

# Real, rational and finite semantics for fuzzy logics

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## Abstract

In the seminal paper of Zadeh about Fuzzy Logic [10], he defined fuzzy sets over the real unit interval but in real systems applications fuzzy sets over rational numbers in  $[0, 1]$  or over finite sets of values of a linguistic variable were also used. In this paper we try to analyze the differences between these different possible sets of truth-values in the setting of residuated multiple-valued logics underlying fuzzy sets. Moreover this analysis also clarifies the differences between t-norms in the strict sense (over  $[0, 1]$ ) and t-norm like over other linearly ordered sets, namely the rational interval and finite chains.

## 1 Introduction

Membership functions for fuzzy sets were originally defined by Zadeh in [10] as functions taking values in the real unit interval. Therefore,  $[0, 1]$  is the most used set of truth-values for fuzzy logics. Nevertheless, in the algebraic approach used in the study of Fuzzy Logic in narrow sense, many classes of algebras have appeared as the algebraic counterpart of these logics. These classes contain as prominent examples algebras over  $[0, 1]$ , but also algebras over other chains of truth-values widely used in the applications of Fuzzy Logic, such as the rational unit interval or finite chains. In this paper, from the general results on the structure and embeddability of these algebras, we present some first results on the relations between the real, the rational and the finite chain semantics.

First we give in the preliminary section, the nec-

essary results on the generalization of the operations over  $[0, 1]$  to other chains, stressing two relevant properties: divisibility (which is related to continuity) and residuation (which is related to left-continuity). This leads to the study of two classes of algebras: BL and MTL-algebras. For these classes of algebras we will study their members defined over the real and the rational unit interval and over finite chains, and their relations.

## 2 Preliminaries on t-norm like operations

First of all we want to clarify the differences between R-implications, as defined by Trillas and Valverde in [9], and residuum as defined in the framework of residuated lattices and in the logic of a t-norm (see the definition of BL in [5] or the definition of MTL in [4]). R-implications were initially defined from a t-norm\* as,

$$R_*(x, y) = \sup\{z \mid x * z \leq y\}$$

while the residuum of a t-norm  $*$  is defined as the operation  $\rightarrow_*$  such that  $*$  and  $\rightarrow$  form an adjoint pair, i.e. they satisfy the condition:

$$x \rightarrow y \geq z \text{ if and only if } x * z \leq y$$

which is equivalent to the following definition of the residuum,

$$x \rightarrow y = \max\{z \mid x * z \leq y\}$$

Notice that R-implication is defined for each t-norm, while residuum not. In fact the residuum exists if, and only if, the t-norm is left-continuous.

Moreover, the residuum seems a better proposal in order to deal with the Modus Ponens (see e.g. [5])

Basic properties of a t-norm and its residuum are:

- *left-continuity*: A t-norm  $*$  has residuum  $\Rightarrow_*$  if, and only if, it is left-continuous.
- *order and residuum*:  $x \leq y$  if, and only if,  $x \Rightarrow_* y = 1$ .
- *Pre-linearity*:  $\max((x \Rightarrow_* y), (y \Rightarrow_* x)) = 1$
- *Definability of max*:  $\max(x, y) = \min((x \Rightarrow_* y) \Rightarrow_* y, (y \Rightarrow_* x) \Rightarrow_* x)$

This leads to the general definition of MTL-algebras (the algebraic counterpart of the Monoidal T-norm based Logic defined in [4]) as algebraic structures  $\mathbf{A} = \langle A, \wedge, \vee, *, \rightarrow, 0, 1 \rangle$  such that:

- $\langle A, \wedge, \vee, 0, 1 \rangle$  is a bounded distributive lattice,
- $\langle A, *, 1 \rangle$  is a commutative monoid with neutral element 1
- $*$  and  $\rightarrow$  form an adjoint pair ( $\rightarrow$  is the residuum of  $*$ )
- for all  $x, y \in A$  then  $(x \rightarrow y) \vee (y \rightarrow x) = 1$  (pre-linearity condition)

If the order defined by the lattice operators is total,  $\mathbf{A}$  we say that  $\mathbf{A}$  is an MTL-chain. Chains play a basic role in the study of MTL-algebras since every MTL-algebra is representable as subdirect product of MTL-chains, and thus equalities valid for all chains are valid for every algebra.

Furthermore, the following result characterizes continuous t-norms.

**Divisibility:** A t-norm is continuous if, and only if, it satisfies divisibility, i.e. if  $x \leq y$ , there exists  $z$  such that  $x * z = y$ .

**Divisibility equation:** A left-continuous t-norm is continuous if, and only if, its corresponding MTL-chain satisfies the divisibility equation, i.e. for all  $x, y \in [0, 1]$ ,  $x * (x \Rightarrow y) = \min(x, y)$ .

This leads to the definition of BL-algebras (the algebraic counterpart of the Basic fuzzy Logic defined in [5]) as residuated lattices satisfying pre-linearity and divisibility or, equivalently, MTL-algebras satisfying divisibility. It is also true that every BL-algebra is representable as subdirect product of BL-chains.

Therefore it seems natural to generalize t-norm to t-norm like  $*$  over a chain  $A$  (which is a distributive lattice with respect to min and max) and generalize left-continuity by being residuated and satisfying pre-linearity, and continuity by being residuated and satisfying pre-linearity and divisibility. Thus to study t-norm like we use the known results about the structure of BL and MTL-chains and their relation with t-norms, i.e. the operation defining BL and MTL chains over the real unit interval.

### 3 Structure of MTL and BL-chains over the real and the rational unit intervals

#### 3.1 Structure of BL-chains

To study the structure of BL-chains we have at disposal two main decomposition theorems for BL-chains. The first one is based on the well-known decomposition of a continuous t-norm as ordinal sum of the three basic continuous t-norms, which in [3] is generalized proving that every BL-chain is an ordinal sum of the three basic types (product, Gödel and MV-chains), or a subalgebra of such an ordinal sum lacking some idempotent elements. The second one is the decomposition as ordinal sum of Wajsberg hoops.

**Definition 1.** A structure  $\mathbf{H} = (H, *, \Rightarrow, 1)$  is a hoop if  $*$  is a commutative operation on  $H$  with the neutral element 1 (i.e.  $x * y = y * x$  and  $1 * x = x$  for all  $x, y$ ) and further  $\Rightarrow$  is a binary operation satisfying

- (i)  $x \Rightarrow x = 1$ ,
- (ii)  $x * (x \Rightarrow y) = y * (y \Rightarrow x)$ ,
- (iii)  $(x * y) \Rightarrow z = x \Rightarrow (y \Rightarrow z)$

for all  $x, y, z$ .

Moreover, we can define an order as:  $x \leq y$  iff  $x \Rightarrow y = 1$ .

A hoop is a **Wajsberg hoop** if it satisfies:

$$(W) (x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x$$

**Definition 2.** Given a family of hoops  $\{H_i \mid i \in I\}$  and being  $I$  a linearly ordered set, the ordinal sum is the hoop whose universe is  $A = (\cup(H_i - \{1_i\})) \cup \{1\}$  whose order is defined by  $x \leq y$  if, and only if:

$x \leq_i y$ , if  $x, y \in H_i - \{1_i\}$ , or  
 $x \in H_i - \{1_i\}$ ,  $y \in H_j - \{1_j\}$ , and  $i < j$ , or  
 $y = 1$ .

and whose  $*$  is defined as,

$$x * y = \begin{cases} x *_i y, & \text{if } x, y \in H_i - \{1_i\} \\ \min(x, y), & \text{otherwise} \end{cases}$$

In a Wajsberg hoop  $A$  the following conditions hold:

- 1) If  $0 \neq x * y = x * z$ , then  $y = z$  (weak cancellation)
- 2) If a linearly ordered Wajsberg hoop  $A$  contains an idempotent element,  $a$ , then  $A$  is decomposable as ordinal sum of the Wajsberg hoops obtained restricting the operations to  $A_a^+ = [a, 1^A]$  and to  $A_a^- = \{x \in A \mid x < a\} \cup \{1^A\}$ .
- 3) If  $A$  is indescomposable and  $a \in A$  is idempotent, then  $a$  has to be a bound of  $A$ .

**Proposition 3.** In a Wajsberg hoop if a monotone sequence  $\{a_i \mid i \in \omega\}$ , has a limit  $\alpha$ , then  $x * \alpha = \bigwedge(x * a_i)$  if  $a_i$  is a non-increasing sequence and  $x * \alpha = \bigvee(x * a_i)$  if  $a_i$  is a non-decreasing sequence.

*Proof:* If  $\alpha$  belongs to the sequence, the result is obvious. If the sequence is increasing then the result is also true because  $*$  is infinite distributive with respect to  $\vee$ . Suppose that the sequence is decreasing,  $\alpha$  does not belong to the sequence and the conclusion is not true, i.e.  $x * \alpha < \bigcap(x * a_i) = d > 0$  (obviously it is impossible to be greater). By divisibility there must exist an element  $\beta$  such that both  $x * \beta = d$  and by weak-cancellation this imply that  $\alpha < \beta < a_i$  for all  $a_i$  belonging to the sequence which is impossible.  $\square$

**Theorem 4.** [1] Any BL-chain is an ordinal sum of (indescomposable) linearly ordered Wajsberg hoops.

### 3.2 BL-chains over the real and the rational unit intervals

An interesting point is the difference between Wajsberg hoops over the real or the rational unit interval. In the real case it is well known that the only Wajsberg hoops are:

- **L**, defined by the Lukasiewicz t-norm and its residuum,
- **C**, defined by product t-norm and its residuum over  $(0, 1]$ ,
- **2**, isomorphic to the two-element Boolean algebra.

However, this is not true over the rational unit interval. The basic difference is the fact that the continuity of the real case implies that each Wajsberg hoop must contain only two Archimedean components, 1 and  $(0, 1]$ .

**Definition 5.** Let  $\mathbf{A}$  be a Wajsberg hoop and let  $\sim$  the following relation:

$$a \sim b \text{ if, and only if, for some } n \in \omega, \text{ either } a^n \leq b \leq a \text{ or } b^n \leq a \leq b$$

Then,  $\sim$  is an equivalence relation and the corresponding classes are called Archimedean components. The component containing the element  $a \in A$  will be denoted by  $[a]_{\sim}$ .

**Proposition 6.** Let  $\mathbf{C}$  be a complete and dense BL-chain and let  $\mathbf{A}$  be a component of the decomposition of  $\mathbf{C}$  as ordinal sum of Wajsberg hoops. Then  $\mathbf{A}$  contains only two Archimedean components which are  $\{1\}$  and  $A - \{1\}$ .

*Proof:* First notice that a indescomposable Wajsberg hoop does not contain idempotent elements different from the bounds (remember that a hoop always has upper bound but not always has lower bound).

Suppose that  $\mathbf{A}$  has  $\{1^A\}$  (the trivial Archimedean component) and another Archimedean component  $X \neq A - \{1\}$ . If  $\inf X = \alpha \in A$  and this is not the infimum of  $A$ , then by Proposition 3 it is clear that  $\alpha$  has to be an idempotent element and this is contradictory with our assumption. On the other

hand if  $\inf X = \inf A$  then by assumption there must exist an element  $b, 1^A \neq b \in A$  such that  $[1^A]_{\sim} > [b]_{\sim} > X$  and thus  $\inf([b]_{\sim}) \in A$  and it is different from the infimum of  $A$ . Thus  $\inf([b]_{\sim})$  must be idempotent which leads to a contradiction.  $\square$

This result is concordant with the fact that over the real unit interval the only indecomposable Wajsberg hoops, possible components of a decomposition of a BL-chain over the real unit interval, are **L**, **C** and **2**. Notice that all of them have exactly two Archimedean components.

As an example take the ordinal sum (in the sense of t-norms)  $L + \Pi + G$  and as ordinal sum of Wajsberg hoops is the ordinal sum  $L + C + (\bigoplus \mathbf{2})$  where the  $\bigoplus$  refers to a sum with as many **2** components as idempotent elements of the initial ordinal sum.

We move now to the case of the rational unit interval. In that case the order of the chain is not complete and thus Proposition 6 is not true. The following examples show that we can find Wajsberg indecomposable hoops over the rational unit interval having more than two Archimedean components which cannot be the restriction of any indecomposable Wajsberg hoop over the real unit interval (that necessarily has only two Archimedean components).

**Example 1.** Take the product hoop **HII** defined over  $(0, 1] \cap \mathbb{Q}$  (the standard product chain restricted to the rationals without zero). Take a cut in the interior of the rational unit interval defining an irrational  $\alpha \in (0, 1)$  and take the translation of **HII** over  $H = \mathbb{Q} \cap (\alpha, 1]$  (denote by  $\times$  the translated t-norm like operation). Take an strictly decreasing involution  $n : \mathbb{Q} \cap [0, 1] \rightarrow \mathbb{Q} \cap [0, 1]$  such that  $n(\mathbb{Q} \cap (\alpha, 1]) = \mathbb{Q} \cap [0, \alpha)$  and define the t-norm like over  $\mathbb{Q} \cap [0, 1]$  as,

$$x * y = \begin{cases} x \times y, & \text{if } x, y \in H \\ n(x \rightarrow_{\times} n(y)), & \text{if } x \in H, \text{ and } y \notin H \\ n(y \rightarrow_{\times} n(x)), & \text{if } x \notin H, \text{ and } y \in H \\ 0, & \text{otherwise} \end{cases}$$

where  $\rightarrow_{\times}$  refers to the residuum of  $\times$ . This corresponds to the disconnected rotation (in the sense of Jenei in [6]) of a cancellative hoop and thus it is an MV-chain (a generalized version of Chang's MV-algebra). The elements of this alge-

bra are divided in two groups: the so-called "infinite" elements (those belonging to  $\mathbb{Q} \cap (\alpha, 1]$ ), which form a filter, and the so-called "infinitesimals" (the elements of  $\mathbb{Q} \cap [0, \alpha)$ ) which are negative and  $x * y = 0$  for all infinitesimals  $x, y$ . Clearly, this Wajsberg hoop is indecomposable and has three Archimedean components,  $\{1\}$ ,  $\mathbb{Q} \cap (\alpha, 1)$  and  $\mathbb{Q} \cap [0, \alpha)$ .

**Example 2.** Take the lexicographic product of the ordered Abelian group of integers ( $\mathbb{Z}$ ) with the sum and the ordered Abelian group of positive rationals ( $\mathbb{Q}^+$ ) with the product operation and denote it by  $\mathbb{Z} \times_{lex} \mathbb{Q}^+$ . The negative cone of it is a (cancellative) Wajsberg hoop having three Archimedean components  $\{(0, 1)\}$ ,  $\{(0, r) \mid r \in \mathbb{Q} \cap [0, 1]\}$  and  $\{(-k, r) \mid k \in \mathbb{N} - \{0\}, r \in \mathbb{Q}^+\}$ . Take finally a cut in  $\mathbb{Q} \cap [0, 1]$  defining a irrational number  $\alpha \in (0, 1)$  and take the transform of the three Archimedean components into the three intervals  $\{1\}$ ,  $\mathbb{Q} \cap (\alpha, 1)$  and  $\mathbb{Q} \cap (0, \alpha)$ . The resulting structure is an indecomposable and cancellative Wajsberg hoop over the rational unit interval that has three Archimedean components.

Notice that the first example is a bounded indecomposable Wajsberg hoop (i.e. an MV-chain), while the second one is a cancellative indecomposable Wajsberg hoop

### 3.3 MTL-chains over the real and the rational unit intervals

As regards to MTL-chains, the situation is completely different from that of BL. On the one hand, a general result describing their structure is not known yet, not even for those defined over the real unit interval. On the other hand, as a positive result, MTL-chains enjoy the following powerful embedding property, which is not true for BL-chains.

**Theorem 7.** Every countable MTL-chain is embeddable into an MTL-chain defined over the real unit interval.

The proof of this theorem is given by Jenei and Montagna by means of a constructive method in [7].

As a consequence we obtain the following corollary.

**Corollary 8.** *Every MTL-chain defined over the rational unit interval is the restriction of an MTL-chain defined over the real unit interval.*

The extension of the t-norm like operation from rationals to reals, is defined in the only possible way that preserves the left-continuity:

For every  $\alpha, \beta \in [0, 1]$   $\alpha \circ \beta := \sup\{x * y : x \leq \alpha, y \leq \beta, x, y \text{ rational numbers}\}$ .

However, this method of completion does not preserve many other properties such as continuity (i.e. divisibility) or cancellation.

As an example consider the chain in Example 1. The completion of this chain in the real interval introduces a new element separating the two non-trivial Archimedean components, say  $\alpha$ . Then for every  $x > \alpha$ ,  $x \circ \alpha = \alpha$  and  $\alpha \circ \alpha = 0$ , and thus the cancellation and the divisibility are not true anymore.

## 4 About finite MTL and BL-chains

Now we focus on finite chains. As in the general case, the structure of such chains is completely known for BL-chains and unknown for MTL-chains. Indeed, since there are no finite product chains (except for the trivial one and **2**), all finite BL-chains are ordinal sums of finite MV and Gödel chains. Taking into account that finite MV-chains are isomorphic to the Łukasiewicz  $n$ -valued chains  $\mathbf{L}_n$ , and the obvious structure of finite Gödel chains  $\mathbf{G}_n$ , we obtain:

**Proposition 9.** *Each finite BL-chain is isomorphic to an ordinal sum of a finite subfamily of  $\{\mathbf{L}_n \mid n \geq 1\} \cup \{\mathbf{G}_n \mid n \geq 1\}$ .*

This result is equivalent to the representation theorem for finite smooth t-norm given in [8].

Since for every  $n \geq 1$ ,  $\mathbf{L}_n$  (resp.  $\mathbf{G}_n$ ) is a subalgebra of the MV-chain (resp. the Gödel chain) over the rational unit interval defined by the Łukasiewicz t-norm (resp. the minimum t-norm), and this is also a subalgebra of  $\mathbf{L}$  (resp.  $\mathbf{G}$ , i.e. the chain over the reals defined by the minimum t-norm), every finite BL-chain is embeddable both in a rational chain and in a real chain.

As regards to MTL-chains, Theorem 7 guaran-

tees that, in particular, all finite ones are embeddable into an MTL-chain over the real unit interval. Nevertheless the constructive method given by Jenei and Montagna in [7] does not prove that these chains are embeddable into a chain over the rationals.

As mentioned before, Jenei and Montagna's embedding does not preserve all properties in general. For instance, when applied to a finite BL-chain which is not a Gödel chain, we do not obtain a BL-chain over the reals (see for example the completion of  $\mathbf{L}_n$  in [2]).

## 5 Conclusions

As a summary, over the rational unit interval there are as many MTL-chains as restrictions of MTL-chains over the reals, while there is a very large set of BL-chains that are not isomorphic restrictions of BL-chains over the unit real interval (recall that by restriction the number of Archimedean components remains invariant). It remains as an open problem the description of the indescomposable Wajsberg hoops over the rationals in order to obtain a full characterization of BL-chains over the rational unit interval.

Regarding finite chains, we have fully explained their structure and embeddability properties in the rational and in the real unit interval, in the BL case. However, for finite MTL-chains we only know that they are embeddable into MTL-chains over the reals. Their structure and embeddability into the rational MTL-chains are also open problems.

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