comparative truth*. In fact, a formula $\varphi \to \psi$ is a logical consequence of a theory T, i.e. if $T \vdash_L \varphi \to \psi$, if the truth degree of φ is at most as high as the truth degree of ψ in any interpretation which is a model of the theory T. Therefore, implications indeed implicitly capture a notion of comparative truth. This is fine, but in some situations one might be also interested to explicitly represent and reason with *partial degrees* of truth. One convenient way to allow for an explicit treatment of degrees of truth is by introducing truth-constants into the language. In fact, if one introduces in the language new constant symbols $\overline{\alpha}$ for suitable values $\alpha \in [0,1]$ and stipulates that $e(\overline{\alpha}) = \alpha$ for all truth-evalutations e, then a formula of the kind $\overline{\alpha} \to \varphi$ becomes 1-true under any evaluation e whenever $\alpha \leq e(\varphi)$.

This approach actually goes back to Pavelka [21] who built a propositional many-valued logical system PL which turned out to be equivalent to the expansion of Łukasiewicz Logic by adding into the language a truth-constant $\overline{r}$ for each *real* $r \in [0,1]$, together with a number of additional axioms. The semantics is the same as Łukasiewicz logic, just expanding the evaluations e of propositional variables in $[0,1]$ to truth-constants by requiring $e(\overline{r}) = r$ for all $r \in [0,1]$. Pavelka proved that his logic is complete for arbitrary theories in a non-standard sense. Namely, he defined the *truth degree* of a formula φ in a theory T as

$$\|\varphi\|_T = \inf\{e(\varphi) \mid e \text{ is a PL-evaluation model of } T\},$$

and the *provability degree* of φ in T as

$$|\varphi|_T = \sup\{r \in [0,1] \mid T \vdash_{\mathrm{PL}} \overline{r} \to \varphi\}$$

and proved that these two degrees coincide, i.e. $\|\varphi\|_T = |\varphi|_T$. This kind of completeness is usually known as Pavelka-style completeness, and strongly relies on the continuity of Łukasiewicz truth functions. Note that $\|\varphi\|_T = 1$ is not equivalent to $T \vdash_{PL} \varphi$, but only to $T \vdash_{PL} \overline{r} \to \varphi$ for all $r < 1$. Later, Hájek [13] showed that Pavelka's logic PL could be significantly simplified while keeping the completeness results. Indeed, he showed that it is enough to extend the language only by a countable number of truth-constants, one for each *rational* in $[0,1]$, and by adding only to the logic the two following additional axiom schemata, called *book-keeping axioms*:

$$\overline{r} \& \overline{s} \leftrightarrow \overline{r *_L s}$$
$$\overline{r} \to \overline{s} \leftrightarrow \overline{r \Rightarrow_L s}$$

for all $r \in [0,1] \cap \mathbb{Q}$, where $*_L$ and $\Rightarrow_L$ are the Łukasiewicz t-norm and its residuum respectively. He called this new system Rational Pavelka Logic, RPL for short. Moreover, he proved that RPL is strong standard complete for finite theories.

On the other hand, Hájek also shows that Gödel logic can be expanded with a *finite* set of truth constants together with a new unary connective Δ while preserving the strong standard completeness. Namely, let $C \subseteq [0,1]$ a finite set containing 1 and 0, and introduce into the language a truth-constant $\overline{r}$ for each $r \in C$, together with the so-called Baaz's projection connective Δ. Truth-evaluations of Gödel logic are extended in an analogous way to RPL as it regards to truth constants and adding the clause

$$e(\Delta\varphi) = \begin{cases} 1, & \text{if } e(\varphi) = 1 \\ 0, & \text{otherwise} \end{cases}$$

Note that despite φ is many-valued, $\Delta\varphi$ is a two-valued formula that is to be understood as a kind of presicification of φ. The introduction of the Δ is due to technical reasons to avoid clashes with the truth-constants. Finally, the axioms and rules of this new logic, denoted $G_\Delta(C)$ are those of Gödel logic G plus the above book-keeping axioms for truth-constants from C and the following axioms for Δ

(Δ1) $\Delta\varphi \vee \neg\Delta\varphi$
(Δ2) $\Delta(\varphi \vee \psi) \rightarrow (\Delta\varphi \vee \Delta\psi)$
(Δ3) $\Delta\varphi \rightarrow \varphi$
(Δ4) $\Delta\varphi \rightarrow \Delta\Delta\varphi$
(Δ5) $\Delta(\varphi \rightarrow \psi) \rightarrow (\Delta\varphi \rightarrow \Delta\psi)$

plus the bookeping axioms

$\Delta\overline{r} \rightarrow_G \overline{0}$ for each $r \in C \setminus \{1\}$

and the *Necessitation* rule for Δ: from φ derive $\Delta\varphi$. Then the following strong completeness result holds: $T \vdash_{G_\Delta(C)} \varphi$ iff $T \models_{G_\Delta(C)} \varphi$, for any theory T and formula φ.

Notation: in the rest of the paper we will write connectives with subindexes G or L, like $\wedge_G$, $\rightarrow_G$, $\rightarrow_L$, $\neg_L$, etc., to differentiate whether they are from Gödel or Łukasiewicz logics.

3 Graded Standard Deontic Logics

As already mentioned, in this section we are going to define two logics to reason about necessity and probability of Standard Deontic Logic formulas. Necessity and probability measures are two outstanding families of plausibility measures

[14]. Given a Boolean algebra F, $\mu : F \rightarrow [0,1]$ is a *plausibility measure* if the following holds:

1. $\mu(\emptyset) = 0$
2. $\mu(W) = 1$
3. If $X, Y \in F$ and $X \subseteq Y$, then $\mu(X) \leq \mu(Y)$

A plausibility measure μ is a *necessity* measure if in addition μ satisfies

$$\mu(X_1 \cap X_2) = \min(\mu(X_1), \mu(X_2)), \text{ for all } X_1, X_2 \subseteq F$$

and μ is a (finitely additive) probability measure if it satisfies

$$\mu(X_1 \cup X_2) = \mu(X_1) + \mu(X_2) \text{ when } X_1 \cap X_2 = \emptyset, \text{ for all } X_1, X_2 \subseteq F$$

Necessity mesures are purely qualitative in the sense that the order is what matters, and they have been widely used to model a notion of ordinal preference [3, 17]. We will mainly assume this last interpretation as intended semantics although we do not exclude other possibilities.

3.1 Necessity-valued Standard Deontic logic

We define a fuzzy modal language over Standard Deontic Logic SDL to reason about the necessity degree of deontic propositions. The language of *Necessity-valued Deontic Logic* (NSDL) results from the union of the language of the logic $G_\Delta(C)$ (Gödel logic extended with the Δ operator and a finite set $C \subset [0,1]$ of truth-constants) and the language of Standard Deontic Logic (SDL), extended with a fuzzy unary operator N. Formulas of $NSDL$ are of two types:

- *Deontic formulas:* they are the formulas of SDL, built in the usual way with the obligation deontic modality O. $\top$ and $\bot$ denote the truth-constants *true* and *false* respectively. It is said that a formula of SDL is *closed* if every propositional variable is in the scope of a modality.
- *N-formulas:* they are built from elementary N-formulas $N\varphi$, where φ is a closed SDL-formula, and truth-constants $\overline{r}$, for each rational $r \in C \subset [0,1]$, using the connectives of Gödel many-valued logic:

  - If $\varphi \in SDL$ is closed, then $N\varphi \in NSDL$
  - If $r \in C \subset [0,1]$ then $\overline{r} \in NSDL$
  - If $\Phi, \Psi \in NSDL$ then $\Phi \rightarrow_G \Psi \in NSDL$ and $\Phi \wedge_G \Psi \in NSDL$ (where $\wedge_G$ and $\rightarrow_G$ correspond to the conjunction and implication of Gödel logic)
  - If $\Phi \in NSDL$ then $\Delta\Phi \in NSDL$

  Other $G_\Delta(C)$ logic connectives for the N-formulas can be defined from $\wedge_G$, $\rightarrow_G$ and $\overline{0}$ in the way described in Section 2.

Since in Gödel Logic $G_\Delta(C)$ the formula $\Phi \to_G \Psi$ is 1-true iff the truth value of Ψ is greater or equal to that of Φ, formulas of the type $\overline{r} \to_G N\varphi$ (where φ is a closed formula of SDL) express that the necessity degree of the norm φ is at least r.

In this language we can express with the formula $\neg_G\neg_G N\varphi$, that the necessity degree of the norm φ is positive[3], and with the formula $\varphi \equiv_G \overline{r}$, that is exactly of degree r. Comparisons of degrees are done by means of formulas of the form $N\varphi \to_G N\psi$.

NSDL Semantics The semantics of our language is given by means of *Necessity-valued Deontic Kripke models* of the following form: $K = (W, R, e, \mu)$, where (W, R, e) is an standard Kripke model of SDL, and μ is a *necessity measure* on some Boolean subalgebra $F \subseteq 2^W$ such that the sets $\{w \mid e(w, \psi) = 1\}$, for every closed SDL-formula ψ, are μ-measurable. Remember that in every standard Kripke model (W, R, e) of SDL, R is a serial binary relation on W (that is, for every $w \in W$ there is $t \in W$ such that $(w, t) \in R$).

The truth value $e(w, \varphi)$ of a SDL formula φ in a world w is defined as usual (either 0 or 1). The truth-value of atomic N-formulas $N\psi$ in the model K is defined as

$$\|N\psi\|_K = \mu(\{w \mid e(w, \psi) = 1\})$$

Then the truth-value $\|\Phi\|_K$ of compound N-formulas Φ is defined by using $G_\Delta(C)$ truth-functions. If Φ is a N-formula, we will write $\models_{NSDL} \Phi$ when $\|\Phi\|_K = 1$ for any model K, and if T is a set of N-formulas, $T \models_{NSDL} \Phi$ when $\|\Phi\|_K = 1$ for all models K such that $\|\Psi\|_K = 1$ for $\Psi \in T$.

NSDL Axioms and Rules Axioms of NSDL are:

1. Axioms of SDL (for SDL-formulas)
2. Axioms of $G_\Delta(C)$ (for N-formulas)
3. Necessity Axioms (where φ and ψ are closed SDL formulas):
   (a) $N(\varphi \to \psi) \to_G (N\varphi \to_G N\psi)$
   (b) $N(\varphi \wedge \psi) \equiv_G N(\varphi) \wedge_G N(\psi)$
   (c) $\neg_G N(\bot)$
   (d) $N\psi$, for every SDL-theorem

Deduction rules for NSDL are Modus Ponens (both for $\to$ of SDL and for $\to_G$ of $G_\Delta(C)$) necessitation for the obligation deontic modality O (from φ derive $O\varphi$, if $\varphi \in SDL$) and necessitation for Δ (from Φ derive $\Delta\Phi$, for N-formulas). Alternatively, instead of Necessity Axiom 3 (d), one may add the rule "from ϕ infer $N\phi$" for a closed SDL-formula, in this way one can obtain a system with finitely-many axiom schemes and rules. We have introduced a recursive Hilbert-style axiom system since provability in SDL is decidable. We will denote by $\vdash_{NSDL}$ the usual notion of proof from the above axioms and rules

[3] Notice that in Gödel Logic, $(x \Rightarrow_G 0) \Rightarrow_G 0 = 1$ iff $x > 0$.

It is worth pointing out that Necessity Axiom 3 (a) ensures that N preserves SDL logical equivalence. Observe that the formula

$$N(\varphi \vee \psi) \equiv_G N(\varphi) \vee_G N(\psi)$$

is neither sound nor provable from the above axioms. On the other hand the following formulas are indeed provable:

1. $N(\varphi \wedge \neg\varphi) \equiv_G \overline{0}$
2. $N(\varphi \vee \neg\varphi) \equiv_G \overline{1}$

Soundness and Completeness Theorems of NSDL

Definition 1 *A set of formulas T is a N-theory if all the formulas in T are N-formulas.*

By definition of the NSDL axioms and rules it is easy to check that for every set of SDL of formulas Σ and every SDL-formula ϕ,

$$\Sigma \vdash_{NSDL} \phi \text{ iff } \Sigma \vdash_{SDL} \phi.$$

Therefore, NSDL is a conservative extension of SDL. Moreover, observe that every SDL-formula provable in a N-theory is a SDL-theorem.

Following Theorem 8.4.9 of [13], a N-theory T can be represented as a theory over the propositional logic $\mathrm{G}_\Delta(C)$. For each closed SDL-formula ϕ we introduce a propositional variable p_ϕ, corresponding to the formula $N\phi$. We define the following translation: $(N\phi)^* = p_\phi$, $(\overline{r})^* = \overline{r}$, for each rational $r \in C \subset [0,1]$ and for every N-formula ϕ and φ, $(\phi \wedge_G \varphi)^* = \phi^* \wedge_G \varphi^*$ and $(\phi \rightarrow_G \varphi)^* = \phi^* \rightarrow_G \varphi^*$. Let T^* be the following set of $\mathrm{G}_\Delta(C)$ formulas:

- Propositional variables p_ϕ, for each closed formula ϕ, theorem of SDL.
- formulas of the form φ^*, for each Necessity Axiom φ.
- α^*, for each formula $\alpha \in T$

Lemma 2 *If T is a N-theory and ϕ a N-formula, then*

$$T \vdash_{NSDL} \phi \textit{ iff } T^* \vdash_{G_\Delta(C)} \phi^*$$

Proof. Assume that $T^* \vdash_{G_\Delta(C)} \phi^*$. Let $\alpha_1^*, \ldots, \alpha_k^*$ be a $\mathrm{G}_\Delta(C)$-proof of ϕ^* in T^*. Then the sequence $\alpha_1, \ldots, \alpha_k$ can be converted in a NSDL-proof of ϕ in T by adding for each formula of the form p_ψ that occurs in $\alpha_1^*, \ldots, \alpha_k^*$, a proof of ψ in SDL and then applying the rule of necessitation for N-formulas.

Conversely, assume $T \vdash_{NSDL} \phi$. Then a $\mathrm{G}_\Delta(C)$-proof of ϕ^* in T^* can be obtained by taking the translation of the formulas of one NSDL-proof of ϕ in T, once the SDL-formulas are deleted. Use the fact that every SDL-formula provable in a N-theory is a SDL-theorem.

From the fact that the Necessity Axioms are 1-true in every Necessity-valued Deontic Kripke model follows the Soundness Theorem:

Lemma 3 ***(Soundness)*** *For every N-theory T over NSDL and every N-formula ϕ, $T \vdash_{NSDL} \phi$ implies $T \models_{NSDL} \phi$.*

Theorem 4 ***(Completeness)*** *For every N-theory T over NSDL and every N-formula ϕ:*

$T \models_{NSDL} \phi$ implies $T \vdash_{NSDL} \phi$.

Proof. By Lemma 2 and the Completeness Theorem of the Logic $G_{\Delta}(C)$ it is enough to prove that

$$T \models_{NSDL} \phi \text{ implies } T^* \models_{G_{\Delta}(C)} \phi^*.$$

Assume $T^* \not\models_{G_{\Delta}(C)} \phi^*$. Let E be a model of T^*, with evaluation v of the propositional variables p_ψ such that $v(\phi^*) < 1$. We show that there is a model K of T that is not a model of ϕ.

Let (W, R, e) be the canonical model of SDL. Observe that for every formula $\phi \in SDL$, the canonical model satisfies:

$$\psi \text{ is valid in } (W, R, e) \text{ iff } \psi \text{ is a theorem of SDL.}$$

Consider now the following Boolean subalgebra $F \subseteq 2^W$:

$$F = \{\{w \mid e(w, \psi) = 1\} : \psi \text{ is a closed formula of SDL}\}$$

let us denote by X_ψ the set $\{w \mid e(w, \psi) = 1\}$. We define a function μ on F in the following way: $\mu(X_\psi) = v(p_\psi)$. Then we can show:

(i) μ is a necessity measure on F.
  1. μ is a well-defined function. Proof: if $X_\alpha = X_\beta$, for α and β, closed SDL-formulas, then $X_{\alpha \equiv \beta} = W$ and $\alpha \equiv \beta$ is valid in the canonical model. Consequently, $\alpha \equiv \beta$ is a theorem of SDL. Since E is a model of T^*, $v(p_{\alpha \equiv \beta}) = 1$. By using the translation by the *-operation of Necessity Axiom 3 (a), $N(\alpha \to \beta) \to_G (N\alpha \to_G N\beta)$, we have $v(p_\alpha) = v(p_\beta)$. Thus, we can conclude that $\mu(X_\alpha) = \mu(X_\beta)$.
  2. It is easy to check with the same kind of argument as before that $\mu(\emptyset) = 0$ and $\mu(W) = 1$.
  3. For every α and β, closed SDL-formulas
  $$\mu(X_\alpha \cap X_\beta) = \min(\mu(X_\alpha), \mu(X_\beta))$$
  Proof: Since E is a model of T^*, E is also a model of the *-translation of the Necessity Axiom 3 (b), $N(\alpha \wedge \beta) \equiv_G N(\alpha) \wedge_G N(\beta)$. Therefore E is a model of the formula $p_{\alpha \wedge \beta} \equiv_G p_\alpha \wedge_G p_\beta$ and thus
  $$v(p_{\alpha \wedge \beta}) = \min(v(p_\alpha), v(p_\beta))$$
  we can conclude that
  $$\mu(X_\alpha \cap X_\beta) = \mu(X_{\alpha \wedge \beta}) = \min(\mu(X_\alpha), \mu(X_\beta))$$

(ii) For every N-formula $\Phi \in NSDL$, $\|\Phi\|_K = v(\Phi^*)$.
Proof: For this, it is enough to show that for every closed formula $\varphi \in SDL$, $\|N\varphi\|_K = v(p_\varphi)$. It is easy to check by induction on the complexity of the N-formulas and by definition of μ.

Let us denote by M_v the model (W, R, e, μ). We have just proved that M_v is a necessity-valued deontic Kripke model of T and $\|\phi\|_{M_v} = v(\phi^*) < 1$.

3.2 Probability-valued Deontic Logic

In a quite similar way to NSDL, we define now a fuzzy modal language over Standard Deontic Logic to reason about the probability degree of deontic propositions. The language of *Probability-valued Deontic Logic* PSDL is defined as follows. Formulas of PSDL are of two types:

- *Deontic formulas:* Formulas of SDL.
- *P-formulas:* they are built from elementary P-formulas $P\varphi$, where φ is a closed SDL-formula, and truth-constants $\overline{r}$, for each rational $r \in [0,1]$, using the connectives of Rational Pavelka Logic.

The semantics of our language is given by means of *Probability-valued Deontic Kripke models* of the following form: $K = (W, R, e, \mu)$, where (W, R, e) is an standard Kripke model of SDL, and μ is a finitely additive probability on some Boolean subalgebra $F \subseteq 2^W$ such that the sets $\{w \mid e(w, \psi) = 1\}$, for every closed SDL-formula ψ, are μ-measurable.

The truth value of an atomic P-formula $P\psi$ in a model K is defined as

$$\|P\psi\|_K = \mu(\{w \mid e(w, \psi) = 1\})$$

and the truth-value of compound P-formulas are computed from the atomic ones using the truth-functions of Łukasiewicz logic. Now, given a P-theory T (a set of P-formulas) one defines the *truth-degree* of a P-formula Φ over T as the value

$$\| \phi \|_T = \inf\{\| \phi \|_K \mid K \text{ is a PSDL-model of } T\}$$

where K is a PSDL-model of T when $\|\Psi\|_K = 1$ for every $\Psi \in T$. We define the *provability degree* of Φ over T as

$$| \phi |_T = \sup\{r \mid T \vdash_{PSDL} \overline{r} \to \phi\}$$

We introduce now a sound and recursive axiom system for PSDL. Axioms of PSDL are:

1. Axioms of SDL (for SDL-formulas)
2. Axioms of RPL (for P-formulas)
3. Probability Axioms (where φ and ψ are closed SDL-formulas):
   (a) $P(\varphi \to \psi) \to_L (P\varphi \to_L P\psi)$
   (b) $P(\varphi \vee \psi) \equiv P\varphi \oplus (P\psi \ominus P(\varphi \wedge \psi))$
   (c) $\neg_L P(\bot)$
   (d) $P\psi$, for every SDL-theorem

where $\Phi \oplus \Psi$ is a shorthand for $\neg_L\Phi \to_L \Psi$ and $\Phi \ominus \Psi$ is a shorthand for $\neg_L(\Phi \to_L \Psi)$ [4]. Deduction rules for PSDL are Modus Ponens (both for $\to$ of SDL and for $\to_L$ of RPL) and necessitation for the obligation modality O.

We can prove in an analogous way we did with the logic NSDL that PSDL is a conservative extension of SDL. Although a completeness theorem analogous to the one for NSDL does not hold for PSDL, similar techniques allows us to prove that the Pavelka-style completeness of RPL extends to PSDL.

[4] Note that in Łukasiewicz Logic $(x \Rightarrow_L 0) \Rightarrow_L y = \min(1, x+y)$ and $(x \Rightarrow_L y) \Rightarrow_L 0 = \max(0, x-y)$.

Theorem 5 *[Pavelka Completeness] Let T a P-theory and Φ a P-formula. Then it holds that $\|\Phi\|_T = |\Phi|_T$.*

4 Examples

Norm preferences. Necessity-valued graded deontic logics allow to attach preference degrees to norms and this may be used to deal with conflicts among norms, in a kind of defeasible logic approach. Consider the following example adapted from [6]. Suppose Company A has the following norms: premium costumers of Company A are entitled to get a discount, but costumers which place special orders are not allowed to get such a discount. Such a policy can be described by the theory

$$\begin{aligned}\Gamma = \{ & \mathit{PremiumCostumer}(x) \rightarrow O\mathit{Discount}(x),\\ & \mathit{SpecialOrder}(x) \rightarrow O\neg \mathit{Discount}(x) \ \}\end{aligned}$$

John is a premium costumer but has placed a special order. In this case, assuming both norms to have the same priority level, the above policy clashes: clearly, $\Gamma \cup \{\mathit{PremiumCostumer}(John), \mathit{SpecialOrder}(John)\} \vdash_{SDL} \bot$.

Let us further assume that the norm regarding special orders has a higher priority than the norm regarding premium costumers. In this case, we can describe the company policy by the following NSDL theory

$$\begin{aligned}\Gamma^* = \{ & \overline{r_1} \rightarrow N(\mathit{PremiumCostumer}(x) \rightarrow O\mathit{Discount}(x)),\\ & \overline{r_2} \rightarrow N(\mathit{SpecialOrder}(x) \rightarrow O\neg \mathit{Discount}(x)) \ \}\end{aligned}$$

where $r_1 < r_2$. Now we have

$$\Gamma^* \cup \{N\mathit{PremiumCostumer}(John), N\mathit{SpecialOrder}(John)\} \vdash_{NSDL} \overline{r_1} \rightarrow N(\bot).$$

But in terms of the non-monotonic consequence relation associated to a necessity-valued logic[5] [3], this leads to

$$\Gamma \cup \{\mathit{PremiumCostumer}(John), \mathit{SpecialOrder}(John)\} \mid\!\sim O\neg \mathit{Discount}(John),$$

this is to say, the norm about special orders prevails.

Norm Compliance. We illustrate the use of Probability-valued Deontic Logics by means of an example in which an agent i evaluates the probability of achieving a certain goal g. Let agent i represent a person with disabilities. Agent i wants to buy a certain product and he can choose between two supermarkets, A and B, for buying the desired product. In order to take this decision, on the one hand, agent i calculates, for each supermarket, the probability of the norm compliance by other clients of the provision of parking places for people with disabilities. On the other hand, he calculates the probability of norm compliance by supermarkets A and B, of the Disability Discrimination Act (1995, extended in 2005), regarding buildings accessibility. We could formalise in SDL sentences expressing these norms, in a simplified way:

[5] Let Γ^* be a theory over NSDL and let $\alpha = \sup\{r \mid \Gamma^* \vdash_{SDL} \overline{r} \rightarrow \bot\}$. Then define $\Gamma \mid\!\sim \varphi$ when $\Gamma^* \vdash_{NSDL} \overline{r} \rightarrow \varphi$ with $r > \alpha$

– It is obligatory for supermarket A (B) to have at least one accessible entrance route $OAcc_A$, ($OAcc_B$, respectively).
– It is prohibited to park in a place reserved for people with disabilities in supermarket A (B) $O\neg Park_A$ ($O\neg Park_B$, respectively)

A graded deontic language will allow us to reason about probabilities in this context of uncertainty. Consider the following sentences of SDL:

1. $OAcc_A \wedge O\neg Park_A$,
2. $OAcc_B \wedge O\neg Park_B$

1. is equivalent to $O(Acc_A \wedge \neg Park_A)$ and 2. is equivalent to $O(Acc_B \wedge \neg Park_B)$. Let T be a set of premises of the graded logic PSDL representing the information we have about norm compliance in this two supermarkets. Then, if the following holds:

$$T \vdash_{PSDL} PO(Acc_A \wedge \neg Park_A) \rightarrow_L PO(Acc_B \wedge \neg Park_B)$$

then agent i would take the decision of going to buy to supermarket B. Observe that the formula of PSDL

$$PO(Acc_A \wedge \neg Park_A) \rightarrow_L PO(Acc_B \wedge \neg Park_B)$$

is 1-true in a model iff the probability degree of the norm compliance of $O(Acc_B \wedge \neg Park_B)$ is greater or equal than the degree of norm compliance of $O(Acc_A \wedge \neg Park_A)$.

5 Related and Future Work

The paper [6] provides a logical analysis of conflicts between informational, motivational and deliberative attitutes. The resolution of conflicts is based on Thomason's idea of prioritization, which is considered in the BOID logic [5] as the order of derivations from different types of attitudes. Thomason's BDP logic (see [24]) is based on Reiter's default logic and extended in the BOID logic with conditional obligations and intentions.

In [11] the authors follow the BOID architecture to describe agents and agent types in Defeasible Logic. Reasoning about agents can be embedded in frameworks based on non-monotonic logics, as one the most interesting problems concerns the cases where the agent's mental attitudes are in conflict or when they are incompatible with obligations and other deontic provisions.

BOID specifies logical criteria (i) to retract agent's attitudes with the changing environment, and so (ii) to settle conflicts by stating different general policies corresponding to the agent type considered. Intentions and beliefs are viewed as constituting the internal constraints of an agent while obligations are its external constraints.

More recently, in [20], M. Nickles proposes a logic-based approach based on the notion of behavioral expectation. In his paper he presents a quantification of the norm adherence of an agent using the measurement of norm deviance. Normative expectations are defined via the degree of resistance to social dynamics in the course of time. Our approach differs from previous ones because our purpose is not to fuzzify deontic logic giving a different interpretation to its modalities. We want to provide a reasoning model for an agent in order to represent how he attaches necessity or probability degrees to norms of a given institution.

Up to now we have applied our framework to SDL. However, our purpose is to provide a general way to define a necessity-valued or probability fuzzy logic over any given deontic logic, allowing to attach a grade to the norms described in the deontic language. Future work will include working with other logics such as Dynamic Logic (see [22] and [15]) Dyadic Deontic Logic (see [29] or [26]), the KARO formalism [27], B-DOING Logic [8], the Logic of "Count-as" [12], Normative ATL [25] or Temporal Logic of Normative Systems [1]. Since different definitions of norm adherence or of probability of norm compliance would give rise to a variety of formal systems, changing or adding new axioms to the basic axiomatization we have introduced here, future work will be devoted to the study of these notions in a multi-institutional setting.

Acknowledgements The authors wish to express their thanks to Carles Sierra and the reviewers of this paper for their helpful comments. Research partially supported by the Generalitat de Catalunya, under the grant 2005-SGR-00093 and the Spanish projects "Agreement Technologies" (CONSOLIDER CSD2007-0022, INGENIO 2010) and TIN2006-15662-C02-01.

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