On Łukasiewicz logic with truth constants

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Abstract. Canonical completeness results for Ł($\mathcal{C}$), the expansion of Łukasiewicz logic Ł with a countable set of truth-constants $\mathcal{C}$, have been recently proved in [5] for the case when the algebra of truth constants $\mathcal{C}$ is a subalgebra of the rational interval $[0, 1] \cap \mathbb{Q}$. The case when $C \not\subseteq [0, 1] \cap \mathbb{Q}$ was left as an open problem. In this paper we solve positively this open problem by showing that Ł($\mathcal{C}$) is strongly canonical complete for finite theories for *any* countable subalgebra $\mathcal{C}$ of the standard Łukasiewicz chain $[0, 1]_{Ł}$.

Keywords: Łukasiewicz logic, truth-constants, canonical standard completeness

1 Introduction

The study of Łukasiewicz infinite-valued logic when adding truth constants goes back to Pavelka [14] who built a propositional many-valued logical system which turned out to be equivalent to the expansion of Łukasiewicz logic by adding into the language a truth-constant $\overline{r}$ for each *real* $r \in [0, 1]$, together with a number of additional axioms. Although the resulting logic is not strongly complete with respect to the intended semantics defined by the Łukasiewicz t-norm, (like the original Łukasiewicz logic), Pavelka proved that his logic, denoted here PL, is complete in a weaker sense. Namely, he defined the *truth degree* of a formula φ in a theory T as $\| \varphi \|_T = \inf\{e(\varphi) \mid e$ is a PL-evaluation model of $T\}$, and the *provability degree* of φ in T as $| \varphi |_T = \sup\{r \mid T \vdash_{\mathrm{PL}} \overline{r} \to \varphi\}$ and proved that these two degrees coincide. This kind of completeness is usually known as Pavelka-style completeness, and strongly relies on the continuity of Łukasiewicz truth functions (see also [7]). Novák extended Pavelka's approach to Łukasiewicz first order logic [10,11]. The approach has been fully developed in the frame of the so-called *fuzzy logic with evaluated syntax* by Novák *et al.* [13].

Hájek [8] showed that Pavelka's logic PL could be simplified while keeping Pavelka-style completeness results. Indeed he showed that it is enough to extend the language only by a countable number of truth-constants, one for each *rational* in $[0,1]$, and by adding to the logic the two following additional axiom schemata, called book-keeping axioms:

$$\overline{r}\&\overline{s} \leftrightarrow \overline{r *_{Ł} s}$$
$$\overline{r} \rightarrow \overline{s} \leftrightarrow \overline{r \Rightarrow_{Ł} s}$$

for each $r, s \in [0,1] \cap \mathbb{Q}$, where $*_{Ł}$ and $\Rightarrow_{Ł}$ are the Łukasiewicz t-norm ($x *_{Ł} y = \max(0, x+y-1)$ and its residuum ($x \Rightarrow_{Ł} y = \min(1, 1-x+y)$) respectively. He called this new system Rational Pavelka logic, RPL for short. Moreover, he proved that RPL is strongly complete for finite theories in the usual sense (not only Pavelka-style) by showing that rational truth-constants can be defined in suitable theories over Łukasiewicz logic. See also [13, Section 4.3.12] and the recent Novák's paper [12] for the possibility of disposing the irrational truth-constants in the framework of fuzzy logic with evaluated syntax.

In some recent papers [6,15,5], a new way to study expansions of logics of a left-continuous t-norm with truth-constants has been developed. If L_* is a logic of (left-continuous) t-norm $*$, and $\mathcal{C}$ is a countable L_*-subalgebra of the standard L_*-algebra $[0,1]_*$, then the logic $L_*(\mathcal{C})$ is defined as follows:

(i) the language of $L_*(\mathcal{C})$ is the one of L_* expanded with a new variable $\overline{r}$ for each $r \in C$

(ii) the axioms of $L_*(\mathcal{C})$ are those of L_* plus the bookeeping axioms

$$\overline{r}\&\overline{s} \leftrightarrow \overline{r * s}$$
$$\overline{r} \rightarrow \overline{s} \leftrightarrow \overline{r \Rightarrow_{*} s}$$

for each $r, s \in C$, where $\Rightarrow_*$ is the residuum of the norm $*$.

The key point is that $L_*(\mathcal{C})$ is an algebraizable logic and thus one can study the varieties of algebras associated to them and completeness results with respect to canonical $L_*(\mathcal{C})$-algebras, i.e. expanded L_*-algebras over [0, 1] where, for each $r \in C$, the truth-constant $\overline{r}$ is interpreted by its own value r. This type of completeness is called *canonical completeness*. Following this approach, the expansion of Gödel (and of some t-norm based logic related to the Nilpotent Minimum t-norm) with rational truth-constants, and the expansion of Product logic with countable sets of truth-constants have been respectively studied in [6] and in [15], while in [5] the authors study the general case of a logic of a continuous t-norm adding truth constants. In fact, in that paper the canonical completeness issue for Ł$(\mathcal{C})$, the expansion of Łukasiewicz logic Ł with a countable set of truth-constants $\mathcal{C}$, is solved for all cases when the algebra of truth constants $\mathcal{C}$ is a subalgebra of the rational interval $[0,1] \cap \mathbb{Q}$ (see [5, Prop. 24]). The case when C contains irrational values, i.e. when $C \not\subseteq [0,1] \cap \mathbb{Q}$, was left as an open problem[3].

[3] Note that, unlike the fuzzy logic with evaluated syntax approach developed in [13], the notion of proof in the Ł$(\mathcal{C})$ logics is finitary, and hence formulas with irrational truth-

In this paper we solve positively this open problem. Namely, we show that Ł$(\mathcal{C})$ is strongly canonical complete for finite theories for *any* countable subalgebra $\mathcal{C}$ of standard Łukasiewicz chain $[0,1]_{\text{Ł}}$. The way of proving this completeness result is by showing that any Ł$(\mathcal{C})$-chain is partially embeddable into the canonical Ł$(\mathcal{C})$-chain $[0,1]_{\text{Ł}(C)}$[4]. And to show this, we use two facts:

(i) the partial embeddability property of Product logic with truth constants proved in [15], and

(ii) a generalization of the observation in [2] showing that the standard MV-algebra $[0,1]_{\text{Ł}}$ can be embedded in a segment of the standard Product algebra $[0,1]_{\Pi}$.

The paper is organized as follows. In the next section we give some basic facts, definitions and known results and in the third section after some antecedents the proof of partial embeddability for Łukasiewicz logic with truth constants is given, and as consequence, the canonical finite strong completeness of the logics Ł$(\mathcal{C})$ is obtained, for any countable $\mathcal{C}$. We conclude with some final remarks about the extension of the completeness results in [5].

As a matter of notation, in this paper we shall use upper case calligraphic letters, e.g. $\mathcal{A}$, to denote algebras while the corresponding roman letters, e.g. A, will be used to denote their universes.

2 ℓ-groups and their relation with MV and product algebras

Through this note by an ℓ-group we will understand a lattice-ordered *commutative* group. Given an ℓ-group $\mathcal{G} = (G, +, -, 0)$, G^- will denote its negative cone, i. e., $G^- = \{x \in G : x \leq 0\}$. It is well known (see, for example [3]) that important examples of product logic algebras (Π- algebras from now on) are the negative cones of ℓ-groups with a bottom added. Indeed, given a ℓ-group $\mathcal{G}$, and $\perp \notin G$, define on the set $G^- \cup \{\perp\}$ the binary operations $\odot$ and $\Rightarrow$ as follows:

$$x \odot y = \begin{cases} x + y & \text{if } x, y \in G^-, \\ \perp & \text{otherwise,} \end{cases}$$

and

$$x \Rightarrow y = \begin{cases} 0 \wedge (y - x) & \text{if } x, y \in G^-, \\ 0 & \text{if } x = \perp, \\ \perp & \text{if } x \in G^- \text{ and } y = \perp. \end{cases}$$

Then it is easy to check that $\langle G^- \cup \{\perp\}, \odot, \Rightarrow, \perp \rangle$ is a Π-algebra, that will be denoted by $\mathcal{P}(\mathcal{G})$. Notice that G^- with the restriction of the operations $\odot$ and $\to$ becomes a *cancellative hoop* as defined in [1].

constants cannot be replaced by infinitely-many formulas with rational truth-constants for completeness purposes.

[4] It is shown in [5, Proposition 11] that the partial embeddability property is not only sufficient but also necessary to prove finite strong completeness.

Moreover if $0 < u \in G$, then it is easy to check that the segment $[-u, 0]$, equipped with the operations

$$x \otimes y = (x + y) \vee -u \quad \text{and} \quad x \to y = (y - x) \wedge 0 \tag{2.1}$$

becomes an MV-algebra that we will denote by $\mathcal{MV}(\mathcal{G}, -u)$. (cf. [1, p. 242]).

On the other hand it was observed in [2] that the standard MV-algebra $[0,1]_Ł = ([0,1], *_Ł, \Rightarrow_Ł, 0)$ can be embedded in a segment of the standard Π-algebra $[0,1]_\Pi = ([0,1], *_\Pi, \Rightarrow_\Pi, 0)$, where $*_\Pi$ is the usual product and $\Rightarrow_\Pi$ its residuum. The authors use this fact to give a faithful interpretation of Łukasiewicz logic into product logic. As a matter of fact, each MV-algebra can be embedded into a segment of a Π-algebra. Indeed, given a MV-algebra $\mathcal{A}$, there is an ℓ-group $\mathcal{G}$ and an order unit u of G such that $\mathcal{A}$ is isomorphic to the MV-algebra $\Gamma(\mathcal{G}, u)$ obtained by defining on the segment $[0, u] = \{x \in G : 0 \leq x \leq u\}$ the operations

$$x \otimes' y = (x + y - u) \vee 0 \text{ and } x \to' y = (u - x + y) \wedge u$$

(see [9,4]). Now the mapping $x \mapsto x - u$ is an isomorphism of $\Gamma(\mathcal{G}, u)$ onto the MV-algebra $\mathcal{MV}(\mathcal{G}, -u)$ obtained by equipping the segment $[-u, 0]$ of G^- with the operations $\otimes$ and $\to$ as defined in (2.1). We shall denote by Γ^- de composition of Γ with mapping $x \mapsto x - u$. It is clear that Γ^-, like Γ, establishes a functorial equivalence between the categories of MV-algebras and ℓ-groups with a distinguished order unit. In what follows we shall use both Γ^- and Γ. In particular, the standard MV-algebra $[0,1]_Ł = \Gamma(\mathbb{R}, 1)$ will be isomorphic to $\Gamma^-(\mathbb{R}, 1)$, i. e., to the MV-algebra defined on the segment $[-1, 0]$ equipped with the operations

$$x \otimes y = \max(x + y, -1), \text{ and } x \to y = \min(y - x, 0).$$

Notice that given an MV-algebra $\mathcal{A}$ we have an ℓ-group $\mathcal{G}$ and an order unit u such that $\mathcal{A} \cong \Gamma^-(\mathcal{G}, u)$ and this is in fact an interval of the product algebra $\mathcal{P}(\mathcal{G})$ defined by the negative cone of $\mathcal{G}$ adding a bottom element.

3 Adding truth constants

Given a subalgebra $\mathcal{C}$ of the standard MV-algebra $\Gamma(\mathbb{R}, 1)$, one can define (see for example [8,5]) the logic Ł$(\mathcal{C})$ as the expansion of Łukasiewicz logic Ł by the set $\overline{C} = \{\overline{c} \mid c \in C\}$ of truth constants and the corresponding book-keeping axioms, i.e., for all $r, s \in C$,

$$\overline{r} \& \overline{s} \equiv \overline{\max(r + s - 1, 0)}$$
$$\overline{r} \to \overline{s} \equiv \overline{\min(1 - r + s, 1)}$$

The only inference rule is modus ponens. The notion of proof is as in Łukasiewicz logic. We will use the notation $\vdash_{Ł(C)}$ to refer to proofs in Ł$(\mathcal{C})$.

As in [5] we define the corresponding MV($\mathcal{C}$)-algebras as the structures $\mathcal{A} = (A, \wedge, \vee, \otimes, \Rightarrow, \{\overline{r}^{\mathcal{A}}\}_{r\in C})$, where $\mathbf{A} = (A, \otimes, \Rightarrow, \overline{0}^{\mathcal{A}})$ is an MV-algebra, and satisfying the following book-keeping equations:

$$\overline{r}^{\mathcal{A}} \otimes \overline{s}^{\mathcal{A}} = \overline{\max(r+s-1,0)}^{\mathcal{A}}$$
$$\overline{r}^{\mathcal{A}} \Rightarrow \overline{s}^{\mathcal{A}} = \overline{\min(1-r+s,1)}^{\mathcal{A}}$$

for any $r, s \in C$.

For each MV($\mathcal{C}$)-algebra $\mathcal{A}$, the set $C^{\mathcal{A}} := \{\overline{r}^{\mathcal{A}} : r \in C\}$ is in fact a subalgebra of $\mathcal{A}$, that we will denote $\mathcal{C}^{\mathcal{A}}$. Since $\mathcal{C}$ is a subalgebra of the standard MV-algebra, any homomorphism from $\mathcal{C}$ into a non-trivial[5] MV-algebra is injective [4, Theorem 3.5.1]. Therefore *the mapping defining the interpretation of the constants* $r \mapsto \overline{r}^{\mathcal{A}}$ *is a bijection from* $\mathcal{C}$ *onto* $C^{\mathcal{A}}$.

The *canonical MV($\mathcal{C}$)-algebra* $[0,1]_{Ł(C)}$ is the MV($\mathcal{C}$)-algebra obtained expanding the standard MV-algebra $\Gamma(\mathbb{R}, 1)$ with the elements of the subalgebra $\mathcal{C}$ as constants.

Moreover given an MV($\mathcal{C}$)-algebra $\mathcal{A}$, an $\mathcal{A}$-evaluation e is just an $\mathbf{A}$-evaluation which is extended by $e(\overline{r}) = \overline{r}^{\mathcal{A}}$ for all $r \in C$. The notions of $\mathcal{A}$-model, $\mathcal{A}$-tautology and logical consequence $\models_{\mathcal{A}}$ are then as in the case of Łukasiewicz logic.

Finally, given a subalgebra $\mathcal{D}$ of the standard product algebra $[0,1]_{\Pi}$, one can define (see for example [5]) the logic $\Pi(\mathcal{D})$ as the expansion of the Product logic Π by the set $\overline{D} = \{\overline{c} \mid c \in D\}$ of truth constants and the corresponding book-keeping axioms. $\Pi(\mathcal{D})$-algebras are defined in an analogous way MV($\mathcal{C}$)-algebras are defined as expansions of MV-algebras. The canonical $\Pi(\mathcal{D})$-algebra defined over the standard product algebra interpreting the constant $\overline{r}$ by r itself will be denoted by $[0,1]_{\Pi(D)}$.

4 Partial embedding

First of all recall a simplified version (suficient for what we need in this paper) of the partial embeddability property for product logic with truth constants given in [15, Theorem 5.3]. Let $\mathcal{C}$ be any countable subalgebra of $[0,1]_{\Pi}$ and let $\mathcal{A}$ a $\Pi(\mathcal{C})$-chain such that the subalgebra $C^{A} = \{\overline{r}^{\mathcal{A}} \mid r \in C\}$ is isomorphic[6] to $\mathcal{C}$. Then the partial embeddability property for $\Pi(\mathcal{C})$ reads as follows.

Proposition 1 ([15]). *Let* $\mathcal{C}$ *be a countable subalgebra of* $[0,1]_{\Pi}$ *and let* $\mathcal{A} = \langle A, \wedge, \vee, \odot, \rightarrow, \{\overline{r}^{\mathcal{A}} : r \in C\}\rangle$ *a* $\Pi(\mathcal{C})$*-chain such that* $\mathcal{C}^{\mathcal{A}}$ *is isomorphic to* $\mathcal{C}$*. For any finite subset* $X \subseteq A$ *there is a partial embedding from* X *to* $[0,1]_{\Pi(C)}$*, i.e. there is a mapping* $f : X \rightarrow [0,1]$ *such that:*

[5] An MV-algebra $\mathcal{A}$ is trivial when it has only one element, i. e., $0^{\mathcal{A}} = 1^{\mathcal{A}}$

[6] It is shown in [15] that for any $\Pi(\mathcal{C})$-chain $\mathcal{A}$, the subalgebra C^{A} is either isomorphic to the 2 element Boolean algebra (when $\overline{r}^{\mathcal{A}} = \overline{1}^{\mathcal{A}}$ for all $r > 0$) or to $\mathcal{C}$ (when $\overline{r}^{\mathcal{A}} \neq \overline{s}^{\mathcal{A}}$ for each $r \neq s$).

– *if $x, y, x \circ y \in X$, then $f(x \circ y) = f(x) \circ' f(y)$ for $\circ = \odot$ and $\circ' = *_\Pi$, or for $\circ = \to$ and $\circ' = \Rightarrow_\Pi$;*
– *for any $r \in C$ such that $\overline{r}^{\mathcal{A}} \in X$, $f(\overline{r}^{\mathcal{A}}) = r$.*

It is well known [4] that if an MV-algebra $\mathcal{A}$ is isomorphic to $\Gamma(\mathcal{G}, u)$ for some ℓ-group $\mathcal{G}$ with strong unit u, and $\mathcal{S}$ is a subalgebra of $\mathcal{A}$, then there is a (unique) sub-ℓ-group $\mathcal{E}$ of $\mathcal{G}$ such that $u \in \mathcal{E}$ and $\mathcal{S} \cong \Gamma(\mathcal{E}, u)$.

Returning to our problem, suppose $\mathcal{C}$ is a countable subalgebra of the standard MV-algebra $\Gamma(\mathbb{R}, 1) = [0,1]_{\text{Ł}}$. Consequently, the subalgebra $\mathcal{C}$ is isomorphic to $\Gamma(\mathcal{H}, 1)$ for a *unique* sub-ℓ-group $\mathcal{H}$ of $\mathbb{R}$ such that $1 \in H$. Moreover the product chain $\mathcal{P}(\mathcal{H})$ is a product subalgebra of $\mathcal{P}(\mathbb{R})$. Notice that, since $\mathbb{R}$ is an archimedean group, each element of the negative cone H^- can be written as $-n + r$, with $r \in C$ and $n \in \mathbb{N}$. The mapping

$$f : \mathcal{P}(\mathbb{R}) \to [0,1]_\Pi$$

defined by $f(x) = e^x$ for $x < 0$ and $f(\bot) = 0$ is indeed an isomorphism of product algebras, and therefore, $C^\star := \{e^{-n+r} : r \in C, n \in \mathbb{N}\} \cup \{0\}$ is a subalgebra of $[0,1]_\Pi$ isomorphic to $\mathcal{P}(\mathcal{H})$. Hence we can consider the expanded logic $\Pi(\mathcal{C}^\star)$ and its canonical $\Pi(\mathcal{C}^\star)$-algebra $[0,1]_{\Pi(C^\star)}$.

Therefore, we have seen that for each countable subalgebra $\mathcal{C}$ of the standard MV-algebra $[0,1]_{\text{Ł}}$, we can define a corresponding countable subalgebra $\mathcal{C}^\star$ of the standard Π-algebra $[0,1]_\Pi$. Hence, we can associate to the canonical MV($\mathcal{C}$)-chain the canonical $\Pi(\mathcal{C}^\star)$-chain.

Now we are ready to prove the partial embeddability property for Łukasiewicz logic with truth constants.

Theorem 1. *For any countable subalgebra $\mathcal{C}$ of $[0,1]_{\text{Ł}}$ and any finite subset X of a Ł(C)-chain $\mathcal{A} = \langle A, \wedge, \vee, \otimes, \to, \{\overline{r}^{\mathcal{A}} : r \in C\}\rangle$ there is a partial embedding from X to $[0,1]_{\text{Ł}(C)}$, i.e. a mapping $f : X \to [0,1]$ such that:*

– *if $x, y, x \circ y \in X$, then $f(x \circ y) = f(x) \circ' f(y)$ for $\circ = \otimes$ and $\circ' = *_{\text{Ł}}$, or for $\circ' = \to$ and $\circ' = \Rightarrow_{\text{Ł}}$;*
– *for any $r \in C$ such that $\overline{r}^{A} \in X$, $f(\overline{r}^{A}) = r$.*

Proof: If $\mathcal{A}$ is a MV($\mathcal{C}$)-algebra, then there is an ℓ-group $\mathcal{G}$, a sub-ℓ-group $\mathcal{L}$ and an order unit u of $\mathcal{G}$ such that $\mathcal{A} \cong \Gamma(\mathcal{G}, u) \cong \Gamma^-(\mathcal{G}, u)$ and $C^{\mathcal{A}} \cong \Gamma(\mathcal{L}, u) \cong \Gamma^-(\mathcal{L}, u)$. Since $C^{\mathcal{A}}$ is isomorphic to a subalgebra of the standard MV-algebra, it follows that $\mathcal{L}$ is isomorphic to a sub-ℓ-group $\mathcal{H}$ of $\mathbb{R}$, and since u is an order unit, all the elements of the negative cone L^- can be written as $-nu + \overline{r}^{\mathcal{A}}$, for $n \in \mathbb{N}$ and $r \in C$. Thus we can consider the product algebra $\mathcal{P}(\mathcal{G})$ as a $\Pi(\mathcal{C}^\star)$-algebra, with $\overline{e^{-n+r}}^{\mathcal{P}(\mathcal{G})} = -nu + \overline{r}^{\mathcal{A}}$.

Let X be a finite subset of A. From now on we identify $\mathcal{A}$ and $\Gamma(\mathcal{G}, u)$ (hence taking $\overline{0}^{\mathcal{A}} = 0_G$ and $\overline{1}^{\mathcal{A}} = u$), and without loosing generality we can assume $u \in X$. Let $i : \Gamma(\mathcal{G}, u) \to \Gamma^-(\mathcal{G}, u)$ be defined by $i(x) = x - u$. By the partial embeddability property of Product logic with constants, the $\Pi(\mathcal{C}^\star)$-chain $\mathcal{P}(\mathcal{G})$ is partially embeddable into the canonical $[0,1]_{\Pi(C^\star)}$. Therefore, considering $i(X)$, as a subset of the $\Pi(\mathcal{C}^\star)$-chain $\mathcal{P}(\mathcal{G})$, there is a partial embedding from $i(X)$ into $[0,1]_{\Pi(\mathcal{C}^\star)}$ such that $\overline{r}^{\mathcal{A}} - u = i(\overline{r}^{\mathcal{A}}) \mapsto e^{r-1}$, for each $\overline{r}^{\mathcal{A}} \in X$. In particular, $-u = i(\overline{0}^{\mathcal{A}}) \mapsto e^{-1}$ and $0_G = i(\overline{1}^{\mathcal{A}}) \mapsto e^0 = 1$, thus all the elements of $i(X)$ go to the segment $[e^{-1}, 1]$. Applying natural logarithms, we obtain a partial embedding of $i(X)$ into $\Gamma^-(\mathbb{R}, 1)$ such that $i(\overline{r}^{\mathcal{A}}) \mapsto r - 1$ for each $\overline{r}^{\mathcal{A}} \in X$. Thus, composing i with this embedding and finally with the isomorphism from $\Gamma^-(\mathbb{R}, 1)$ to $\Gamma(\mathbb{R}, 1)$ sending $r - 1 \mapsto r$, we obtain a partial embedding of $X \subset \mathcal{A}$ into the canonical Ł$(\mathcal{C})$-chain $[0,1]_{\text{Ł}(C)}$. This ends the proof. □

Following the notation introduced in [15,5] where canonical completeness means that the logic is complete with respect to intended semantics, i.e. in our case with respect to the (canonical) algebra $[0,1]_{\text{Ł}(C)}$, the above partial embeddability property leads to the following canonical completeness result.

Corollary 1. *For any countable subalgebra $\mathcal{C}$ of $[0,1]_{\text{Ł}}$, the logic Ł$(\mathcal{C})$ is canonically finite strong complete.*

Proof: By [5, Theorem 26], Ł$(\mathcal{C})$ is finite strong complete with respect to the class of Ł$(\mathcal{C})$-chains over the standard $[0,1]_{\text{Ł}}$. But this class contains only the canonical Ł$(\mathcal{C})$-chain since $\mathcal{C}$ is simple (it has no non-trivial filters). □

5 Final remarks

The general completeness results in [5] for the expansions of the logics of a continuous t-norm adding truth-constants were restricted by the condition (called (C3) in [5]) that every truth-constant $r \in C$ belonging to a Łukasiewicz component of the t-norm generate a finite MV-chain, or equivalently, r should be a rational in the isomorphic copy of the component on $[0,1]$. Checking the proofs in [5], the result of this paper allows to remove such a restriction while keeping all the completeness results.

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