Towards the generalization of Mundici's Γ functor to IMTL algebras: the linearly ordered case*

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Abstract

Mundici's Γ functor establishes a categorical equivalence between MV-algebras and lattice-ordered Abelian groups with a strong unit. In this short note we present a first step towards the generalization of such a relationship when we replace MV-algebras by weaker structures obtained by dropping the divisibility condition. These structures are the so-called involutive monoidal t-norm based algebras, IMTL-algebras for short. In this paper we restrict ourselves to linearly ordered IMTL-algebras, for which we show a one-to-one correspondence with a kind of ordered grupoid-like structures with a strong unit. A key feature is that the associativity property in such a new structure related to a IMTL-chain is lost as soon the IMTL-chain is no longer a MV-chain and the strong unit used in Mundici's Γ functor is required here to have stronger properties. Moreover we define a functor between the category of such structures and the category of IMTL algebras that is a generalization of Mundici's functor Γ and, restricted to their linearly ordered objects, a categorical equivalence.

1 Introduction

Completeness results for Lukasiewicz infinitely-valued logic have been obtained by Rose and Rosser and by Chang in the fifties. But it was Chang who gave in [4] an algebraic proof based on the study of the MV-algebras, that constitute the algebraic counterpart of the logic. Chang construction [3] associates to each linearly ordered MV-algebra a linearly ordered Abelian group with strong order unit and conversely. Mundici generalized this early result in [11] to a categorical equivalence (given by what is known as the Γ functor) between the category of MV-algebras and the category of lattice ordered Abelian groups with a strong

*Dedicated to the Renaissance man, great scientist and good friend Daniele Mundici in the occasion of his 60th anniversary unit. This categorical equivalence has also been extended by Dvurečenski in [6] to non-commutative MV-algebras (called pseudo-MV-algebras) and arbitrary lattice ordered groups with a strong unit, and even it has been further extended by Galatos and Tsinakis in [8] in a very general context of non-integral, non-commutative and unbounded residuated lattices.

In this note our aim is to study the extension of Chang-Mundici's construction in the algebraic setting of t-norm based fuzzy logics, whose biggest variety is the one of MTL-algebras, i.e. of integral, bounded, commutative and prelinear residuated lattices. Actually, one of the key points in the Chang-Mundici construction is the fact that MV-algebras have an involutive negation, which allows to define a addition-like operation (strong disjunction) as the De Morgan dual of the monoidal operation (strong conjunction). In fact MV-algebras can be axiomatized using only the negation and the strong conjunction. Therefore it seems reasonable, in order to generalize the Γ functor, to try to do it in the frame of (bounded, integral, commutative) residuated lattices with an involutive negation (hence including MV-algebras) and so, having a non-trivial strong disjunction. Moreover, as a first step and for the sake of simplicity, in this paper we shall also restrict ourselves to linearly ordered structures.

Indeed, after reviewing some basic facts about involutive residuated lattices in the next section, and following Chang's construction, in Section 3 we first introduce a type of (possibly) non-associative group-like structures linked to linearly ordered involutive MTL algebras, IMTL-chains for short, and after we show that there exists a one-to-one correspondence between these two classes of algebraic structures that particularizes to Chang-Mundici construction when restricted to linearly ordered MV-chains (in other words, to divisible IMTL-chains). Moreover we define a functor between the category of such algebraic structures and the category of IMTL algebras extending the Γ functor, and when restricted to the subcategories of the corresponding linearly ordered structures provides a categorical equivalence.

2 About involutive residuated lattices

In this paper, following [10], by *residuated lattice* we shall mean a lattice ordered residuated commutative integral monoid, as it is defined next.

Definition 1. *An structure $\mathcal{A} = \langle A, \wedge, \vee, *, \rightarrow, 0, 1\rangle$ is a residuated lattice if:*

(L) $\langle A, \wedge, \vee, 0, 1\rangle$ *is a bounded lattice,*

(M_*) $\langle A, *, 1\rangle$ *is a commutative monoid with neutral element 1,*

(R) Residuation: $*$ and $\rightarrow$ form an adjoin pair, i.e., $x \rightarrow y \geq z$ if and only if $x * z \leq y$,

In residuated lattices, a negation operation $\neg$ can be defined as usual by stipulating $\neg x = x \rightarrow 0$.

Residuated lattices form a variety, and the subvarieties we are interested in are those of *involutive residuated lattices*, that is, residuated lattices whose negation operation $\neg$ is involutive, hence satisfying the following condition:

(INV) $\neg\neg x = x$.

Having an involutive negation allows us to define in an involutive residuated lattice a strong disjunction operation $\oplus$ as the dual of the monoidal operation (or strong conjunction) with respect to $\neg$, i.e. $x \oplus y = \neg(\neg x * \neg y)$. Distinguished subvarieties of involutive residuated lattices are:

- the full variety of involutive residuated lattices **IRL**, whose algebras will be denoted IRL-algebras[1];
- the variety **IMTL** of involutive monoidal t-norm based algebras (IMTL-algebras for short) introduced in [7],
- the variety **MV** of MV-algebras, and
- the variety **B** of Boolean algebras.

Note that IMTL-algebras are IRL-algebras satisfying the prelinearity condition

(PL) $(x \to y) \vee (y \to x) = 1$,

while MV-algebras are IMTL-algebras verifying the divisibility condition

(DV) $x * (x \to y) = x \wedge y$ [2].

Finally, Boolean algebras are MV-algebras satisfying the excluded middle law $x \vee \neg x = 1$.

Remark 1. An easy computation shows that all linearly ordered IRL-algebras satisfy the prelinearity condition and thus they are in fact IMTL-algebras. Moreover in [7] it is proved that all IMTL-algebras are subdirect product of linearly ordered IMTL-algebras. Therefore the varieties **IRL** and **IMTL** have the same linearly ordered algebras and hence **IMTL** is the variety generated by linearly ordered IRL-algebras.

It is well known that the **B** and **MV** varieties can be (term-wise) equivalently defined using the operations $\oplus$, $\neg$ and 0. Analogously, IRL and IMTL-algebras can be alternatively defined using the operations $\wedge$, $\vee$, $\oplus$, $\neg$ and 0. Next propositions provide such axiomatizations.[3]

[1] Called *Girard Monoids* in [10] and proved in [1] to be equivalent to the so-called *Grising algebras* as defined in [9].

[2] Or equivalently, verifying the equation $x \vee y = (x \to y) \to y$.

[3] The authors are indebted to Roberto Cignoli for pointing us property (C) that is the key property in this axiomatization and which was also used in [1] to axiomatize the $\{*, \neg\}$-fragment of **IRL**.

Proposition 2. *$\mathcal{A} = \langle A, \wedge, \vee, *, \rightarrow, 0, 1\rangle$ is a IRL-algebra if, and only if, the following conditions and equations hold:*

(SL) $\langle A, \wedge, 0, 1\rangle$ *is a bounded* $\wedge$*-semilattice,*

($M_\oplus$) $\langle A, \oplus, 0\rangle$ *is a commutative monoid with neutral element 0,*

(OW) $x \oplus (y \wedge z) = (x \oplus y) \wedge (x \oplus z)$,

(INV) $\neg\neg x = x$,

(C) $x \oplus \neg(x \oplus y) \geq \neg y$.

where the new operations are defined as follows:

$$\neg x = x \rightarrow 0$$
$$x \oplus y = \neg(\neg x * \neg y).$$

Conversely, a structure $\mathcal{A}' = \langle A, \wedge, \vee, \oplus, \neg, 0, 1\rangle$ satisfies the above conditions and equations if, and only if, $\mathcal{A} = \langle A, \wedge, \vee, *, \rightarrow, 0, 1\rangle$ is a IRL-algebra, where the new operations are defined as follows:

$$x * y = \neg(\neg x \oplus \neg y),$$
$$x \rightarrow y = \neg x \oplus y \ .$$

Proof: On the one hand, any IRL-algebra defining $\oplus$ as the dual of $*$ wrt $\neg$ satisfies the set properties of this proposition. The fulfillment of (SL), ($M_\oplus$), (OW) and (INV) is obvious. To prove (C) first observe that $x \oplus y = 1$ if and only if $y \geq \neg x$. Then, simply the following equivalences hold: $y \geq \neg x$ is equivalent to $\neg y \leq \neg\neg x = \neg x \rightarrow 0$, and by residuation, it is equivalent to $\neg y * \neg x = 0$, and thus it is also equivalent to $x \oplus y = \neg(\neg x * \neg y) = \neg 0 = 1$. From this equivalence, property (C) is an obvious consequence of the following equality: $y \oplus (x \oplus \neg(x \oplus y)) = (x \oplus y) \oplus \neg(x \oplus y) = 1$.

On the other hand, if $\mathcal{A}' = \langle A, \wedge, \vee, \oplus, \neg, 0\rangle$ satisfies (SL), ($M_\oplus$), (OW), (INV) and (C), we will prove that the set A with the lattice operations plus $*$, $\rightarrow$ and the constants 0 and 1 is an IRL-algebra. The (L) and (M_*) conditions are obviously satisfied. Notice that by condition (OW) the operation $\oplus$ is monotone with respect to the order. To prove the residuation property (R), first we prove that $x \oplus y = 1$ if and only if $x \geq \neg y$. To this end we will follow the proof in [1] with the obvious modification. From $x \oplus y = 1$ and condition (C) we obtain $x \oplus \neg 1 = x \geq \neg y$. Suppose now that $y \geq \neg x$. Then $x \oplus y \geq (x \oplus y) \wedge (x \oplus \neg x) = x \oplus (y \wedge \neg x) = x \oplus \neg x = x \oplus \neg(0 \oplus x) \geq \neg 0 = 1$. From this property and definition of $\rightarrow$, residuation is easy since, $x \rightarrow y \geq z = \neg\neg z$ is equivalent then to $\neg z \oplus (\neg x \oplus y) = 1$, and by associativity and commutativity, it is equivalent to $(\neg x \oplus \neg z) \oplus y = 1$, that is, by definition of $*$, equivalent to $\neg(x * z) \oplus y = 1$, and finally by condition (C), this is equivalent to $y \geq x * z$. □

Corollary 3. *An structure $\mathcal{A} = \langle A, \wedge, \vee, *, \rightarrow, 0, 1\rangle$ is a IMTL algebra if, and only if, besides of ($M_\oplus$), (OW), (INV) and (C), the following condition and equation hold:*

(DL) $\langle A, \vee, \wedge, 0, 1\rangle$ *is a bounded distributive lattice,*

(OV) $x \oplus (y \vee z) = (x \oplus y) \vee (x \oplus z)$,

Conversely, a structure $\mathcal{A}' = \langle A, \wedge, \vee, \oplus, \neg, 0, 1\rangle$ *satisfies the above properties if, and only if,* $\mathcal{A} = \langle A, \wedge, \vee, *, \to, 0, 1\rangle$ *is a IRL-algebra, where the new operations are defined as in Proposition 2.*

Proof: Proposition 2.3 in [10] proves that in any integral, residuated, commutative l-monoid, the property (PL) of prelinearity is equivalent to the distributivity of $*$ with respect to $\wedge$, or equivalently to the distributivity of $\oplus$ with respect to $\vee$. Since IMTL-algebras are prelinear IRL-algebras, the proof is completed. □

As a consequence of these characterizations, in the rest of the paper we shall indistinctly refer to IRL- and IMTL-algebras as structures in the language $(\wedge, \vee, *, \to)$ or in the language $(\wedge, \vee, \oplus, \neg)$.

Remark 2. In the above definitions of IRL and IMTL algebras we have used the two lattice operations $\wedge$ and $\vee$ and the two constants 0 and 1. In fact only one operation and one constant would be actually needed since the other ones could be defined from those two using the negation.

3 Chang's construction and IMTL chains

Mundici's Γ functor (see [11] for the original paper and [5] for a nice proof of the Γ-functor) is a well known functor that gives a categorical equivalence between the category of MV-algebras and the category of lattice ordered Abelian groups with a strong unit. In fact, it is a deep generalization of a Chang result associating to each MV chain a linearly ordered abelian group (see [3, 4]).

In this section we generalize Chang's construction to linearly ordered IRL- (or IMTL-) algebras. In fact we shall work in the IMTL setting, i.e. using the properties of IMTL-algebras given in Corollary 3, because it seems more natural and a generalization to the non-linearly ordered case seems more plausible for **IMTL** than for **IRL**. The latter appears to be more difficult since the **IRL** variety is not generated by its linearly ordered members, moreover in a IRL-algebra the lattice may be not distributive and the $\oplus$ operation may not be a morphism with respect to $\vee$. From now on, the linearly ordered IMTL-algebras will be called IMTL-chains for short.

3.1 The partially associative structure defined by an IMTL chain

Extending Chang's construction for MV-chains, we will associate to each IMTL-chain an algebraic structure, which is partially non-associative in the general case.

Definition 4. *Let* $\mathcal{A} = \langle A, \wedge, \vee, \oplus, \neg, 0_A, 1_A\rangle$ *be a linearly ordered IMTL-algebra. The linearly ordered algebraic structure associated to* $\mathcal{A}$ *is the structure* $\mathcal{S}(\mathcal{A}) = \langle S(A), +, -, (0, 0_A), \leq\rangle$ *of type (2,1,0) where:*

- $S(A) = \{(m, x) \mid m \in \mathbb{Z}, x \in A - \{1_A\}\}$

- $(m, x) + (n, y) = \begin{cases} (m+n, x \oplus y), & \text{if } x \oplus y < 1_A \\ (m+n+1, x * y), & \text{otherwise} \end{cases}$

- $-(m, x) = \begin{cases} (-m, 0_A), & \text{if } x = 0_A \\ (-(m+1), \neg x), & \text{otherwise} \end{cases}$

- *$\leq$ is the lexicographic order, that is, $(n, x) \leq (m, y)$ if and only if $n < m$ or $n = m$ and $x \leq y$.*

From this definition it is easy to prove the following properties of $\mathcal{S}(\mathcal{A})$, where we use $\wedge$ and $\vee$ to denote the glb and lub with respect to the lexicographic order $\leq$.

Proposition 5. *If $\mathcal{A}$ is a linearly ordered IMTL-algebra, the linearly ordered algebraic structure $\mathcal{S}(\mathcal{A}) = \langle S(A), +, -, (0, 0_A), \leq\rangle$ satisfies the following properties:*

1) $+$ is commutative, monotone w.r.t. the order and $(0, 0_A)$ is neutral,

2) $(S(A), \wedge, \vee)$ is a distributive lattice,

3) $+$ is distributive with respect to $\vee$ and $\wedge$,

4) $-(n, x)$ is the inverse element of (n, x) with respect to $+$,

5) $-$ is involutive, i.e. $-(-(n, x)) = (n, x)$, and is a morphism with respect to $+$, i.e. $-((n, x) + (k, y)) = -(n, x) + (-(k, y))$,

6) for all $k \in \mathbb{Z}$, $(k, 0_A)$ is an associative element, i.e. for any $a, b, c \in S(A)$ it holds that $a + (b + c) = (a + b) + c$ whenever at least one of a, b and c is of the form $(k, 0_A)$.

7) for all $x, y \in A$, $(0, x) + [(1, 0_A) + (-((0, x) + (0, y)))] \geq (1, 0_A) + (-(0, y))$, i.e. property (C)

8) for all $x, y, z \in A$, $[(0, x) + ((0, y) + (0, z))] \wedge (1, 0_A) = [((0, x) + (0, y)) + (0, z)] \wedge (1, 0_A)$,

9) $(1, 0_A)$ is a strong unit in the following sense: for any $(m, x) > (0, 0_A)$ there is a natural k such that $k(1, 0_A) > (m, x)$ where $k(1, 0_A) = (1, 0_A) + \overset{k}{\ldots} + (1, 0_A)$.

These properties are straightforward consequences of the properties of the IMTL chains given in Corollary 3.

Remark 3. The algebraic ordered structure $\mathcal{S}(\mathcal{A})$ is not associative in general. In fact, as it is known, it is associative if, and only if, the initial IMTL-algebra $\mathcal{A}$ is a MV-algebra. As an example of non-associativity, take $\mathcal{A}$ as the linearly ordered NM-algebra defined by the nilpotent minimum and its residuum over the real interval $[0,1]$. Recall that in that case, $*$ and $\oplus$ are defined as follows:

$$x * y = \begin{cases} \min(x,y), & \text{if } x > 1-y \\ 0, & \text{otherwise} \end{cases}$$

$$x \oplus y = \begin{cases} \max(x,y), & \text{if } x < 1-y \\ 1, & \text{otherwise} \end{cases}$$

Then, if we take $\beta > \alpha > \frac{1}{2}$, we have, for any $m, n, k \in \mathbb{Z}$,

$$((m,\alpha) + (n, 1-\alpha)) + (k,\beta) = (m+n+1, 0) + (k,\beta) = (m+n+k+1, \beta)$$

while

$$(m,\alpha) + ((n, 1-\alpha) + (k,\beta)) = (m,\alpha) + (n+k+1, 1-\alpha) = (m+n+k+2, 0)$$

Remark 4. Another important property that is lost in the above algebraic ordered structure associated to an IMTL-chain (related to non-associativity) is the cancellation law. For example, taking the standard NM chain (the same as in the previous remark), it is clear that $(m,\alpha) + (n,\beta) = (m,\alpha) + (n,\gamma)$ for $\gamma < \beta \leq 1-\alpha$.

The initial IMTL-algebra $\mathcal{A}$ can indeed be recovered from the structure $\mathcal{S}(\mathcal{A})$. Actually, it can be identified with the interval $[(0,0),(1,0)]$ of the structure $\mathcal{S}(\mathcal{A})$ with suitably adapted operations.

Proposition 6. *Let $\mathcal{S}(\mathcal{A})$ be the algebraic ordered structure associated to the linearly ordered IMTL-algebra $\mathcal{A}$, and let $\Phi(\mathcal{S}(\mathcal{A})) = \langle A^+, \oplus, \neg, \min, \max, (0, 0_A), (1, 0_A)\rangle$ be the algebra defined by:*

1) $A^+ = \{(m,x) \mid (0,0_A) \leq (m,x) \leq (1,0_A)\}$

2) for all $(m,x),(n,y) \in A^+$, $(m,x) \oplus (n,y) = ((m,x)+(n,y)) \wedge (1,0_A)$, that is, $\oplus$ is the bounded sum.

3) for all $(n,x) \in A^+$, $\neg(n,x) = (1,0_A) + (-(n,x))$

4) the order is the restriction to A^+ of the order on $\mathcal{S}(\mathcal{A})$.

Then $\Phi(\mathcal{S}(\mathcal{A}))$ is a IMTL-chain which is isomorphic to the initial IMTL-algebra $\mathcal{A}$.

Proof: An easy computation shows that the mapping $f : A^+ \longrightarrow A$ defined by $f(0,x) = x$ and $f(1,0_A) = 1_A$ is an isomorphism of IMTL-algebras. □

3.2 Representation theorem

We have seen in the previous subsection how to associate to each linearly ordered IMTL-algebra $\mathcal{A}$ the pair $(\mathcal{S}(\mathcal{A}), (1,0))$ formed by the algebraic ordered structure associated to $\mathcal{A}$ and the element $(1,0)$. Conversely, given a *suitable* pair formed by an ordered algebraic structure with a distinguished element (not necessarily linearly ordered), we show next that we can build a IMTL-algebra generalizing what Chang did for linearly ordered Abelian groups with a strong unit (see, for example [4]). Inspired in the results of previous section, first we need to introduce the definition of an algebraic structure generalizing the notion of Abelian groups with strong unit.

Definition 7. *A pair $(\mathcal{G}, u)$ formed by an algebra $\mathcal{G} = (G, \wedge, \vee, +, -, 0_G)$ of type (2,2,2,1,0) and by an element $u \in \mathcal{G}$ is called (lattice ordered)* a partially associative Abelian groupoid with strong associative unit u *if the following properties are satisfied :*

i) $(G, \wedge, \vee)$ *is a distributive lattice*

ii) $(G, +, 0)$ *is an Abelian grupoid with neutral element* 0_G*,*

iii) *For all* $x \in G$, $+_x : G \to G$ *defined by* $+_x(y) = x+y$ *is a lattice morphism,*

iv) *For each* $a \in G$, $-a$ *is the inverse of* a*, i.e.* $a + (-a) = 0_G$*,*

v) $-$ *is involutive and a morphism with respect to* $+$*,*

vi) *the strong associative unit* $u \in G$ *satisfies the following conditions:*

  (a) *for all* $x, y \in G$, $u+(x+y) = (u+x)+y$ *(u is an associative element),*

  (b) *for all* $x, y, z \in G$, $[x^* + (y^* + z^*)] \wedge u = [(x^* + y^*) + z^*] \wedge u$*, where* $x^* = (x \wedge u) \vee 0_G$*,*

  (c) *for all* $x, y \in G$, $[x^* + (u + (-((x^* + y^*) \wedge u)))] \wedge u \geq u + (-y^*)$*, where* x^* *is defined as in the previous item ,*

  (d) *for any* $x > 0_G$ *there is a natural* k *such that* $ku > x$*, where* $ku = u + ..k.. + u$ *(strong unit).*

It easily follows from the previous definition that any partially associative Abelian groupoid with strong associative unit $(\mathcal{G}, u)$ fulfills the next three properties:

1) u is a cancellative element;

2) any triple containing an element of the form ku, for some $k \in \mathbb{Z}$, is associative;

3) $-$ is an order-reversing lattice isomorphism of $(G, \min, \max)$;

Moreover, it is easy to check that, for any IMTL-chain $\mathcal{A}$, the pair $(\mathcal{S}(\mathcal{A}),(1,0_A))$ is indeed a linearly ordered partially associative grupoid with strong associative unit.

We can show now that the interval $[0_G,u]$ of a lattice ordered partially associative Abelian groupoid with strong associative unit $(\mathcal{G},u)$ can be endowed with a structure of IMTL-algebra.

Theorem 8. *Let $(\mathcal{G},u)$ be a lattice ordered partially associative abelian groupoide with strong associative unit u. Then $\Phi(G,u)=([0_G,u],\wedge,\vee,\oplus,\neg,0)$ is a IMTL-algebra where the lattice operation are the restriction of the lattice operation of G, $\oplus$ is the bounded sum defined by $x\oplus y=(x+y)\wedge u$ and $\neg x=u+(-x)$ for all $x\in[0_G,u]$.*

Proof: We have to prove that $\Phi(G,u)=\langle[0_G,u],\wedge,\vee,\oplus,\neg,0_G,u\rangle$ satisfies the properties of Corollary 3. Property (DL) is obvious and (OW) and (OV) are consequences of the fact that $+_x$ is a lattice morphism ((iii) of definition 7). Moreover using ii), v) and (vi-a) from Definition 7, $\neg\neg(x)=u+(-(u+(-x)))=u+((-u)+x)=(u+(-u))+x=x$. Finally property (C) is an easy consequence of condition $(vi-c)$ of the strong associative unit u according to Definition 7, taking into account that $x^*=x$ for any $x\in[0_G,u]$. □

Finally, for linearly ordered structures we can complete the representation with the following theorem.

Theorem 9. *The following statements hold:*

(i) Any linearly ordered IMTL-algebra $\mathcal{A}$ is isomorphic to $\Phi(\mathcal{S}(\mathcal{A}),(1,0_A))$

(ii) Any linearly ordered partially associative Abelian groupoid with strong associative unit $(\mathcal{G},u)$ is isomorphic to $(\mathcal{S}(\Phi(\mathcal{G},u)),(1,0_G))$.

Proof. (i) is proved in Proposition 6. To prove (ii), we define the following mapping:

$$f:\mathcal{G}\to\mathcal{S}(\Phi(\mathcal{G},u))$$

by $f(x)=(k_x,x^*)$, where k_x is the integer such that $x^*=x+(-k_xu)\in[0_G,u]$. By construction, the mapping is well defined since, for any $x\in\mathcal{G}$, k_x exists and it is unique. Moreover f is a morphism with respect to $+$ since:

1. If $x^*+y^*<u$ then $k_{x+y}=k_x+k_y$, and thus obviously $f(x)+f(y)=(k_x,x^*)+(k_y,y^*)=(k_x+k_y,x^*+y^*)=f(x+y)$.
2. If $x^*+y^*\geq u$ then $k_{x+y}=k_x+k_y+1$, since $x^*+y^*<2u$. Thus,
$$f(x+y)=(k_{x+y},(x+y)^*)=(k_x+k_y+1,(x+y)^*)$$
and
$$f(x)+f(y)=(k_x,x^*)+(k_y,y^*)=(k_x+k_y+1,x^*\ast_{\Phi(G)}y^*).$$
Both expressions actually coincide because
$$x^*\ast_{\Phi(G)}y^*=\neg_{\Phi(G)}(\neg_{\Phi(G)}x\oplus_{\Phi(G)}(\neg_{\Phi(G)}y)=u-((u-x)\oplus(u-y)),$$
and taking into account that $(u-x)\oplus(u-y)=(u-x)+(u-y)<u$, this is equal to $u-((u-x)+(u-y))=(x+y)-u=(x+y)^*$.

Obviously f is also morphism for $-$ and clearly $f(u) = (1, 0_G)$. Finally notice that f is a bijection. Indeed, if $f(x) = f(y)$ then $k_x = k_y$ and $x^* = y^*$, and thus $x = k_x u + x^* = y$ as well. And for any $(m, x) \in \mathcal{S}(\Phi(\mathcal{G}, u))$, a simple computation shows that $f(y) = (m, x)$ for $y = mu + x \in \mathcal{G}$. □

Summarizing, we have seen that each IMTL-chain can be seen as an interval algebra of some l.o. partially associative Abelian groupoid with an associative strong unit, and conversely, each of such structures can be generated by a suitable IMTL-chain.

4 Towards the generalization of the Mundici's functor Γ

Let us consider the category $\mathcal{PAG}$ where the objects are (lattice ordered) partially associative Abelian groupoids with strong associative unit $(\mathcal{G}, u)$, and given two objects $(\mathcal{G}, u)$ and $(\mathcal{G}', u')$, a homomorphism $\varphi : (\mathcal{G}, u) \to (\mathcal{G}', u')$ is a morphism of groupoids $\varphi : \mathcal{G} \to \mathcal{G}'$ such that $\varphi(u) = u'$. The category of IMTL-algebras $\mathcal{IMTL}$ is defined in the obvious way, i.e. the objects are IMTL-algebras and the homomorphisms are the IMTL-alegbra morphisms.

Then, as in the MV case, the mapping Φ defined in Theorem 8 actually defines a functor from the category $\mathcal{PAG}$ into the category $\mathcal{IMTL}$. Indeed, over objects, Φ is defined as in Theorem 8, i.e. if $\mathcal{G} = (G, \wedge, \vee, +, -, 0_G)$ is a (lattice ordered) partially associative Abelian groupoid with strong associative unit u, then $\Phi(\mathcal{G}, u) = ([0_G, u], \wedge, \vee, \oplus, \neg, 0)$ with $x \oplus y = (x+y) \wedge u$ and $\neg x = u + (-x)$ for all $x \in [0_G, u]$. Now, given a homomorphism $\varphi : (\mathcal{G}, u) \to (\mathcal{G}', u')$ of $\mathcal{PAG}$, we define $\Phi(\varphi) : \Phi(\mathcal{G}, u) \to \Phi(\mathcal{G}', u')$ as the restriction of φ to $[0_G, u]$. So, defined, it is easy to prove that Φ is a functor from the category $\mathcal{PAG}$ into the category $\mathcal{IMTL}$. To this end we have to check the following two conditions:

(A) Φ transforms homomorphisms of $\mathcal{PAG}$ into homomorhisms of $\mathcal{IMTL}$.

(B) Φ preserves the composition and the identities.

In fact, taking into account that operations of in the IMTL-algebra $\Phi(\mathcal{G}, u)$ are defined from the operations of $\mathcal{G}$, it is not difficult to prove condition (A). Moreover since Φ restricts grupoid morphisms, condition (B) is also easily proved.

To conclude, observe that if $\mathcal{G}$ is a lattice-ordered Abelian group, then $\Phi(\mathcal{G}, u)$ is an MV-algebra since an easy computation shows that $\Phi(\mathcal{G}, u) = \Gamma(\mathcal{G}, u)$. Therefore, the functor Φ is actually a generalization of the functor Γ. Finally, Theorem 9 shows that the functor Φ provides a categorical equivalence between the subcategories of linearly ordered objects of the categories $\mathcal{PAG}$ and $\mathcal{IMTL}$.

5 Concluding remarks

This paper contains some initial ideas towards the generalization of Mundici's functor Γ to functor Φ from the categories of (lattice ordered) partially associative lattice abelian groupoids to the category of IMTL-algebras. The interest of the deep result by Mundici is to relate MV-algebras to a very well known and studied class of algebraic structures such as lattice-ordered Abelian groups. This is not the case for IMTL-algebras since the algebraic structures that can be associated to them are not well known and till now we have not proved that Φ is a categorical equivalence. Actually, the algebraic structures associated to IMTL-algebras which are not MV-algebras are necessarily non associative. The interest of this research, if any, would probably be in the converse sense, that is, available results about IMTL-algebras can perhaps help in knowing something more about some particular (partially) non-associative structures. To prove or disprove that Φ is a categorical equivalence is left to be accomplished in future work.

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References

[1] F. Bou, A. Garcia-Cerdaña and V. Verdú. On some Substructural Aspects of t-Norm Based Logics. In *Proc. of IPMU'04*, Perugia (Italy), July 2004 pp. 545–552.

[2] F. Bou, A. Garcia-Cerdaña and V. Verdú. On two fragments with negation and without implication of the logic of residuated lattices. *Archive for Mathematical Logic* 45 (2006), pp. 615–647.

[3] C.C. Chang. Algebraic analysis of many-valued logics. *Transactions of the American Mathematical Society* 88 (1958), pp. 467–490.

[4] C.C. Chang. A new proof of the completeness of the Lukasiewicz axioms. *Transactions of the American Mathematical Society* 93 (1959), pp. 74–90.

[5] R. Cignoli, I.M.L. D'Ottaviano and D. Mundici. *Algebraic Foundations of Many-valued Reasoning*, Kluwer, 2000.

[6] A. Dvurečenskij. Pseudo MV-algebras are intervals in l-groups. *J. Austr. Math. Soc.* 72 (3) (2002) 427?445.

[7] F. Esteva and L. Godo. Monoidal t-norm based Logic: Towards a logic for left-continuous t-norms. *Fuzzy Sets and Systems* 124 (2001), pp. 271–288.

[8] N. Galatos and C. Tsinakis. Generalized MV-algebras. *Journal of Algebra* 283 (2005) 254?291.

[9] V.N. Grišin. Predicate and set theoretic calculi on logic without contraction. *Math. USSR Izv.* 18 (1) (1982) 42–59.

[10] U. Höhle. Commutative, residuated l-monoids. In *Non-Classical Logics and Their Applications to Fuzzy Subsets* (U. Höhle and E.P. Klement eds.), Kluwer Acad. Publ., Dordrecht, 1995, pp. 53-106.

[11] D. Mundici. Interpretation of AF C*-algebras in Lukasiewicz sentential calculus. *Journal of Functional Analysis* 65 (1986), pp. 15-63.