# On weakly cancellative fuzzy logics

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**Abstract**

Starting from a decomposition result of MTL-chains as ordinal sums, we focus our attention on a particular kind of indecomposable semihoops, namely weakly cancellative semihoops. Weak cancellation property is proved to be the difference between cancellation and pseudocomplementation, so it gives a new axiomatization of Product logic and ΠMTL. By adding this property, some new fuzzy logics (propositional and first-order) are defined and studied obtaining some results about their (finite) strong standard completeness and other logical and algebraic properties.

**Keywords:** Algebraic Logic, Fuzzy logics, Left-continuous t-norms, MTL-algebras, Non-classical logics, Residuated lattices, Standard completeness, Substructural logics, Varieties, Weak cancellation, WCMTL-algebras.

## 1 Introduction

Hájek introduced in [27] the logic BL as a common generalization of the three main fuzzy logics: Łukasiewicz logic, Product logic and Gödel logic, semantically defined from a continuous t-norm (the Łukasiewicz t-norm, the product of reals and the minimum, respectively). In particular, Product logic (see [29]) was proved to be the axiomatic extension of BL obtained by adding:

$\neg\neg\chi \to ((\varphi * \chi \to \psi * \chi) \to (\varphi \to \psi))$ (Π1),

and

$\varphi \wedge \neg\varphi \to \bar{0}$ (Π2),

where first one is the law of cancellativity and the second one is the law of pseudocomplementation.

Actually, Hájek conjectured that BL was complete with respect to the semantics given by continuous t-norms and their residua. This was proved by Cignoli, Esteva, Godo and

Torrens in [11]. Also in [27] an algebraic semantics was given for BL logic based on the variety of BL-algebras (bounded integral commutative prelinear divisible residuated lattices). The algebraic semantics for Łukasiewicz logic, Product logic and Gödel logic (MV-algebras, product algebras and Gödel algebras, respectively) were obtained as subvarieties of the variety of all BL-algebras.

Nevertheless, the necessary and sufficient condition for a t-norm to have a residuated implication is not the continuity, but the left-continuity. For that reason, Esteva and Godo in [16] defined a logic weaker than BL, which they called MTL (for Monoidal T-norm based Logic) aiming to capture the logic of all left-continuous t-norms and their residua. Jenei and Montagna proved in [35] that MTL was indeed complete with respect to the semantics given by the class of all left-continuous t-norms and their residua. This kind of completeness with respect to a class of left-continuous t-norms and their residua has been called *standard completeness*. Later on, in [15] the standard completeness was proved also for some other axiomatic extensions of MTL (namely SMTL and IMTL). However, the standard completeness of ΠMTL, the extension of MTL obtained by adding pseudocomplementation and cancellation (the analogue of Product logic in the non-divisible case) was left in [15] as an open problem. It remained unsolved until Horčík proved this logic to be standard complete in [32].

Esteva and Godo gave also an algebraic semantics for MTL based on MTL-algebras (bounded integral commutative prelinear residuated lattices). This class is a variety that contains the class of BL-algebras as a proper subvariety and it is possible to prove that in fact it is an equivalent algebraic semantics for MTL logic in the sense of Blok and Pigozzi [6]. Therefore, MTL is an algebraizable logic,[1] i.e. it belongs to the class of logics which is better studied by Abstract Algebraic Logic and for which this discipline gives a lot of important results. In particular, we have an order-reversing isomorphism between the lattice of subvarieties of MTL-algebras and the lattice of axiomatic extensions of MTL, which implies that the study of such extensions is equivalent to the study of varieties of MTL-algebras, and this gives a correspondence between logical and algebraic properties. The structure of BL-algebras is well-known and some important parts of their lattice of subvarieties have been completely described (see for instance [37], [2], [20]), but in the framework of MTL, i.e. when the property of divisibility is not assumed, few algebraic studies have been done till now (see [10], [25], [26], [32], [33], [41] and [42], and, in a more general framework, see [4] and [23]).

This paper is devoted to the investigation of some varieties of MTL-algebras, or equivalently to some axiomatic extensions of MTL. We focus our attention on the so called weakly cancellative MTL-algebras (WCMTL-algebras for short) and on their logic, WCMTL. WCMTL-algebras are MTL-algebras in which the monoidal operation is either cancellative or has 0 as a result. The interest of this variety and of its corresponding logic is motivated as follows:

- Both MV-algebras and Product algebras are weakly cancellative, hence WCMTL-algebras are obtained from the join of the varieties of MV-algebras and of Product algebras by removing divisibility. Moreover, it will turn out that ΠMTL-algebras are exactly WCMTL-algebras without zero divisors, and that MV-algebras are exactly the involutive WCMTL-algebras.

- While the structure of involutive MTL-algebras seems to be very hard to describe (every

[1] Actually, it has been recently proved in [22] that all the logics of a much bigger family (which includes MTL and its axiomatic extensions) are algebraizable.

MTL-algebra generates an involutive one by disconnected rotation [41], so involutive MTL-algebras can contain the zero-free reduct of any MTL-algebra), the structure of WCMTL-algebras, although not easy, seems to be more accessible. Moreover some techniques introduced by Horčík for the study of ΠMTL-algebras can be successfully applied to WCMTL-algebras.

- WCMTL-chains are either indecomposable as ordinal sums or are the ordinal sum of a two-element chain and a cancellative (hence indecomposable) residuated lattice. So they constitute an interesting example of indecomposable (or almost indecomposable) MTL-algebras. This also suggests the investigation of the variety $\Omega(\mathbb{WCMTL})$ generated by all ordinal sums of zero-free subreducts of WCMTL-algebras. Interestingly, the divisible $\Omega$(WCMTL)-algebras are precisely the BL-algebras.

The paper is organized as follows. After some preliminaries, we prove that, as in the case of BL-algebras (see [2]), all MTL-chains have a maximum decomposition as ordinal sum of indecomposable totally ordered semihoops. Then, we introduce weak cancellation to obtain a class of those indecomposable semihoops. Moreover, some interesting properties of weak cancellation are proved, obtaining a new axiomatization for the cancellative fuzzy logics (Product logic and ΠMTL) and defining a new hierarchy of fuzzy logics. We study some properties of those logics and their corresponding algebraic semantics, namely finite embedding property, finite model property and standard completeness. We finish with some concluding remarks and open problems.

## 2 Preliminaries

In [27] the logic BL is defined as a Hilbert-style calculus in the language $\mathcal{L} = \{*, \to, 0\}$ of type $\langle 2, 2, 0\rangle$ where the only inference rule is *Modus Ponens* and the axiom schemata are the following (taking $\to$ as the least binding connective):

(A1) $(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$
(A2) $\varphi * \psi \to \varphi$
(A3) $\varphi * \psi \to \psi * \varphi$
(A4) $\varphi * (\varphi \to \psi) \to \psi * (\psi \to \varphi)$
(A5a) $(\varphi \to (\psi \to \chi)) \to (\varphi * \psi \to \chi)$
(A5b) $(\varphi * \psi \to \chi) \to (\varphi \to (\psi \to \chi))$
(A6) $((\varphi \to \psi) \to \chi) \to (((\psi \to \varphi) \to \chi) \to \chi)$
(A7) $0 \to \varphi$

Some other connectives are defined as follows:
$\varphi \wedge \psi := \varphi * (\varphi \to \psi)$;

$\varphi \vee \psi := ((\varphi \to \psi) \to \psi) \wedge ((\psi \to \varphi) \to \varphi)$;

$\varphi \leftrightarrow \psi := (\varphi \to \psi) * (\psi \to \varphi)$;

$\neg\varphi := \varphi \to 0$;

$1 := \neg 0$.

Łukasiewicz logic Ł is the extension of BL with the law of involution:

$\neg\neg\varphi \to \varphi$ (Inv),

Gödel logic G can be obtained by adding to BL the contraction axiom schema:

$\varphi \to \varphi * \varphi$ (Con),

and Product logic Π can be obtained by adding to BL the following two axiom schemata:

$\neg\neg\chi \to ((\varphi * \chi \to \psi * \chi) \to (\varphi \to \psi))$ (Π1),

and

$\varphi \wedge \neg\varphi \to 0$ (Π2),

where the first one is the law of cancellativity and the second one expresses the law of pseudocomplementation (it will be called (PC) in the paper).

In [19] another important axiomatic extension of BL, SBL, is introduced by adding to BL the axiom schema (PC). This logic is weaker than Gödel and Product logic.

MTL is also presented by means of a Hilbert-style calculus in [16] but now in the enriched language $\mathcal{L} = \{*, \to, \wedge, 0\}$ of type $(2, 2, 2, 0)$. The only inference rule is again *Modus Ponens* and the axiom schemata are the following:

| | |
|---|---|
| (B1) | $(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$ |
| (B2) | $\varphi * \psi \to \varphi$ |
| (B3) | $\varphi * \psi \to \psi * \varphi$ |
| (B4) | $\varphi \wedge \psi \to \varphi$ |
| (B5) | $\varphi \wedge \psi \to \psi \wedge \varphi$ |
| (B6) | $\varphi * (\varphi \to \psi) \to \varphi \wedge \psi$ |
| (B7a) | $(\varphi \to (\psi \to \chi)) \to (\varphi * \psi \to \chi)$ |
| (B7b) | $(\varphi * \psi \to \chi) \to (\varphi \to (\psi \to \chi))$ |
| (B8) | $((\varphi \to \psi) \to \chi) \to (((\psi \to \varphi) \to \chi) \to \chi)$ |
| (B9) | $0 \to \varphi$ |

The connectives $\vee, \leftrightarrow, \neg$ and 1 are defined as in BL.

BL is proved to be the axiomatic extension of MTL obtained by adding the divisibility axiom:

$\varphi \wedge \psi \to \varphi * (\varphi \to \psi)$ (Div)

Some other axiomatic extensions of MTL are introduced in [16]. Namely, IMTL is obtained by adding the axiom schema (Inv), SMTL is obtained by adding (PC), and ΠMTL is obtained by adding (PC) and (Π2).

Let $Fm_{\mathcal{L}}$ be the set of $\mathcal{L}$-formulas built over a countable set of variables. Given $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}$, we write $\Gamma \vdash_{\mathrm{MTL}} \varphi$ if, and only if, $\varphi$ is provable from $\Gamma$ in the system MTL.

**Definition 2.1** ([16]). *Let $\mathcal{A} = \langle A, *, \to, \wedge, \vee, 0, 1\rangle$ be an algebra of type $\langle 2, 2, 2, 2, 0, 0\rangle$. $\mathcal{A}$ is an* MTL*-algebra iff it is a bounded integral commutative residuated lattice satisfying the prelinearity equation:*

$$(x \to y) \vee (y \to x) \approx 1$$

*The negation operation is defined as* $\neg a = a \rightarrow 0$. *If the lattice order is total we will say that* $\mathcal{A}$ *is an* MTL-*chain. The* MTL-*chains defined over the real unit interval* $[0,1]$ *(with the usual order) are those where* $*$ *is a left-continuous t-norm*[2] *and they are called standard* MTL-*chains. If* $\circ$ *is a left-continuous t-norm,* $[0,1]_\circ$ *will denote the standard chain given by* $\circ$.

The zero-free subreducts of MTL-algebras are term equivalent to *prelinear semihoops*, which are defined as follows:

**Definition 2.2** ([18]). *An algebra* $\mathcal{A} = \langle A, *, \rightarrow, \wedge, 1\rangle$ *of type* $\langle 2,2,2,0\rangle$ *is a prelinear semihoop*[3] *iff:*

- $\mathcal{A} = \langle A, \wedge, 1\rangle$ *is an inf-semilattice with upper bound.*
- $\langle A, *, 1\rangle$ *is a commutative monoid isotonic w.r.t. the inf-semilattice order.*
- *For every* $a, b \in A$, $a \leq b$ *iff* $a \rightarrow b = 1$.
- *For every* $a, b, c \in A$, $a * b \rightarrow c = a \rightarrow (b \rightarrow c)$.
- *For every* $a, b, c \in A$, $(a \rightarrow b) \rightarrow c \leq ((b \rightarrow a) \rightarrow c) \rightarrow c$.

*An operation* $\vee$ *is defined as:* $a \vee b = ((a \rightarrow b) \rightarrow b) \wedge ((b \rightarrow a) \rightarrow a)$. *If in addition it has a minimum element, then it is a bounded prelinear semihoop (i.e. term equivalent to an* MTL-*algebra).*

For the discussion of the paper we will need some usual notions defined for MTL-algebras and prelinear semihoops. We write them here for the reader's convenience.

**Definition 2.3** ([2]). *Let* $\langle I, \leq\rangle$ *be a totally ordered set. For all* $i \in I$, *let* $\mathcal{A}_i$ *be a totally ordered semihoop (hence prelinear) such that for* $i \neq j$, $A_i \cap A_j = \{1\}$. *Then* $\bigoplus_{i\in I} \mathcal{A}_i$ *(the ordinal sum of the family* $\{\mathcal{A}_i : i \in I\}$*) is the structure whose universe is* $\bigcup_{i\in I} A_i$ *and whose operations are:*

$$x * y = \begin{cases} x *^{\mathcal{A}_i} y & \text{if } x, y \in A_i, \\ y & \text{if } x \in A_i \text{ and } y \in A_j \setminus \{1\} \text{ with } i > j, \\ x & \text{if } x \in A_i \setminus \{1\} \text{ and } y \in A_j \text{ with } i < j. \end{cases}$$

$$x \rightarrow y = \begin{cases} x \rightarrow^{\mathcal{A}_i} y & \text{if } x, y \in A_i, \\ y & \text{if } x \in A_i \text{ and } y \in A_j \text{ with } i > j, \\ 1 & \text{if } x \in A_i \setminus \{1\} \text{ and } y \in A_j \text{ with } i < j. \end{cases}$$

*For every* $i \in I$, $\mathcal{A}_i$ *is called a component of the ordinal sum.*

*If in addition* $I$ *has a minimum, say* $i_0$, *and* $\mathcal{A}_{i_0}$ *is bounded, then the ordinal sum* $\bigoplus_{i\in I} \mathcal{A}_i$ *forms an* MTL-*chain.*

[2]A t-norm is a binary operation $\circ : [0,1]^2 \rightarrow [0,1]$ which is associative, commutative, isotonic and has 1 as a neutral element (see [36]).

[3]These algebras are sometimes also called MTLH-algebras.

**Definition 2.4.** *Let $\mathcal{A}$ be an* MTL*-chain or a totally ordered semihoop. We define a binary relation $\sim$ on $A$ by letting for every $a, b \in A$, $a \sim b$ if, and only if, there is $n \geq 1$ such that $a^n \leq b \leq a$ or $b^n \leq a \leq b$. It is easy to check that $\sim$ is an equivalence relation. Its equivalence classes are called* Archimedean classes. *Given $a \in A$, its Archimedean class is denoted as $[a]_\sim$.*

A filter in an MTL-algebra is defined in [16] as any subset $F$ such that:

- $1 \in F$
- If $a \in F$ and $a \leq b$, then $b \in F$
- If $a, b \in F$, then $a * b \in F$.

$F(a)$ will denote the principal filter generated by the element $a$. It can be described as follows: $F(a) = \{b : a^n \leq b \text{ for some } n \geq 1\}$. There is the usual correspondence between filters and congruences in MTL-algebras:

**Proposition 2.5.** *Let $\mathcal{A}$ be an* MTL*-algebra. For every filter $F \subseteq A$ we define $\Theta(F) := \{\langle a, b\rangle \in A^2 : a \leftrightarrow b \in F\}$, and for every congruence $\theta$ of $\mathcal{A}$ we define $Fi(\theta) := \{a \in A : \langle a, 1\rangle \in \theta\}$. Then, $\Theta$ is an order isomorphism from the set of filters onto the set of congruences and $Fi$ is its inverse.*

Given a filter $F$ and an element $a$, $[a]_F$ will denote the equivalence class of $a$ w.r.t. to the congruence $\Theta(F)$.

**Theorem 2.6** ([16])**.** MTL*-algebras are representable as a subdirect product of* MTL*-chains.*

$\mathbb{MTL}$ will denote the class of all MTL-algebras. It is well known that this class is definable by equations; hence it is a variety.[4] Given $\Gamma \cup \{\varphi\} \subseteq Fm_\mathcal{L}$ and $\mathcal{A} \in \mathbb{MTL}$, we write $\Gamma \models_\mathcal{A} \varphi$ if, and only if, $v(\varphi) = 1^\mathcal{A}$ whenever $v$ is an evaluation of the formulas in $\mathcal{A}$ such that $v[\Gamma] \subseteq \{1^\mathcal{A}\}$. With this notation we can express the completeness of MTL with respect to the semantics given by $\mathbb{MTL}$ as follows:

For every $\Gamma \cup \{\varphi\} \subseteq Fm_\mathcal{L}$, $\Gamma \vdash_{\mathrm{MTL}} \varphi$ iff $\Gamma \models_\mathcal{A} \varphi$ for every $\mathcal{A} \in \mathbb{MTL}$.

Furthermore, it is possible to prove that MTL is an algebraizable logic in the sense of Blok and Pigozzi [6] and $\mathbb{MTL}$ is its equivalent algebraic semantics. This implies that there is an order-reversing isomorphism between axiomatic extensions of MTL and subvarieties of $\mathbb{MTL}$. If $\Sigma \subseteq Fm_\mathcal{L}$ and L is the logic obtained by adding to MTL the formulas of $\Sigma$ as axiom schemata, then the equivalent algebraic semantics of L is the subvariety of $\mathbb{MTL}$ axiomatized by the equations $\{\varphi \approx 1 : \varphi \in \Sigma\}$. We denote this variety by $\mathbb{L}$ and we call its members *L-algebras*. We will do two exceptions to that rule: the algebras associated to Ł are called *MV-algebras* following the terminology of Chang in [9], and the algebras associated to the Classical Propositional Calculus (CPC for short) are called, of course, *Boolean algebras* (we will use $\mathcal{B}_2$ to denote the Boolean algebra of two elements, and $\mathbb{BA}$ to denote the variety of Boolean algebras).

All the logics so far mentioned are propositional logics. However, already in [27], the first-order version for BL and its axiomatic extensions is introduced and in [16] it is generalized to MTL and its axiomatic extensions.

[4] For any unexplained notion on Universal Algebra see [8].

Given a first-order language $\mathcal{J}$ and some axiomatic extension L of MTL[5], the first-order version of L in the language $\mathcal{J}$, L$\forall$, is defined by means of a Hilbert-style calculus where the axioms are all formulas resulting from the axioms of L by substituting the propositional variables for arbitrary formulas of $\mathcal{J}$, plus the following axiom schemata for the quantifiers:

($\forall$1) $\forall x\, \varphi(x) \to \varphi(t)$ (where $t$ is a term substitutable for $x$ in $\varphi(x)$)

($\exists$1) $\varphi(t) \to \exists x\, \varphi(x)$ (where $t$ is a term substitutable for $x$ in $\varphi(x)$)

($\forall$2) $\forall x(\nu \to \varphi) \to (\nu \to \forall x\, \varphi)$ (where $x$ is not free in $\nu$)

($\exists$2) $\forall x(\nu \to \varphi) \to (\exists x\, \varphi \to \nu)$ (where $x$ is not free in $\nu$)

($\forall$3) $\forall x(\nu \vee \varphi) \to (\forall x\, \varphi \vee \nu)$ (where $x$ is not free in $\nu$)

and the inference rules are Modus Ponens and Generalization: $\frac{\varphi}{\forall x\, \varphi}$.

The semantics for L$\forall$ logic is defined in the following way. Given an L-chain $\mathcal{A}$, an $\mathcal{A}$-structure for $\mathcal{J}$ is defined as $\mathcal{M} = \langle M, (r_P)_P, (m_c)_c \rangle$, where $M \neq \emptyset$, for each n-ary predicate $P$ of $\mathcal{J}$, $r_P$ is a function from $M^n$ to $A$, and for each constant symbol $c$ of $\mathcal{J}$, $m_c$ is an element of $M$. An $\mathcal{M}$-evaluation is a mapping $v$ assigning to each object variable $x$ of $\mathcal{J}$, an element $v(x) \in M$. Given two $\mathcal{M}$-evaluations $v$ and $v'$, and an object variable $x$, $v \equiv_x v'$ means that $v(y) = v'(y)$ for every $y \neq x$.

The values of the terms given by $\mathcal{M}$ and $v$ are defined as: $||x||_{\mathcal{M},v} = v(x)$ and $||c||_{\mathcal{M},v} = m_c$. The truth value of a formula $\varphi$ is an element $||\varphi||^{\mathcal{A}}_{\mathcal{M},v} \in A$ and it is defined inductively as:

$$||P(t_1, \ldots, t_n)||^{\mathcal{A}}_{\mathcal{M},v} = r_P(||t_1||_{\mathcal{M},v}, \ldots, ||t_n||_{\mathcal{M},v})$$
$$||\varphi * \psi||^{\mathcal{A}}_{\mathcal{M},v} = ||\varphi||^{\mathcal{A}}_{\mathcal{M},v} *^{\mathcal{A}} ||\psi||^{\mathcal{A}}_{\mathcal{M},v}$$
$$||\varphi \wedge \psi||^{\mathcal{A}}_{\mathcal{M},v} = ||\varphi||^{\mathcal{A}}_{\mathcal{M},v} \wedge^{\mathcal{A}} ||\psi||^{\mathcal{A}}_{\mathcal{M},v}$$
$$||\varphi \to \psi||^{\mathcal{A}}_{\mathcal{M},v} = ||\varphi||^{\mathcal{A}}_{\mathcal{M},v} \to^{\mathcal{A}} ||\psi||^{\mathcal{A}}_{\mathcal{M},v}$$
$$||0||^{\mathcal{A}}_{\mathcal{M},v} = 0^{\mathcal{A}}$$
$$||1||^{\mathcal{A}}_{\mathcal{M},v} = 1^{\mathcal{A}}$$
$$||\forall x\, \varphi||^{\mathcal{A}}_{\mathcal{M},v} = \inf\{||\varphi||^{\mathcal{A}}_{\mathcal{M},v'} : v \equiv_x v'\}$$
$$||\exists x\, \varphi||^{\mathcal{A}}_{\mathcal{M},v} = \sup\{||\varphi||^{\mathcal{A}}_{\mathcal{M},v'} : v \equiv_x v'\}$$

if the suprema and the infima exist in $\mathcal{A}$; otherwise the truth value of the formula is undefined. The structure $\mathcal{M}$ is called $\mathcal{A}$-safe when the truth values are defined for every formula and every $\mathcal{M}$-evaluation.

Given an $\mathcal{A}$-structure $\mathcal{M}$ and a $\mathcal{J}$-formula $\varphi$, the truth value of $\varphi$ in $\mathcal{M}$ is defined as $||\varphi||^{\mathcal{A}}_{\mathcal{M}} = \inf\{||\varphi||^{\mathcal{A}}_{\mathcal{M},v} : v\, \mathcal{M}\text{-evaluation}\}$. Let $T$ be a set of $\mathcal{J}$-formulas. $\mathcal{M}$ is an $\mathcal{A}$-model of $T$ if $||\varphi||^{\mathcal{A}}_{\mathcal{M}} = 1^{\mathcal{A}}$ for every $\varphi \in T$. With this notation, we can write the completeness theorem for all first-order fuzzy logics (cf. [13]).

**Theorem 2.7.** *Let $\mathcal{J}$ be a first-order language and* L *an axiomatic extension of* MTL. *Then, for every $\mathcal{J}$-formula $\varphi$ and every set of $\mathcal{J}$-formulas $T$, $T \vdash_{\mathrm{L}\forall} \varphi$ if, and only if, for each* L*-chain $\mathcal{A}$ and each $\mathcal{A}$-safe model of $T$, $||\varphi||^{\mathcal{A}}_{\mathcal{M}} = 1^{\mathcal{A}}$.*

We also need to recall some relevant properties of Universal Algebra.

**Definition 2.8.** *A class $\mathbb{K}$ of algebras is locally finite (LF, for short) if, and only if, for every $\mathcal{A} \in \mathbb{K}$ and for every finite set $B \subseteq A$, the subalgebra generated by $B$, $\langle B \rangle_{\mathcal{A}}$, is also finite.*

[5]We consider MTL as a trivial axiomatic extension of itself.

**Definition 2.9.** *Let $\mathcal{A} = \langle A, \langle f_i : i \in I\rangle\rangle$ be an algebra and let $B \subseteq A$ be a non-empty set. The partial subalgebra $\mathcal{B}$ of $\mathcal{A}$ with domain $B$ is the partial algebra $\langle B, \langle f_i : i \in I\rangle\rangle$, where for every $i \in I$, $f_i$ n-ary, $b_1, \ldots, b_n \in B$,*

$$f_i^{\mathcal{B}}(b_1, \ldots, b_n) = \begin{cases} f_i^{\mathcal{A}}(b_1, \ldots, b_n) & \text{if } f_i^{\mathcal{A}}(b_1, \ldots, b_n) \in B, \\ undefined & \text{otherwise.} \end{cases}$$

Given a class $\mathbb{K}$ of algebras, $\mathbb{K}_{fin}$ will denote the class of its finite members.

**Definition 2.10.** *A class $\mathbb{K}$ of algebras has the finite embeddability property (FEP, for short) if, and only if, every finite partial subalgebra of some member of $\mathbb{K}$ can be embedded in some algebra of $\mathbb{K}_{fin}$.*

**Definition 2.11.** *A class $\mathbb{K}$ of algebras of the same type has the strong finite model property (SFMP, for short) if, and only if, every quasiequation that fails to hold in every algebra of $\mathbb{K}$ can be refuted in some member of $\mathbb{K}_{fin}$.*

**Definition 2.12.** *A class $\mathbb{K}$ of algebras of the same type has the finite model property (FMP, for short) if, and only if, every equation that fails to hold in every algebra of $\mathbb{K}$ can be refuted in some member of $\mathbb{K}_{fin}$.*

A variety has the FMP if, and only if, it is generated by its finite members and a quasivariety has the SFMP if, and only if, it is generated (as a quasivariety) by its finite members. In [7] it is proved that for classes of algebras of finite type closed under finite products (hence, in particular, for varieties of MTL-algebras) the FEP and the SFMP are equivalent. Moreover, it is clear that for every class of algebras $\mathbb{L}$ which is the equivalent algebraic semantics of a logic L, we have:

- If $\mathbb{L}$ is locally finite, then it has the FEP.
- If $\mathbb{L}$ has the FEP, then it has the FMP.
- If $\mathbb{L}$ has the FMP, then L is decidable.

None of these implications can be inverted.

$\mathbb{G}$ is locally finite, so it has all the properties. The FEP is true for MV-algebras (see [5]), BL-algebras and SBL-algebras (see [1] and [39]). The FEP also holds in $\mathbb{MTL}$, $\mathbb{IMTL}$ and $\mathbb{SMTL}$ (proved by Ono, private communication). However, since there are no finite $\Pi$-chains and $\Pi$MTL-chains (except for the trivial one and $\mathcal{B}_2$), the FMP (hence also the FEP) fails for $\Pi$MTL and $\Pi$. Nevertheless, Product logic is decidable (see [27]), but the decidability of $\Pi$MTL is an open question.

Finally, we recall the notion of standard completeness. If a logic L is an axiomatic extension of MTL, we say that L enjoys *(finite) strong standard completeness* if, and only if, for every (finite) set of formulas $T \subseteq Fm_{\mathcal{L}}$ and every formula $\varphi$, $T \vdash_L \varphi$ iff $T \models_{\mathcal{A}} \varphi$ for every standard L-algebra $\mathcal{A}$. We will call this property (F)SSC, for short. The three main fuzzy logics enjoy FSSC; it is proved in [31] for Ł, in [29] for Product logic and in [14] for Gödel logic. But only for the last one SSC is true. SSC for MTL, IMTL and SMTL is proved in [35] and [15]. FSSC is also true for BL and SBL (proved in [11]) and for $\Pi$MTL (proved in [32]).

A first-order fuzzy logic L∀ enjoys *(finite) strong standard completeness* (again we use (F)SSC, for short) if, and only if, for every (finite) set of formulas $T \cup \{\varphi\}$, $T \vdash_{\text{L}\forall} \varphi$ iff

$||\varphi||^{\mathcal{A}}_{\mathcal{M}} = 1^{\mathcal{A}}$ for every model $\mathcal{M}$ of $T$ over a standard L-chain $\mathcal{A}$. G∀ and MTL∀ enjoy SCC (see [27] and [40], respectively). On the other hand, Ł∀, Π∀ and BL∀ do not enjoy FSSC (see [28] and [38]).

# 3 Main results

## 3.1 Decomposition of MTL-chains as ordinal sums of totally ordered semihoops

**Definition 3.1.** *A totally ordered semihoop is indecomposable if, and only if, it is not isomorphic to any ordinal sum of two non-trivial totally ordered semihoops.*

**Theorem 3.2.** *For every* MTL*-chain $\mathcal{A}$, there is a maximum decomposition as ordinal sum of indecomposable totally ordered semihoops, with the first one bounded.*

*Proof.* First we need to define the set $\mathcal{D}$ of decompositions of $\mathcal{A}$. For every $F \subseteq \mathcal{P}(A \setminus \{1\})$, $F \in \mathcal{D}$ if, and only if, $F$ is a partition of $A \setminus \{1\}$ such that for every $B \in F$, $B \cup \{1\}$ is a subuniverse of the zero-free reduct of $\mathcal{A}$ (hence the universe of a totally ordered semihoop $\mathcal{B}$) and $\mathcal{A} = \bigoplus\{\mathcal{B} : B \in F\}$. A partial order $\preceq$ is defined in $\mathcal{D}$ in the following way:
for every $F, G \in \mathcal{D}$, $F \preceq G$ if, and only if, for each $B \in G$ there is a $B' \in F$ such that $B \subseteq B'$, i.e. the decomposition $G$ is finer than $F$.

We will use Zorn's Lemma to show that the partially ordered set $\langle \mathcal{D}, \preceq\rangle$ has some maximal element. Suppose that $\mathcal{C} = \{D_k : k \in K\}$ is a chain of $\langle \mathcal{D}, \preceq\rangle$. Then, we define the following equivalence relation on $A \setminus \{1\}$:
For every $a, b \in A \setminus \{1\}$, $a \equiv b$ if, and only if, $a$ and $b$ belong to the same class of $D_k$ for every $k \in K$. Let $[a]_\equiv$ denote the equivalence class of $a$ w.r.t $\equiv$. We will prove that $\{[a]_\equiv : a \in A \setminus \{1\}\} \in \mathcal{D}$ and it is an upper bound of $\mathcal{C}$. Take $a \in A \setminus \{1\}$. It is straightforward to check that $[a]_\equiv \cup \{1\}$ is closed under $*$ and $\rightarrow$. Now take $a, b \in A \setminus \{1\}$ such that $a < b$ and $[a]_\equiv \neq [b]_\equiv$. Then, there is some $k \in K$ such that $a$ and $b$ are not in the same component of $D_k$, thus $a * b = a$. Therefore, $\{[a]_\equiv : a \in A \setminus \{1\}\} \in \mathcal{D}$. Now take arbitrary $k \in K$ and $a \in A \setminus \{1\}$. Then, by the definition of $\equiv$ all the elements of $[a]_\equiv$ must be in the same component of $D_k$, so $D_k \preceq \{[a]_\equiv : a \in A \setminus \{1\}\}$.

Therefore, by Zorn's Lemma for every $F \in \mathcal{D}$, there exists a maximal decomposition $M \in \mathcal{D}$ such that $F \preceq M$. Finally, we will prove that there is a maximum one, i.e. there cannot be two different maximal decompositions. Suppose that $M_1, M_2 \in \mathcal{D}$ are two different maximal elements. Then there is $A \in M_1$ which is not included in any element of $M_2$. Moreover, $A$ is indecomposable so it is not a union of elements of $M_2$, thus there is $B \in M_2$ such that $A \cap B \neq \emptyset$ and $B \not\subseteq A$. Then it is easy to see that $A$ could be decomposed as ordinal sum of $A \cap B$ and $A \setminus B$, a contradiction. □

**Corollary 3.3.** *Let $\mathcal{A}$ be an* MTL*-chain. If the partition $\{[a]_\sim : a \in A \setminus \{1\}\}$ given by the Archimedean classes gives a decomposition as ordinal sum, then it is the maximum one.*

*Proof.* With the notation of the previous proof, take an arbitrary $F \in \mathcal{D}$. For every $a \in A\setminus\{1\}$, there is some $B \in F$ such that $[a]_\sim \subseteq B$, since the elements of $F$ are closed under $*$. Therefore, $F \preceq \{[a]_\sim : a \in A \setminus \{1\}\}$. So if $\{[a]_\sim : a \in A \setminus \{1\}\} \in \mathcal{D}$, then it is the maximum. □

Unfortunately, the class of indecomposable totally ordered semihoops is really big. For instance, as the following proposition proves, all involutive MTL-chains are indecomposable.

**Proposition 3.4.** *All* IMTL-*chains are indecomposable.*

*Proof.* Let $\mathcal{A}$ be an IMTL-chain. If $\mathcal{A} \cong \mathcal{B}_2$, it is clearly indecomposable. Suppose that $\mathcal{A} \ncong \mathcal{B}_2$ and it is decomposable as ordinal sum of two non-trivial totally ordered semihoops, i.e. $\mathcal{A} \cong \mathcal{C}_1 \oplus \mathcal{C}_2$. Then, there is $a \in C_2 \setminus \{1\}$ and it satisfies $\neg\neg a = 1$, but this contradicts the fact that the negation is involutive. □

### 3.2 The property of weak cancellation

In this section we will study a different class of indecomposable semihoops that seems more accessible than the class of all IMTL-chains. These semihoops are defined by considering a generalization of the property of cancellation that we will call *weak cancellation*.

An MTL-chain $\mathcal{A}$ is said to be cancellative if, and only if, for every $a, b, c \in A$ if $a \neq 0$ and $a * b = a * c$, then $b = c$. This property is typically satisfied by the product of real numbers. The axiom ($\Pi$1) was proposed to express the law of cancellation in order to axiomatize the logic of the product t-norm. Nevertheless, ($\Pi$1) is proved to be equivalent to the property of cancellativity in the presence of the axiom (PC), i.e. it is equivalent to the cancellativity for SMTL-chains. Now we propose an alternative axiom that is equivalent to the cancellativity for all MTL-chains:

$\neg\psi \vee ((\psi \to \varphi * \psi) \to \varphi)$ (C).

**Proposition 3.5.** *The variety generated by cancellative* MTL-*chains is axiomatized by the equation corresponding to axiom (C), i.e.* $\neg y \vee ((y \to x * y) \to x) \approx 1$.

*Proof.* Let $\mathcal{A}$ be an MTL-chain. We have to prove that $\mathcal{A} \models \neg y \vee ((y \to x * y) \to x) \approx 1$ if, and only if, $\mathcal{A}$ is cancellative. First, suppose that the equation is valid in $\mathcal{A}$ and take $a, b, c \in A$ such that $a \neq 0$ and $a * b = a * c$. Then, using the equation we have: $(a \to b * a) \to b = (a \to c * a) \to c = 1$, hence $a \to b * a = b$ and $a \to c * a = c$, so $b = c$. Conversely, suppose that the chain is cancellative and let's check that for any pair of elements $a, b \in A$ we have $\neg b \vee ((b \to a * b) \to a) = 1$. If $b = 0$, it is obviously true. Otherwise, using the cancellativity we obtain $b \to a * b = a$, so the equation is also true. □

Therefore, in the axiomatization of Product logic and $\Pi$MTL we could replace the axiom ($\Pi$1) by (C). But, in fact, the law of cancellation implies the pseudocomplementation as the following lemma shows.

**Lemma 3.6.** *Let $\mathcal{A}$ be an* MTL-*chain. If* $\mathcal{A} \models \neg y \vee ((y \to x * y) \to x) \approx 1$*, then* $\mathcal{A} \models x \wedge \neg x \approx 0$.

*Proof.* If there exists $a \in A$ such that $a \wedge \neg a \neq 0$, then $a, \neg a \neq 0$. Thus, applying cancellation, from $a * \neg a = a * 0$ we obtain $\neg a = 0$, a contradiction. □

**Corollary 3.7.** $\Pi$ *is the axiomatic extension of* BL *obtained by adding (C) and* $\Pi$MTL *is the axiomatic extension of* MTL *obtained by adding (C).*

In particular, we have found a new axiomatization for Product logic that is also different from the one proposed[6] by Cintula in [12].

[6] This axiomatization was also obtained by adding only one axiom with two variables to BL. In fact, it was proved in the same paper that it cannot be done with one axiom in one variable only.

Therefore, cancellativity (C) is a very strong axiom for the axiomatization of Product logic and ΠMTL which makes (PC) superfluous. We may wonder if there is an axiom which does not imply (C) but, added to SBL (resp. SMTL) gives an axiomatization of Π (resp. ΠMTL). We will prove that the answer to this question is provided by the following weaker form of cancellativity:

**Definition 3.8.** *Let $\mathcal{A}$ be an* MTL*-chain. We say that $\mathcal{A}$ is weakly cancellative if, and only if, for every $a, b, c \in A$ if $a * b = a * c \neq 0$, then $b = c$.*

Analogously to Proposition 3.5 we can give an equivalent equation for this property:

**Proposition 3.9.** *Let $\mathcal{A}$ be an* MTL*-chain. Then, $\mathcal{A} \models \neg(x * y) \vee ((y \to x * y) \to x) \approx 1$ if, and only if, $\mathcal{A}$ is weakly cancellative.*

We will refer to the corresponding axiom schema as *axiom of weak cancellation* (WC):
$\neg(\varphi * \psi) \vee ((\psi \to \varphi * \psi) \to \varphi)$ (WC)
This axiom turns out to be the difference between pseudocomplementation and cancellation that we were looking for:

**Proposition 3.10.** *Let $\mathcal{A}$ be an* MTL*-chain. Then the following are equivalent:*

*(i)* $\mathcal{A} \models x \wedge \neg x \approx 0$ *and* $\mathcal{A} \models \neg(x * y) \vee ((y \to x * y) \to x) \approx 1$

*(ii)* $\mathcal{A} \models \neg y \vee ((y \to x * y) \to x) \approx 1$

*Proof.* $(ii) \Rightarrow (i)$: It follows from lemma 3.6. $(i) \Rightarrow (ii)$: Suppose that $a * b = a * c$ for some $a, b, c \in A$ with $a \neq 0$. If $a * b \neq 0$, then by weak cancellation $b = c$. Suppose now that $a * b = 0$, i.e. $a \leq \neg b$. If $b \neq 0$, then $\neg b = 0$ (by pseudocomplementation), hence $a = 0$, a contradiction. Thus $b = 0$ and analogously $c = 0$, so $b = c$. □

Another interesting fact about weak cancellation is that $(WC)$ added to IMTL axiomatizes Łukasiewicz logic. Recall that an MTL-algebra satisfying $x \vee y \approx (x \to y) \to y$ is already an MV-algebra.

**Proposition 3.11.** *Let $\mathcal{A}$ be an* IMTL*-chain. Then, $\mathcal{A} \models x \vee y \approx (x \to y) \to y$ if, and only if, $\mathcal{A} \models \neg(x * y) \vee ((y \to x * y) \to x) \approx 1$.*

*Proof.* One direction follows from the fact that all MV-algebras are weakly cancellative. For the other one, suppose that $\mathcal{A}$ is a weakly cancellative IMTL-chain and take a pair of arbitrary elements $a, b \in A$. We have to check that $a \vee b = (a \to b) \to b$. If $a \leq b$, it is obvious. Suppose $a > b$, i.e. $\neg b * a \neq 0$. Then, $(a \to b) \to b = \neg b \to \neg(a \to b) = \neg b \to a * \neg b = a = a \vee b$, by weak cancellation. □

**Corollary 3.12.** *Łukasiewicz logic is the axiomatic extension of* IMTL *obtained by adding the axiom schema (WC).*

Therefore, in the involutive case the property of weak cancellation is not giving any new logic. But in the general case we obtain a new logic and a new variety of MTL-algebras. Let WCMTL be the axiomatic extension of MTL obtained by adding (WC). Of course its equivalent algebraic semantics is the variety of weakly cancellative MTL-algebras, that are called WCMTL-algebras. We will now axiomatize their zero-free subreducts, the weakly cancellative prelinear semihoops.

**Proposition 3.13.** *The class of zero-free subreducts of* WCMTL*-algebras is the variety of prelinear semihoops satisfying the equation:*
$(x * y \to z) \vee ((y \to x * y) \to x) \approx 1$.

*Proof.* Let $\mathcal{A}$ be a totally ordered semihoop. We have to check that $\mathcal{A} \models (x * y \to z) \vee ((y \to x * y) \to x) \approx 1$ if, and only if, $\mathcal{A}$ is a zero-free subreduct of some WCMTL-chain $\mathcal{C}$. First suppose that $\mathcal{A} \models (x*y \to z) \vee ((y \to x*y) \to x) \approx 1$. If there is a minimum element in $\mathcal{A}$, say $m$, then we define $\mathcal{C}$ as the $\mathcal{L}$-expansion of $\mathcal{A}$ where 0 is interpreted as $m$. It is obvious that $\mathcal{C}$ satisfies the equation of weak cancellation for MTL-algebras. If $\mathcal{A}$ has no minimum, then define $\mathcal{C} := \mathcal{B}_2 \oplus \mathcal{A}$. It is clear that $\mathcal{C}$ is an MTL-chain and $\mathcal{A}$ is one of its zero-free subreducts. To check that it satisfies the equation of weak cancellation for MTL-algebras, take an arbitrary pair of elements $a, b \in C$ such that $a*b \neq 0$ (hence $a, b \in A$). Then, since there is no minimum in $\mathcal{A}$, there is some $c < a*b$, hence $a*b \to c \neq 1$, which implies $(b \to a*b) \to a = 1$. Conversely, suppose that $\mathcal{A}$ is a zero-free subreduct of some WCMTL-chain $\mathcal{C}$. Then for every $a, b, c \in A$, we have $(a * b \to c) \vee ((b \to a * b) \to a) \geq (a * b \to 0) \vee ((b \to a * b) \to a) = 1$. □

We will show now that this kind of semihoops gives some examples of indecomposable totally ordered semihoops.

**Proposition 3.14.** *Let $\mathcal{A}$ be a weakly cancellative totally ordered semihoop. Then:*

*(1) If $\mathcal{A}$ is unbounded, then it is indecomposable.*

*(2) Suppose that $\mathcal{A}$ is bounded.*

*(2.1) If $\mathcal{A}$ has no zero divisors, then it is a* ΠMTL*-chain and it is decomposable as $\mathcal{A} \cong \mathcal{B}_2 \oplus \mathcal{C}$, where $\mathcal{C}$ is the zero-free subreduct whose domain is $A \setminus \{0\}$.*

*(2.2) If $\mathcal{A}$ has zero divisors, then it is indecomposable.*

*Proof.* First suppose that $\mathcal{A}$ is unbounded and decomposable as $\mathcal{A} \cong \mathcal{C}_1 \oplus \mathcal{C}_2$. Then, take $a \in C_1 \setminus \{1\}$ and $b \in C_2 \setminus \{1\}$. Since it is unbounded there is some $c < a$. Then, the equation of weakly cancellative semihoops would not hold because $a * b \to c = a \to c \neq 1$ and $(a \to a * b) \to b = (a \to a) \to b = b \neq 1$. Now suppose that $\mathcal{A}$ is bounded and has no zero divisors. This means that it is pseudocomplemented, hence, by Proposition 3.10 it is cancellative, i.e. a ΠMTL-chain. Clearly, it is decomposable as $\mathcal{A} \cong \mathcal{B}_2 \oplus \mathcal{C}$, where $C = A \setminus \{1\}$. Suppose that $\mathcal{A}$ is bounded, it has zero divisors and it is decomposable as $\mathcal{A} \cong \mathcal{C}_1 \oplus \mathcal{C}_2$. Then, the existence of zero divisors implies that $\mathcal{C}_1 \ncong \mathcal{B}_2$. Take $a \in C_1 \setminus \{0, 1\}$ and $b \in C_2 \setminus \{1\}$. Then, $a * b \to 0 = a \to 0 \neq 1$ and $(a \to a * b) \to b = (a \to a) \to b = b \neq 1$, so $\mathcal{A}$ cannot be weakly cancellative. □

Given an MTL-chain $\mathcal{A}$ and an element $a \in A$, the truncation of $\mathcal{A}$ with respect to $a$ is the algebra $\mathcal{A}[a] = \langle \{x \in A : a \leq^{\mathcal{A}} x \leq^{\mathcal{A}} 1^{\mathcal{A}}\}, *_a^{\mathcal{A}}, \to_a^{\mathcal{A}}, \leq^{\mathcal{A}}, a, 1^{\mathcal{A}} \rangle$ where $*_a^{\mathcal{A}}$ is defined as $x *_a^{\mathcal{A}} y = (x *^{\mathcal{A}} y) \vee a$, and $\to_a^{\mathcal{A}}$ is its residuum (i. e. the restriction of $\to^{\mathcal{A}}$ to $\{x \in A : a \leq^{\mathcal{A}} x \leq^{\mathcal{A}} 1^{\mathcal{A}}\}$).

It can be easily checked that any truncation of a ΠMTL-chain is a WCMTL-chain. It is well known (see [3]) that each MV-chain is isomorphic to a truncation of some Π-chain, i.e. given an MV-chain $\mathcal{A}$ there is a Π-chain $\mathcal{B}$ and an element $b \in B$ such that $\mathcal{A} \cong \mathcal{B}[b]$. It seems natural to ask whether the same kind of result is true in the general non-divisible case, i.e. whether each WCMTL-chain is isomorphic to a truncation of some ΠMTL-chain. We will end the section giving a negative answer to such question by using an example of a totally ordered monoid defined in [21].

For any $a, b, c, d \in \mathbb{N}$, $\langle a, b, c\rangle$ will denote the submonoid of $\mathbb{N}$ generated by $a$, $b$, $c$, and $\langle a, b, c\rangle/d$ will denote the totally ordered monoid obtained by identifying with $\infty$ all elements of $\langle a, b, c\rangle$ that are greater than or equal to $d$.

Let $S = \{32^*\}\cup\langle 9, 12, 16\rangle/30$ denote the totally ordered monoid obtained from $\langle 9, 12, 16\rangle/30$ by adding one additional element, denoted by $32^*$. This element satisfies $16 + 16 = 32^*$, $32^* + z = \infty$ for $z \neq 0$, and the whole monoid is to be ordered as follows:

$$0 < 9 < 12 < 16 < 18 < 21 < 24 < 25 < 27 < 28 < 32^* < \infty .$$

All the relations that do not involve $32^*$ are as in $\langle 9, 12, 16\rangle/30$, so we have to only check that $x \leq y$ implies $x + z \leq y + z$ when some of the terms attain the value $32^*$. If $x$ or $y$ or $z$ is equal to $32^*$ then it is easy to see. If $x + z = 32^*$ and $x, z \neq 32^*$ then $x = z = 16$. Thus $32^* = 16 + 16 \leq y + 16$ because if $y > x$ then $y + 16 = \infty$.

Now since we want to make from this monoid an MTL-chain $\mathcal{A}$, we reverse the order:

$$0 > 9 > 12 > 16 > 18 > 21 > 24 > 25 > 27 > 28 > 32^* > \infty .$$

It is clear that a residuum exists since $A$ is finite and linearly ordered. Even the weak cancellation is satisfied. Suppose that $x + z = y + z \neq \infty$. Then if $x + z = y + z \neq 32^*$ then you can cancel like in $\mathbb{N}$. If $x + z = y + z = 32^*$ then there are three possibilities: (1): $x = y = z = 16$; (2): $x = 0$, $z = 32^*$, and $y = 0$; (3): $x = 32^*$, $z = 0$, and $y = 32^*$. Thus $\mathcal{A} = \langle A, +, \rightarrow, \leq, \infty, 0\rangle$ is a WCMTL-chain.

Now let us introduce the following identity:

$$(x_1 * z_1 \rightarrow y_1 * z_2) \vee (x_2 * z_2 \rightarrow y_2 * z_1) \vee (y_1 * y_2 \rightarrow x_1 * x_2) \approx 1 \qquad (1)$$

This identity is not valid in $\mathcal{A}$. Indeed, let

$$\begin{array}{lll} x_1 = 16\,, & y_1 = 18\,, & z_1 = 16\,, \\ x_2 = 12\,, & y_2 = 9\,, & z_2 = 12\,. \end{array}$$

Then we get the following:

$$\begin{aligned} x_1 + z_1 \rightarrow y_1 + z_2 &= 32^* \rightarrow \infty = 9\,, \\ x_2 + z_2 \rightarrow y_2 + z_1 &= 24 \rightarrow 25 = 9\,, \\ y_1 + y_2 \rightarrow x_1 + x_2 &= 27 \rightarrow 28 = 9\,. \end{aligned}$$

Thus

$$(x_1 + z_1 \rightarrow y_1 + z_2) \vee (x_2 + z_2 \rightarrow y_2 + z_1) \vee (y_1 + y_2 \rightarrow x_1 + x_2) = 9 \neq 0\,.$$

On the other hand, we claim that given any ΠMTL-chain $\mathcal{B} = \langle B, *, \rightarrow, \leq, 0, 1\rangle$, every truncation $\mathcal{B}[a] = \langle B[a], *_a, \rightarrow_a, \leq, a, 1\rangle$, satisfies the identity (1). There are four cases.

1. It is clear that if one of the inequalities $x_1 *_a z_1 \leq y_1 *_a z_2$, $x_2 *_a z_2 \leq y_2 *_a z_1$, $y_1 *_a y_2 \leq x_1 *_a x_2$ is valid then the identity (1) is obviously valid.

2. Let $y_1$ or $y_2$ equals $a$. Then $y_1 *_a y_2 = a \leq x_1 *_a x_2$.

3. Let $z_1$ or $z_2$ equals $a$. Then either $x_1 *_a z_1 = a \leq y_1 *_a z_2$ or $x_2 *_a z_2 = a \leq y_2 *_a z_1$

4. Suppose that $x_1 *_a z_1 > y_1 *_a z_2$, $x_2 *_a z_2 > y_2 *_a z_1$, and $y_1, y_2, z_1, z_2 > a$. Then we have $x_1 * x_2 * z_1 * z_2 > y_1 * y_2 * z_1 * z_2$ in the original ΠMTL-chain $\mathcal{B}$. By cancellativity of $\mathcal{B}$ we get $x_1 * x_2 > y_1 * y_2$ in $L$. After truncation we obtain that $x_1 *_a x_2 \geq y_1 *_a y_2$. Thus the identity (1) is valid in this case as well.

Summing up, the identity is valid in all truncations of any ΠMTL-chain, but it is not valid in the WCMTL-chain $\mathcal{A}$. Thus, $\mathcal{A}$ cannot be isomorphic to any truncation of a ΠMTL-chain.

### 3.3 The logics of weakly cancellative chains and their ordinal sums

In the previous section we have defined the logic of weakly cancellative MTL-chains, WCMTL. Now we will consider the logic of ordinal sums of weakly cancellative totally ordered semihoops. This can be done with any axiomatic extension of MTL, so it is worth formulating first this process in an abstract way.

**Definition 3.15.** *Let L be an axiomatic extension of* MTL*. We define* $\Omega(\mathbb{L})$ *as the variety of* MTL*-algebras generated by all the ordinal sums of zero-free subreducts of L-chains with the first bounded, and we denote by* $\Omega(L)$ *its corresponding logic.*

Some well known subvarieties of $\mathbb{MTL}$ are closed under this operator, for instance:

- $\Omega(\mathbb{G}) = \mathbb{G}$
- $\Omega(\mathbb{BL}) = \mathbb{BL}$
- $\Omega(\mathbb{SBL}) = \mathbb{SBL}$
- $\Omega(\mathbb{SMTL}) = \mathbb{SMTL}$
- $\Omega(\mathbb{MTL}) = \mathbb{MTL}$

In some other cases they are not closed but we obtain an already known variety:

- $\Omega(\mathbb{BA}) = \mathbb{G}$
- $\Omega(\mathbb{MV}) = \mathbb{BL}$

But sometimes the operator $\Omega$ gives new varieties (and hence new fuzzy logics) as we will show now for $\Omega$(WCMTL) and $\Omega$(ΠMTL).

**Definition 3.16.** *Let* $\mathbb{K}$ *be the variety of* MTL*-algebras such that letting* $x \prec y = x \to x * y$ *and* $I(x) = x \to x^2$*, satisfy the following conditions:*
*(1)* $(x \wedge y \to x * y) \vee I(x * y) \vee ((x \to x * y) \to y) = 1$
*(2)* $(x \prec y) * (z \to x) \leq z \prec y$
*(3)* $(x \prec y) * (x \to z) * (z \to y) \leq (z \prec y) \vee (x \prec z) \vee I(x * y)$

We will prove that $\mathbb{K} = \Omega(\mathbb{WCMTL})$.

Note that $x \prec y = 1$ if $x \leq y$ and $x * y = x$. In an ordinal sum of weakly cancellative totally ordered semihoops, this happens if either $x$ is the minimum of the component which $y$ belongs to or $y = 1$ or $x < y$ and $x$ and $y$ belong to different components. Moreover $I(x) = 1$ iff $x$ is an idempotent. Thus the intuitive meaning of (1) is that either $x * y = x$ or $x * y = y$ or

$x * y$ is an idempotent or $x$ and $y$ belong to the same component and satisfy the cancellation law. The intuitive meaning of (2) is the following: suppose that $x < y$, that $x$ and $y$ are not in the same component and that $z \leq x < y$. Then $z$ and $y$, are in different components. The complementary property is true in all totally ordered semihoops: if $x < y \leq z$ and $x * y = x$, then $x * z \geq x * y = x$, so $x * z = x$. Finally (3) means that if $x * y = x$ and $x$ is not an idempotent, then for any $z$ with $x \leq z \leq y$ we must have either $x * z = x$ or $z * y = z$.

**Lemma 3.17.** *Equations (1), (2) and (3) hold in any ordinal sum of weakly cancellative totally ordered semihoops whose first component is bounded.*

*Proof.* This is not completely trivial because we have to verify that the equations hold also when the lefthand side is not 1. In the sequel we write $x \ll y$ to mean that $x < y$ and $x$ and $y$ are not in the same component. We also write $x \equiv y$ to mean that $x$ and $y$ are in the same component.

We start from equation (1). The equation clearly holds if $x \not\equiv y$ or if $x*y$ is an idempotent. If $x \equiv y$ and $x * y$ is not an idempotent, then $x * y$ must satisfy the cancellation law and the third disjunct is 1.

Now consider equation (2). The equation clearly holds if $z \prec y = 1$, hence a fortiori if $z \ll y$. The equation also holds if $z \leq x$ and $x \prec y = 1$, because then either $z = x$ or $z \ll y$, and in both cases $z \prec y = 1$. The equation also holds if $x \leq z$, because then $(x \to x * y) * (z \to x) \leq z \to z * y$. It remains to consider the case where $z < x$ and either $y \ll x$ or $x \equiv y$. If $z < x$ and $y \ll x$ then $x \prec y = y$, and (2) becomes $y \leq z \prec y$, which is clearly satisfied. Finally suppose $z < x$ and $x \equiv y$. Without loss of generality we can also suppose $z \equiv x \equiv y$, otherwise $z \ll y$ and $z \prec y = 1$. Thus (2) becomes $x \prec y \leq z \prec y$. If $z * y$ is not an idempotent, then $x \prec y = z \prec y = y$ and (2) holds. If $x * y$ is an idempotent, then $x * y = z * y$ is the minimum $m$ of the component which $x, y, z$ belong to, and (2) becomes $x \to m \leq z \to m$, which clearly holds as $z < x$. Finally if $z * y = m$ is an idempotent and $x * y$ is not (so $x * y > m$), then $x \prec y = y$ and $z \prec y = z \to m$. Now from $z * y = m$ by residuation we derive $y \leq z \to m$ and the claim is proved.

We verify (3). Note that (3) holds (in any ordinal sum of WCMTL semihoops) if either $x * y$ is an idempotent or $z \prec y = 1$ or $x \prec z = 1$ (thus in particular if $z \ll y$ or $x \ll z$). Thus we suppose that none of the above conditions holds. If $y \ll z$ then $z \to y = y$ and (3) holds. If $z \ll x$ then $x \to z = z$ and (3) holds. It remains to consider the case where $x \equiv z \equiv y$. In this case, since we have excluded that $x * y$ is an idempotent, we have $x \prec y = y$. Now let $C$ be the component which $x, y, z$ belong to. If either $C$ has no minimum or $z * y$ is not the minimum of $C$, then $x \prec y = z \prec y = y$, and (3) is verified. If $C$ has a minimum $m$ and $z * y = m$, then $x \prec y = y \leq z \to m = z \prec y$ and once again (3) is verified. □

**Lemma 3.18.** *Let $\mathcal{A}$ be an* MTL*-chain which satisfies (1), (2) and (3). Then $\mathcal{A}$ is the ordinal sum of an ordered family of weakly cancellative totally ordered semihoops, whose first component is bounded.*

*Proof.* By Theorem 3.2 any linearly ordered MTL-algebra can be decomposed as an ordinal sum of sum-indecomposable totally ordered semihoops, with the first bounded. So it is sufficient to prove that a sum-indecomposable linearly ordered semihoop satisfying (1), (2) and (3) is weakly cancellative. Let $\mathcal{C}$ be such a semihoop. We claim that $\mathcal{C}$ has no idempotent elements except from its maximum and its minimum (if such a minimum exists). Suppose by contradiction that $u$ is idempotent and that there are $a, b \in C$ with $a < u < b$. Then

$x * u = u$ for all $x \geq u$, and by (2), for all $z \leq u \leq v$ one has $z * v = z$. Then $\mathcal{C} = \mathcal{C}_1 \oplus \mathcal{C}_2$ where $C_2 = \{z : z \geq u\}$ and $C_1 = (C \setminus C_2) \cup \{1\}$, contradicting our assumption that $\mathcal{C}$ is sum-indecomposable. We now prove that if both $x$ and $y$ are not idempotent, then $x * y < x \wedge y$. The claim is obvious if $x = y$ so we can assume without loss of generality that $x < y$. The claim is also obvious if $x * y$ is the minimum $m$ of $C$, because $m$ is an idempotent and $x, y$ are not such, so $m = x * y < x \wedge y$. Thus suppose by contradiction that there is $z \in C$ such that $z < x * y = x \wedge y = x < y$. Since $x * y$ is not an idempotent, by axiom (3), for any $u$ with $x \leq u \leq y$ we have either $x * u = x$ or $u * y = u$. Now let $C_1 = \{u : u * y = u\} \cup \{1\}$ and $C_2 = (C \setminus C_2) \cup \{1\}$. $C_1 \setminus \{1\}$ is downwards closed, so for all $w \in C_2$ and for all $z \in C_1 \setminus \{1\}$ we have $z \leq w$. We claim that for all $w \in C_2$ and for all $z \in C_1 \setminus \{1\}$ we have $z * w = z$. This implies that $\mathcal{C} = \mathcal{C}_1 \oplus \mathcal{C}_2$, which is impossible. Thus let $w \in C_2$ and $z \in C_1 \setminus \{1\}$. We can assume without loss of generality that $z$ is not an idempotent, otherwise $z$ is the minimum of $\mathcal{C}$ and the claim is trivial. Moreover by the definition of $\mathcal{C}_1$ we have that $z * y = z$. So if $w \geq y$, we have $z * w = z$ as desired. If $w < y$, then since $z * y = z$ is not an idempotent, by axiom (3) with $x$ replaced by $z$ we have that either $w * y = w$ or $z * w = z$. But $w * y = w$ is excluded, because $w \in C_2$. So $z * w = z$ and the proof is complete. □

Thus we obtain a finite axiomatization for the variety generated by those ordinal sums:

**Theorem 3.19.** *$\mathbb{K}$ is the variety generated by the ordinal sums of weakly cancellative totally ordered semihoops (with the first bounded), i.e. $\mathbb{K} = \Omega(\mathbb{WCMTL})$.*

Now consider the variety $\Omega(\Pi\mathbb{MTL})$. Adapting slightly the axiomatization and the proof of the last theorem we obtain the following result.

**Theorem 3.20.** *The variety $\Omega(\Pi\mathbb{MTL})$ generated by ordinal sums of cancellative semihoops (with the first bounded) is axiomatized by:*
*(1')* $(x \wedge y \to x * y) \vee I(x) \vee ((x \to x * y) \to y) = 1$
*(2)* $(x \prec y) * (z \to x) \leq z \prec y$
*(3)* $(x \prec y) * (x \to z) * (z \to y) \leq (z \prec y) \vee (x \prec z) \vee I(x * y)$

Accordingly, we define the corresponding logics. The logic $\Omega$(WCMTL) is the axiomatic extension of MTL obtained by adding the following schemata:

(a) $(\varphi \wedge \psi \to \varphi * \psi) \vee I(\varphi * \psi) \vee ((\varphi \to \varphi * \psi) \to \psi)$

(b) $(\varphi \prec \psi) * (\chi \to \varphi) \to \chi \prec \psi$

(c) $(\varphi \prec \psi) * (\varphi \to \chi) * (\chi \to \psi) \to (\chi \prec \psi) \vee (\varphi \prec \chi) \vee I(\varphi * \psi)$

and the logic $\Omega(\Pi$MTL) is the axiomatic extension of MTL obtained by adding the following schemata:

(a') $(\varphi \wedge \psi \to \varphi * \psi) \vee I(\varphi) \vee ((\varphi \to \varphi * \psi) \to \psi)$

(b) $(\varphi \prec \psi) * (\chi \to \varphi) \to \chi \prec \psi$

(c) $(\varphi \prec \psi) * (\varphi \to \chi) * (\chi \to \psi) \to (\chi \prec \psi) \vee (\varphi \prec \chi) \vee I(\varphi * \psi)$

Let (OS) be the conjunction of the schemata (a), (b) and (c), and let (OS') be the conjunction of the schemata (a'), (b) and (c). Adding combinations of the schemata (WC), (PC), (OS), (OS'), (Div) and (Inv) to MTL we obtain the hierarchy of logics depicted in figure 1, where CPC is the Classical Propositional Calculus and the following two new logics appear:

- SΩ(WCMTL) is Ω(WCMTL) plus (PC).
- WCBL is BL plus (WC).

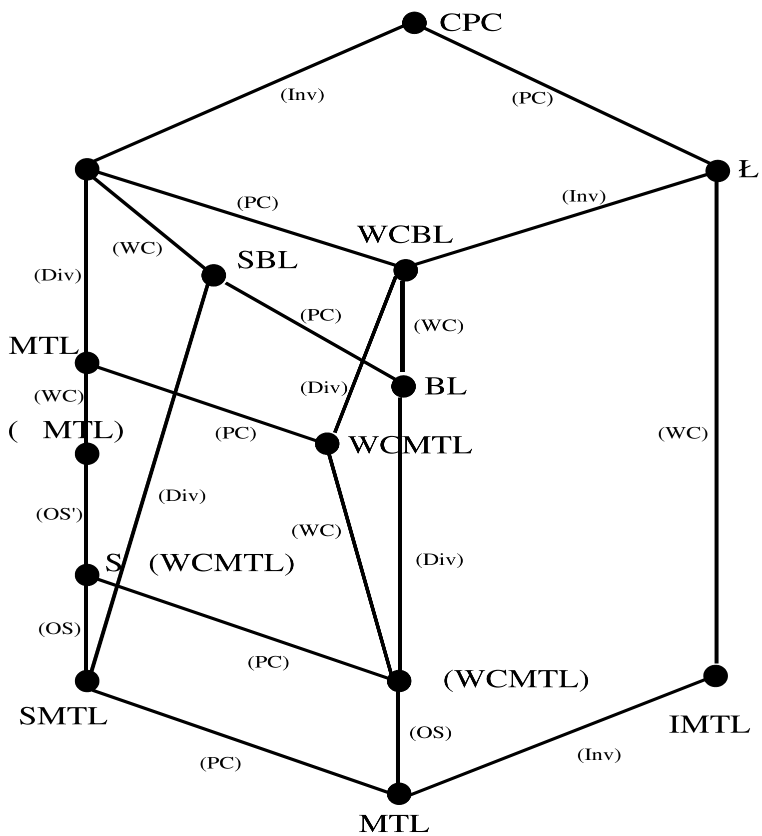


Figure 1: Graphic of axiomatic extensions of MTL obtained by adding combinations of the schemata (WC), (PC), (OS), (Div) and (Inv). All the depicted inclusions are proper.

## 3.4 LF, FEP and FMP in weakly cancellative fuzzy logics

We will study some properties of these logics and their corresponding varieties of MTL-algebras.

**Lemma 3.21.** *Let* $\mathcal{A}$ *be an* MTL*-chain. Then,* $\mathcal{A}$ *is a* WCBL*-chain if, and only if, it is an* MV*-chain or a* Π*-chain.*

*Proof.* One direction is trivial. For the other one, let $\mathcal{A}$ be a WCBL-chain and consider its decomposition as ordinal sum of Wajsberg hoops (with the first bounded), $\mathcal{A} \cong \bigoplus_{i\in I} \mathcal{C}_i$. If $|I| = 1$, then $\mathcal{A} \cong \mathcal{C}_{i_0}$ is an MV-chain. If $|I| > 1$ it must be of the form $\mathcal{A} \cong \mathcal{B}_2 \oplus \mathcal{C}$, with $\mathcal{C}$ cancellative (otherwise the weak cancellation would not be satisfied), hence it is a Π-chain. □

**Proposition 3.22.** WCBL *is the infimum of* Π *and* Ł *in the lattice of axiomatic extensions of* MTL*. Thus,* $\mathbb{WCBL} = \mathbf{V}([0,1]_L, [0,1]_\Pi)$ *and* WCBL *enjoys FSSC.*

*Proof.* It follows directly from the previous lemma. □

Therefore, WCBL is the logic ŁΠ defined in [11] for which we have found now an alternative axiomatization.

**Corollary 3.23.** WCBL *does not have the finite model property.*

*Proof.* Suppose WCBL has the FMP. Then, $\mathbb{WCBL}$ would be generated as a variety by the finite WCBL-chains, but since there are no finite Π-chains with more than two elements, it would be generated by finite MV-algebras, so $\mathbb{WCBL} = \mathbb{MV}$, a contradiction. □

Thus $\mathbb{WCBL}$ lacks also the FEP. Nevertheless, WCBL logic is still decidable, since it is the infimum of Π and Ł and those logics are decidable.

**Proposition 3.24.** *Let $\mathcal{A}$ be an* MTL*-chain. Then, $\mathcal{A}$ is an* SΩ(WCMTL)*-chain if, and only if, it is an ordinal sum of totally ordered weakly cancellative semihoops such that the first one is a* ΠMTL*-chain.*

*Proof.* One direction is trivial. For the other one, let $\mathcal{A}$ be a SΩ(WCMTL)-chain. In particular it is a Ω(WCMTL)-chain, so it is decomposable as an ordinal sum of totally ordered weakly cancellative semihoops with the first one bounded. Then, it is obvious that the axiom (PC) implies that the first component must be an SMTL-chain, hence a ΠMTL-chain. □

**Proposition 3.25.** *Let $\mathbb{K} \subseteq \mathbb{MTL}$ be a variety. If for every $n \geq 2$, $\mathbb{K} \not\models x^n \approx x^{n-1}$, then $\mathbb{K}$ is not locally finite.*

*Proof.* For every $n \geq 2$, there is $\mathcal{A}_n \in \mathbb{K}$ and $a_n \in A_n$ such that $a_n^n < a_n^{n-1}$. Consider the algebra $\prod_{n\geq 2} \mathcal{A}_n$ and the element $a = \langle a_2, a_3, a_4, \dots, \rangle \in \prod_{n\geq 2} A_n$. Then for every $n \geq 2$, we have $a^n < a^{n-1}$, thus the subalgebra generated by $a$ is infinite. □

**Corollary 3.26.** *All the logics depicted in figure 1 (except for Gödel logic and the Classical Propositional Calculus) have a non locally finite equivalent algebraic semantics.*

*Proof.* Notice that for every $n \geq 2$, $[0,1]_L \not\models x^n \approx x^{n-1}$ and $[0,1]_\Pi \not\models x^n \approx x^{n-1}$, and all the corresponding varieties contain one of these algebras, hence they all satisfy the condition of the last proposition. □

Finally, we will prove that the FMP fails for all logics between Ω(WCMTL) and ΠMTL (both included). First we need some lemmas.

**Lemma 3.27.** *Each finite* WCMTL*-chain $\mathcal{A}$ is Archimedean, i.e. for any $0 < x < y < 1$ there is $n$ such that $y^n \leq x$.*

*Proof.* Suppose not. Then $x < y^n$ for all $n$. Since $y^n \neq 0$ for all $n$, we have $y > y^2 > y^3 > \dots$ by weak cancellativity. Thus $\mathcal{A}$ must be infinite, a contradiction. □

**Lemma 3.28.** *Let $\mathcal{A}$ be an* MTL*-chain and $p, q \in A$. If $p \to q = q$ then $q = \max\, [q]_{F(p)}$.*

*Proof.* Assume that $p \to q = q$. Suppose that $z \in [q]_{F(p)}$. Then $z \to q \in F(p)$. Thus there exists $n \in \omega$ such that $p^n \leq z \to q$. By residuation we get $z \leq p^n \to q$. Since we assume that $p \to q = q$, we have $p^n \to q = p^{n-1} \to (p \to q) = p^{n-1} \to q = q$. Thus we obtain that $z \leq q$. Hence $q = \max\, [q]_{F(p)}$. □

**Lemma 3.29.** *Let $\mathcal{A}$ be an Archimedean* MTL*-chain. Then $\mathcal{A}$ is either a* BL*-chain or it has a co-atom.*

*Proof.* Suppose that there is no co-atom. Then we will show that the divisibility condition, $a \wedge b = a * (a \to b)$, holds in $\mathcal{A}$. If $a \leq b$ or $a$ equals 1, then the equality trivially holds. If $a \to b = 0$ then $b = 0$ and the equality again holds. Thus suppose that $a > b$, $a, b \neq 1$, and $a \to b > 0$. By residuation we get $a * (a \to b) \leq b$. Suppose that $a * (a \to b) < b$. Let $M = A \setminus \{1\}$. Clearly $\bigvee M = 1$ because there is no co-atom. Since $\mathcal{A}$ is Archimedean, we get that for each $r \in M$ there exists $k_r \in \omega$ (possibly 0) such that

$$r^{k_r+1} \leq a \to b < r^{k_r}\,.$$

Thus we obtain for all $r \in M$:

$$a * r^{k_r+1} \leq a * (a \to b) < b < a * r^{k_r}\,.$$

The last inequality holds since $a \to b$ is the maximal solution of the inequality $a * x \leq b$ and $a \to b < r^{k_r}$.

Further, from the existence of residuum we get $\bigvee_{r \in M}(b * r) = b * \bigvee M = b$. Hence there must be an $s \in M$ such that $a * (a \to b) < b * s$. Thus we obtain

$$a * s^{k_s+1} \leq a * (a \to b) < b * s \leq a * s^{k_s+1}\,,$$

a contradiction. □

**Lemma 3.30.** *In each Archimedean* WCMTL*-chain $\mathcal{A}$ the identity*

$$((p \to q) \to q)^2 \leq p \vee q \vee \neg q \tag{2}$$

*is valid.*

*Proof.* If there is no co-atom, then by Lemma 3.29, $\mathcal{A}$ is a WCBL-chain hence either a Π-chain or an MV-chain. But in any Π-chain or MV-chain the identity (2) is valid.

Thus suppose that there is a co-atom $a$. The only interesting case is for $1 > p > q > 0$. We can also assume that $p \to q > 0$ otherwise $q = 0$. Since $\mathcal{A}$ is Archimedean, there is $n \in \omega$ such that

$$a^{n+1} \leq p \to q < a^n\,.$$

Since $a^n > p \to q$, we get $a^n \to q < p$ (if $p \leq a^n \to q$ then $a^n \leq p \to q$). It follows that

$$(p \to q) \to q \leq a^{n+1} \to q = a \to (a^n \to q) \leq a \to p\,.$$

Thus $(p \to q) \to q \leq a \to p$.

Now we claim that $(p \to q) \to q \leq a$. If not then $(p \to q) \to q = 1$, i.e. $p \to q = q$. Thus by Lemma 3.28 we have $q = \max\, [q]_{F(p)}$. Since $\mathcal{A}$ is Archimedean, $F(p)$ equals either to $A$ or to $A \setminus \{0\}$. Thus $q \in F(p)$ and $q = 1$. But we assume that $1 > p > q > 0$. Hence $(p \to q) \to q \leq a$.

Finally, we get

$$((p \to q) \to q)^2 \leq a * ((p \to q) \to q) \leq a * (a \to p) \leq p \leq p \vee q \vee \neg q\,.$$

□

Let $\varphi = (q \to (p * q)) \to p$, $\psi = (p \to q) \to q$, and $\chi = p \vee q \vee \neg q$.

**Lemma 3.31.** *In any finite* $\Omega(\mathrm{WCMTL})$*-chain* $\mathcal{A}$ *the identity* $\varphi \wedge \psi^2 \leq \chi$ *is valid.*

*Proof.* If $p \leq q$ then $\psi = q$ and $\varphi \wedge \psi^2 = \varphi \wedge q^2 \leq \chi$. Thus let us suppose that $p > q$.

First, let $p, q$ belong to different components. Then $\varphi = (q \to q) \to p = p$. Thus $\varphi \wedge \psi^2 \leq \varphi = p \leq \chi$.

Second, let $p, q$ be in the same component. This component is a zero-free subreduct of a finite WCMTL-chain $\mathcal{W}$. By Lemma 3.27 we know that $\mathcal{W}$ is Archimedean. Thus by Lemma 3.30 we get that $\psi^2 \leq \chi$ is valid in $\mathcal{W}$. Since $\mathcal{W}$ is a subalgebra of $\mathcal{A}$, we get that $\varphi \wedge \psi^2 \leq \psi^2 \leq \chi$ is valid in $\mathcal{A}$. □

**Lemma 3.32.** *There is a* $\Pi\mathrm{MTL}$*-chain* $\mathcal{A}$ *such that* $\varphi \wedge \psi^2 \leq \chi$ *is not valid in* $\mathcal{A}$.

*Proof.* Consider the algebra $\mathcal{A}$ defined as follows:

- The domain of $\mathcal{A}$ is $\{\langle 0, 0\rangle\} \cup ((0, 1] \times (0, 1])$.
- The lexicographic order $\leq_{lex}$ defines the lattice structure.
- Multiplication is defined componentwise.
- Implication $\Rightarrow$ is defined as follows: if $\langle a, b\rangle \leq_{lex} \langle c, d\rangle$, then $\langle a, b\rangle \Rightarrow \langle c, d\rangle = \langle 1, 1\rangle$; if $\langle a, b\rangle \neq \langle 0, 0\rangle$, then $\langle a, b\rangle \Rightarrow \langle 0, 0\rangle = \langle 0, 0\rangle$; if $a, b, c, d > 0$ and $a \geq c$ and $b \geq d$, then $\langle a, b\rangle \Rightarrow \langle c, d\rangle = \langle \frac{c}{a}, \frac{d}{b}\rangle$; if $a > c$ and $b \leq d$, then $\langle a, b\rangle \Rightarrow \langle c, d\rangle = \langle \frac{c}{a}, 1\rangle$.

It is readily seen that $\mathcal{A}$ is a $\Pi$MTL-algebra. For $e(p) = \langle 1, \frac{1}{2}\rangle$ and $e(q) = \langle \frac{1}{2}, 1\rangle$, we have $e(\varphi) = e(\psi^2) = \langle 1, 1\rangle$ and $e(\chi) \neq \langle 1, 1\rangle$. □

Thus we get the following theorem.

**Theorem 3.33.** *If* $\mathbb{K}$ *is a variety such that* $\Pi\mathbb{MTL} \subseteq \mathbb{K} \subseteq \Omega(\mathbb{WCMTL})$*, then* $\mathbb{K}$ *has not the FMP (and hence also the FEP is false in* $\mathbb{K}$*).*

*Proof.* Let $\mathcal{A}$ be the chain defined in the previous lemma. Therefore, $\mathcal{A}$ is an infinite chain of $\mathbb{K}$ where $\varphi \wedge \psi^2 \leq \chi$ fails, but by Lemma 3.31, the equation is valid in all the finite chains of $\mathbb{K}$. □

## 3.5 On standard completeness theorems

In this section we discuss the standard completeness of the logics introduced so far and of their first-order extensions.

**Theorem 3.34.** WCMTL *enjoys FSSC.*

*Proof.* We will prove it by following the method used in [32] and its modification from [33] for the FSSC of ΠMTL, so we will not check again the details that are already done there. Take a finite set $T \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}$ such that $T \nvdash_{WCMTL} \varphi$. Then, there is a WCMTL-chain $\mathcal{A} = \langle A, *, \rightarrow, \wedge, \vee, 0^{\mathcal{A}}, 1^{\mathcal{A}} \rangle$ and an evaluation $e : Fm_{\mathcal{L}} \rightarrow \mathcal{A}$ such that $e[T] \subseteq \{1^{\mathcal{A}}\}$ and $e(\varphi) \neq 1^{\mathcal{A}}$. Consider the set $G := \{e(\psi) : \psi$ is a subformula of some formula of $T \cup \{\varphi\}\}$. $G$ is finite because $T$ is. Let $\mathcal{S}$ be the submonoid of $\mathcal{A}$ generated by $G$. As in [32], $\mathcal{S}$ is residuated and the residuum is given by: $a \rightarrow b = \max\{z \in S : a * z \leq b\}$. Thus, the enriched submonoid $\mathcal{S} = \langle S, *, \rightarrow, \wedge, \vee, 0^{\mathcal{A}}, 1^{\mathcal{A}} \rangle$ is a countable MTL-chain. Moreover, since its monoidal operation is just the restriction of the monoidal operation of $\mathcal{A}$, it is clear that it is also weakly cancellative, hence a WCMTL-chain. Now we consider the evaluation $e' : Fm_{\mathcal{L}} \rightarrow \mathcal{S}$ such that for every propositional variable $v$,

$$e'(v) = \begin{cases} e(v) & \text{if } v \text{ appears in some formula of } T \cup \{\varphi\} \\ 0^{\mathcal{A}} & \text{otherwise.} \end{cases}$$

One can prove by induction that $e'(\psi) = e(\psi)$ for every $\psi$ a subformula of some formula of $T \cup \{\varphi\}$. Furthermore, since $\mathcal{S}$ is generated from a finite set by using the monoidal operation, then it has only a finite number of Archimedean classes.

Now define a new chain over the set $S' := \{\langle s, r \rangle : s \in S \setminus \{0^{\mathcal{A}}\}, r \in (0, 1]\} \cup \{\langle 0^{\mathcal{A}}, 1 \rangle\}$, with the lexicographical order $\leq_{lex}$ and the following operations:

$$\langle a, x \rangle *' \langle b, y \rangle = \begin{cases} \langle 0^{\mathcal{A}}, 1 \rangle & \text{if } a * b = 0^{\mathcal{A}}, \\ \langle a * b, xy \rangle & \text{otherwise.} \end{cases}$$

$$\langle a, x \rangle \rightarrow' \langle b, y \rangle = \begin{cases} \langle a \rightarrow b, 1 \rangle & \text{if } a * (a \rightarrow b) < b, \\ \langle a \rightarrow b, \min\{1, y/x\} \rangle & \text{otherwise.} \end{cases}$$

$\mathcal{S}' = \langle S', *', \rightarrow', \leq_{lex}, \langle 0^{\mathcal{A}}, 1 \rangle, \langle 1^{\mathcal{A}}, 1 \rangle \rangle$ is an MTL-chain with a finite number of Archimedean classes, and there is an embedding $\Psi : \mathcal{S} \rightarrow \mathcal{S}'$ defined by $\Psi(a) = \langle a, 1 \rangle$. Moreover $\mathcal{S}'$ is weakly cancellative. Indeed, if $\langle a, x \rangle, \langle b, y \rangle, \langle c, z \rangle \in S'$ are such that $\langle a, x \rangle *' \langle b, y \rangle = \langle a, x \rangle *' \langle c, z \rangle \neq \langle 0^{\mathcal{A}}, 1 \rangle$, then $\langle a * b, xy \rangle = \langle a * c, xz \rangle \neq \langle 0^{\mathcal{A}}, 1 \rangle$. Thus, $a * b = a * c \neq 0^{\mathcal{A}}$ and $xy = xz \neq 0$ which, using the weak cancellation of $\mathcal{A}$ and the cancellation of the product of reals, implies $b = c$ and $y = z$.

Finally, as in [33] the set $S'$ is order isomorphic to the real unit interval $[0, 1]$, so there is a standard WCMTL-chain $\mathcal{B}$ and an isomorphism $\Phi : \mathcal{S}' \rightarrow \mathcal{B}$. This standard chain and the evaluation $\Phi \circ \Psi \circ e'$ are a countermodel for the derivation of $\varphi$ from $T$. □

**Theorem 3.35.** Ω(WCMTL)*,* SΩ(WCMTL) *and* Ω(Π*MTL) enjoy FSSC.*

*Proof.* Consider first the Ω(WCMTL) case. The first part of the proof runs parallel to the previous one. Take a finite set $T \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}$ such that $T \nvdash_{\Omega(WCMTL)} \varphi$. Then, there is a Ω(WCMTL)-chain $\mathcal{A} = \langle A, *, \rightarrow, \wedge, \vee, 0^{\mathcal{A}}, 1^{\mathcal{A}} \rangle$ and an evaluation $e : Fm_{\mathcal{L}} \rightarrow \mathcal{A}$ such that $e[T] \subseteq \{1^{\mathcal{A}}\}$ and $e(\varphi) \neq 1^{\mathcal{A}}$. Consider the set $G := \{e(\psi) : \psi$ is a subformula of some formula of $T \cup \{\varphi\}\}$. $G$ is finite because $T$ is. Let $\mathcal{S}$ be the submonoid of $\mathcal{A}$ generated by $G$. Again it is residuated, so we have an enriched submonoid $\mathcal{S} = \langle S, *, \rightarrow, \wedge, \vee, 0^{\mathcal{A}}, 1^{\mathcal{A}} \rangle$ such that is a countable MTL-chain (with a finite number of Archimedean classes). Moreover, since its monoidal operation is just the restriction of the monoidal operation of $\mathcal{A}$, it is clear that it is also an ordinal sum of weakly cancellative totally ordered semihoops with the first bounded, hence a $\Omega$(WCMTL)-chain. Since it is finitely generated, this ordinal sum must have a finite number of components, say $\mathcal{S} = \bigoplus_{i<k} \mathcal{C}_i$ for some natural number $k$. Now we consider the evaluation $e' : Fm_{\mathcal{L}} \rightarrow \mathcal{S}$ such that for every propositional variable $v$,

$$e'(v) = \begin{cases} e(v) & \text{if } v \text{ appears in some formula of } T \cup \{\varphi\} \\ 0^{\mathcal{A}} & \text{otherwise.} \end{cases}$$

Again, by induction, it is provable that $e'(\psi) = e(\psi)$ for every $\psi$ a subformula of some formula of $T \cup \{\varphi\}$.

Finally, applying to every weakly cancellative totally ordered semihoop of the ordinal sum the construction of the proof of the previous theorem, we have for every $i < k$ an embedding $\mathcal{C}_i \hookrightarrow [0,1]_{*_i}$ into a standard WCMTL-chain. Therefore, there is an embedding $f : \mathcal{S} \hookrightarrow \bigoplus_{i<k}[0,1]_{*_i}$. It is clear that $\bigoplus_{i<k}[0,1]_{*_i}$ is isomorphic to a standard $\Omega$(WCMTL)-chain. This standard $\Omega$(WCMTL)-chain with the evaluation $f \circ e'$ gives the desired countermodel for the derivation of $\varphi$ from $T$.

For the cases of S$\Omega$(WCMTL) and $\Omega$($\Pi$MTL) the proof is similar. For the first one we only need to realize that the first component of the ordinal sum of $\mathcal{S}$ now is a $\Pi$MTL-chain and it will be embedded into a standard $\Pi$MTL-chain, so in the end we will get a standard S$\Omega$(WCMTL)-chain. For the second one, notice that all the components of $\mathcal{S}$ are cancellative so they embed into standard $\Pi$MTL-chains, so in the end a standard $\Omega$($\Pi$MTL)-chain is obtained. □

Furthermore, taking into account that in the proofs of the last two theorems the standard chains that are built have only finitely many Archimedean classes, we can improve the finite standard completeness results by considering only the semantics given by standard chains with a finite number of Archimedean classes.

**Corollary 3.36.** *If L is a logic from the set* {WCMTL, $\Omega$(WCMTL), $\Omega$($\Pi$MTL), S$\Omega$(WCMTL)}, *then for every finite set of formulas* $T \cup \{\varphi\}$ *we have:*

$T \vdash_L \varphi$ *if, and only if,* $T \models_{\mathcal{A}} \varphi$ *for every standard L-chain* $\mathcal{A}$ *with finitely many Archimedean classes.*

Now we will prove that no logic between $\Omega$(WCMTL) and $\Pi$ (both included) enjoys SSC. Consider the following set $\Gamma$ of sentences in a language whose propositional variables are $p_0, ..., p_n, ..., p_\omega$:

1. $p_i \leftrightarrow p_{i+1}^2$ $(i \in \omega)$.
2. $\neg\neg p_0$.
3. $p_i \rightarrow p_\omega$ $(i \in \omega)$.

**Claim (A)**. For any standard $\Omega$(WCMTL)-algebra $\mathcal{A}$ one has:
$\Gamma \models_{\mathcal{A}} p_0 \rightarrow p_\omega * p_0$

*Proof of Claim (A).* Suppose that all formulas of $\Gamma$ are satisfied in $\mathcal{A}$ under some evaluation $e$. Let, for $k = 0, 1, ..., n, ..., \omega$, $a_k = e(p_k)$. Then by (2), $a_0 \neq 0$ and by (1) and (3), for all $k \in \omega$ we have $a_{k+1}^2 = a_k$ and $a_k \leq a_\omega$. So all $a_i$ with $i < \omega$ are in the same component.

If $a_\omega = 1$ the result is obvious. Suppose $a_\omega < 1$. Let $a = \sup\{a_k : k \in \omega\}$ (such a supremum exists by the completeness of $[0, 1]$). Then $a \leq a_\omega$. Moreover by the left-continuity of the monoidal operation $\cdot$, we have $a^2 = \sup\left\{a_{k+1}^2 : k \in \omega\right\} = \sup\{a_k : k \in \omega\} = a$. So $a$ is an idempotent, between $a_0$ and $a_\omega$. It follows that $a_\omega$ and $a_0$ are in different components, therefore $a_0 \cdot a_\omega = a_0$, and the claim is proved.

**Claim (B)**. There are a product algebra $\mathcal{B}$ and an evaluation $e$ in $\mathcal{B}$ such that $e(A) = 1$ for all $A \in \Gamma$ and $e(p_\omega \cdot p_0) < e(p_0)$.

*Proof of Claim (B).* Let $\mathcal{B} = \{\langle 0, 0\rangle\} \cup \{\langle 1, p\rangle : 0 < p \leq 1\} \cup ((0, 1) \times (0, +\infty))$, ordered by the lexicographic order $\leq_{lex}$ (thus if $0 < a < b \leq 1$ then for any $c, d \in (0, +\infty)$, one has $\langle a, c\rangle <_{lex} \langle b, d\rangle$) and having ordinary product (defined componentwise) as monoidal operation. Thus our algebra consists of $\langle 0, 0\rangle$ plus the negative cone of the multiplicative group $(0, +\infty)^2$ ordered lexicographically. Here the identity is $\langle 1, 1\rangle$, therefore *negative* means *less than* $\langle 1, 1\rangle$. In other words, it is a product algebra. Now define inductively $e(p_0) = \left\langle \frac{1}{2}, 1\right\rangle$, $e(p_{i+1}) = \left\langle \sqrt{e(p_i)}, 1\right\rangle$. Further, define $e(p_\omega) = \left\langle 1, \frac{1}{2}\right\rangle$. It is immediate to verify that $e(A) = \langle 1, 1\rangle$ for any $A \in \Gamma$ and that $e(p_\omega \cdot p_0) = \left\langle \frac{1}{2}, \frac{1}{2}\right\rangle <_{lex} \left\langle \frac{1}{2}, 1\right\rangle = e(p_0)$. This concludes the proof of Claim (B).

**Theorem 3.37.** *No propositional logic between* $\Omega$(WCMTL) *and Product logic* $\Pi$ *(both included) enjoys SSC.*

*Proof.* Let L be such a logic and $\mathbb{L}$ the corresponding variety. Then the standard elements of $\mathbb{L}$ are standard $\Omega$(WCMTL)-algebras and the algebra $\mathcal{B}$ in Claim B is in $\mathbb{L}$. Hence $\Gamma \models_{\mathcal{A}} p_0 \to p_\omega * p_0$ holds in any standard algebra $\mathcal{A}$ in $\mathbb{L}$, but not in all algebras in $\mathbb{L}$ ($\mathcal{B}$ is a counterexample). It follows that $\Gamma \not\vdash_L p_0 \to p_\omega * p_0$. □

**Corollary 3.38.** *The following logics do not enjoy SSC:* $\Pi$, WCBL, SBL, BL, $\Pi$MTL, $\Omega(\Pi$MTL$)$, WCMTL, S$\Omega$(WCMTL) *and* $\Omega$(WCMTL)*.*

We now prove that if L is any (recursively enumerable) logic between $\Omega$(WCMTL) and $\Pi$ (both included), its first-order extension L$\forall$ does not enjoy FSSC. Indeed the set of finite consequence relations valid in all standard models of L is not recursively enumerable.

We denote with $\mathbf{I\Sigma_1}$ the fragment of arithmetic with induction only for $\Sigma_1$-formulas (with the order in the language). Recall that it is finitely axiomatizable (see [30, Theorem 2.52, p. 78]). Let $\Gamma$ be the finite set consisting of:

1. All axioms of $\mathbf{I\Sigma_1}$, with functions represented as predicates (in the obvious manner).

2. All axioms of the form $\forall x_1 ... \forall x_n(P(x_1, ..., x_n) \vee \neg P(x_1, ..., x_n))$, $P$ a predicate symbol of $\mathbf{I\Sigma_1}$.

3. The axiom $\neg\neg U(0) \wedge \forall x \forall y(S(x, y) \to (U(y)^2 \leftrightarrow \exists z((z \preceq x) \wedge U(z)))$, where $S$ is a predicate symbol of $\mathbf{I\Sigma_1}$ such that $S(x, y)$ represents the relation $y = x + 1$, $\preceq$ is a binary symbol of $\mathbf{I\Sigma_1}$ representing the order of natural numbers, and $U$ is a unary predicate not in the language of $\mathbf{I\Sigma_1}$.

4. The axiom $\forall x \forall y(x \approx y \to ((U(x) \to U(y)))$.

Let $\mathcal{B}$ be a standard WCMTL-algebra, and let $\mathcal{A}$ be a first-order structure over $\mathcal{B}$ with a domain of individuals $D$. Given a formula $\alpha(x_1, \ldots, x_n)$ and $c_1, \ldots, c_n \in D$, let us write $e_{\mathcal{A},\mathcal{B}}(\alpha(c_1, \ldots, c_n))$ for $||\alpha(x_1, \ldots, x_n)||^{\mathcal{B}}_{\mathcal{A},v}$ where $v$ is an evaluation of object variables such that for $i = 1, \ldots, n$, $v(x_i) = c_i$. In the sequel we omit the subscript $\mathcal{A}, \mathcal{B}$ when it is clear from the context. Now assume that $e(B) = 1$ for all $B \in \Gamma$. Define, for $c, d \in D$, $c \equiv d$ iff $e(c \approx d) = 1$. Then $\equiv$ is an equivalence relation (remember that by axioms of the form (2), every formula of $\mathbf{I\Sigma_1}$ has a crisp value). Let for $d \in D$, $[d]$ denote its equivalence class modulo $\equiv$. We define a model $\mathcal{N}^{A,e}$ of $\mathbf{I\Sigma_1}$ as follows:

- The domain of $\mathcal{N}^{A,e}$ is $D/\equiv$, that is, $\{[d] : d \in D\}$.
- For every $n$-ary predicate symbol $P$ we define $P^{\mathcal{N}^{A,e}} = \{\langle [d_1], ..., [d_n] \rangle \in (\mathcal{N}^{A,e})^n : e(P(d_1, ..., d_n)) = 1\}$.

Note that for every sentence $\phi$ in the language of $\mathbf{I\Sigma_1}$ one has: $\mathcal{N}^{A,e} \models \phi$ iff $e(\phi) = 1$. Thus since all axioms of $\mathbf{I\Sigma_1}$ are in $\Gamma$, $\mathcal{N}^{A,e}$ is a model of $\mathbf{I\Sigma_1}$.

Let for every universal sentence in prenex normal form $\eta = \forall x_1 ... \forall x_n \beta(x_1, ..., x_n)$ with $\beta$ open, $\eta^U = \neg U(0) \vee \forall x_1 ... \forall x_n((U(0) \to (U(x_1) * U(0))) \vee ... \vee (U(0) \to (U(x_n) * U(0)) \vee \beta(x_1, ..., x_n))$.

**Theorem 3.39.** *For any universal sentence $\eta$ in the language of $\mathbf{I\Sigma_1}$, the following are equivalent:*
*(a) $\eta$ is true in the standard model.*
*(b) For every first-order structure $\mathcal{A}^+$ on a standard $\Omega$(WCMTL)-algebra $\mathcal{A}$, one has $\Gamma \models_{\mathcal{A}^+} \eta^U$.*
*(c) For every first-order structure $\mathcal{A}^+$ on a standard $\Pi$-algebra $\mathcal{A}$, one has $\Gamma \models_{\mathcal{A}^+} \eta^U$*

*Proof.* Since (b) trivially implies (c) we only prove $(a) \Rightarrow (b)$ and $(c) \Rightarrow (a)$.

$(a) \Rightarrow (b)$. Suppose that $\eta$ is true in the standard model of natural numbers and let $\mathcal{A}^+$ be a first-order structure over a standard $\Omega$(WCMTL)-algebra $\mathcal{A}$ such that $e(\phi) = 1$ for all $\phi \in \Gamma$. If $e(U(0)) = 0$, then $e(\eta^U) \geq e(\neg U(0)) = 1$. Suppose $e(U(0)) > 0$. Consider the function $f$ from the domain $D$ of individuals into $\mathcal{A}$ defined by $f(d) = e(U(d))$. Note that by the validity of axiom (4) of $\Gamma$, if $c \equiv d$, then $f(c) = f(d)$. Thus we can consider $f$ as a function from $\mathcal{N}^{A,e}$ into $\mathcal{A}$. Since $\mathcal{N}^{A,e}$ is a model of $\mathbf{I\Sigma_1}$, it contains an isomorphic copy of the standard model $\mathcal{N}$ of natural numbers, so we will identify every natural number $n$ with its copy in $\mathcal{N}$. Note that $f(0) > 0$ and the validity of axiom (3) of $\Gamma$ implies that for every $n$ one has $f(n) = f(n+1)^2$, therefore as in the proof of Claim (A) we see that $\sup\{f(n) : n \in \omega\}$ is an idempotent element of $\mathcal{A}$. Moreover, again by axiom (3) of $\Gamma$, the function $f([d])$ is weakly increasing in $[d]$. Thus for any non-standard $[d]$, $f([d]) \geq \sup\{f(n) : n \in \omega\}$, and there is an idempotent element between $f(0)$ and $f([d])$. So $f(0) \cdot f([d]) = f(0)$, and $e(U(d) * U(0)) = e(U(0))$. Now consider $[d_1], ..., [d_n] \in \mathcal{N}^{A,e}$. If they are all standard numbers then $\mathcal{N}^{A,e} \models \beta([d_1], ..., [d_n])$ as $\forall x_1 ... \forall x_n \beta(x_1, ..., x_n)$ is true in the standard model. Hence $e(\beta(d_1, ..., d_n)) = 1$. If at least one of the $[d_i]$ is non-standard, then $e(U(d_i) * U(0)) = e(U(0))$ and $e(U(0) \to (U(0) * U(d_i)) = 1$. By the arbitrariness of $d_1, ..., d_n$ we have that $e(\eta^U) = 1$.

$(c) \Rightarrow (a)$. Suppose that $\eta$ is false in the standard model $\mathcal{N}$. Let $k_1, ..., k_n$ be natural numbers such that $\beta(k_1, ..., k_n)$ is false in $\mathcal{N}$. Let $[0,1]_\Pi$ be the standard product algebra. Define a first-order structure $\mathcal{A}$ on $[0,1]_\Pi$ as follows: the domain $D$ of interpretation of object variables is the set $\mathbf{N}$ of natural numbers; moreover, writing $e$ for $e_{\mathcal{A},[0,1]_\Pi}$, define:

- For every predicate $P(x_1, ..., x_n)$ in the language of $\mathbf{I\Sigma_1}$ and for $r_1, ..., r_n \in \mathbf{N}$, define $e(P(r_1, ..., r_n)) = 1$ iff $P(r_1, ..., r_n)$ is true in $\mathcal{N}$ and $e(P(r_1, ..., r_n)) = 0$ otherwise.

- Finally define inductively $e(U(0)) = \frac{1}{2}$ and $e(U(n+1)) = \sqrt{e(U(n))}$.

Then we get $e(U(0) \to (U(k_i) * U(0))) < 1$, $e(\beta(k_1, ..., k_n)) = 0$ and $e(\neg U(0)) = 0$, therefore $e(\eta^U) < 1$. □

**Theorem 3.40.** *Let L be a logic between* $\Omega$(WCMTL) *and Product logic* $\Pi$ *(both included). Then the consequence relation for L$\forall$ with respect to the standard semantics for L is not recursively enumerable. Thus if L$\forall$ is recursively enumerable, then it has no FSSC.*

*Proof.* By the previous result we can recursively associate to every universal sentence $\eta$ of arithmetic a finite set $\Gamma$ of sentences and a sentence $\eta^U$ such that for any class $\mathbb{K}$ of standard algebras contained in the class of all standard $\Omega$(WCMTL)-algebras and such that $[0,1]_\Pi \in \mathbb{K}$ one has that $\eta$ is true in $\mathbf{N}$ iff $\Gamma \models \eta^U$ is valid in all first-order structures over algebras in $\mathbb{K}$. Since the set of universal sentences which are true in $\mathbf{N}$ is not recursively enumerable, the set of finite consequence relations which are valid in $\mathbb{K}$ is in turn not recursively enumerable. Taking the class of standard algebraic models of L as $\mathbb{K}$ we get the claim. □

# 4 Concluding remarks

We have introduced and studied the property of weak cancellation and we have obtained the following results:

- We have proved a theorem of representation of MTL-chains as ordinal sums of indecomposable totally ordered semihoops. A characterization of such indecomposable semihoops is still not known, but weak cancellation gives a big and interesting class of indecomposable totally ordered semihoops.

- Weak cancellation gives a new way to define Łukasiewicz logic from IMTL.

- Weak cancellation is exactly the difference between cancellation and pseudocomplementation, so it gives an alternative axiomatization of $\Pi$ and $\Pi$MTL and allows to define a new hierarchy of fuzzy logics.

- The ordinal sums of weakly cancellative totally ordered semihoops define a new logic, $\Omega$(WCMTL), that it is analogous to BL, in the sense that all BL-chains are decomposable as ordinal sums of Wajsberg hoops (hence weakly cancellative).

- We have studied some properties of these weakly cancellative fuzzy logics, but some important problems remain open. These properties are gathered in the following table:

| | LF | FEP | FMP | Decidable | FSSC | SSC |
|---|---|---|---|---|---|---|
| MTL | No | Yes | Yes | Yes | Yes | Yes |
| IMTL | No | Yes | Yes | Yes | Yes | Yes |
| SMTL | No | Yes | Yes | Yes | Yes | Yes |
| $\Omega$(WCMTL) | No | No | No | ? | Yes | No |
| S$\Omega$(WCMTL) | No | No | No | ? | Yes | No |
| WCMTL | No | No | No | ? | Yes | No |
| $\Omega$($\Pi$MTL) | No | No | No | ? | Yes | No |
| $\Pi$MTL | No | No | No | ? | Yes | No |
| BL | No | Yes | Yes | Yes | Yes | No |
| SBL | No | Yes | Yes | Yes | Yes | No |
| WCBL | No | No | No | Yes | Yes | No |
| $\Pi$ | No | No | No | Yes | Yes | No |
| G | Yes | Yes | Yes | Yes | Yes | Yes |
| Ł | No | Yes | Yes | Yes | Yes | No |

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