On weakening the Deduction Theorem and strengthening Modus Ponens

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This paper studies, with techniques of Abstract Algebraic Logic, the effects of putting a bound on the cardinality of the set of side formulas in the Deduction Theorem, viewed as a Gentzen-style rule, and of adding additional assumptions inside the formulas present in Modus Ponens, viewed as a Hilbert-style rule. As a result, a denumerable collection of new Gentzen systems and two new sentential logics have been isolated. These logics are weaker than the positive implicative logic. We have determined their algebraic models and the relationships between them, and have classified them according to several standard criteria of Abstract Algebraic Logic. One of the logics is protoalgebraic but neither equivalential nor weakly algebraizable, a rare situation where very few natural examples were hitherto known. In passing we have found new, alternative presentations of positive implicative logic, both in Hilbert style and in Gentzen style, and have characterized it in terms of the restricted Deduction Theorem: it is the weakest logic satisfying Modus Ponens and the Deduction Theorem restricted to at most 2 side formulas. The algebraic part of the work has lead to the class of quasi-Hilbert algebras, a quasi-variety of implicative algebras introduced by Pla and Verdú in 1980, which is larger than the variety of Hilbert algebras. Its algebraic properties reflect those of the corresponding logics and Gentzen systems.

Introduction

The Deduction-Detachment Theorem (DDT) is the metalogical property, which a sentential logic $\mathcal{S}$ may or may not have, that for all formulas φ, ψ and all sets of formulas Γ,

(DDT) $\quad \Gamma, \varphi \vdash_{\mathcal{S}} \psi \quad$ if and only if $\quad \Gamma \vdash_{\mathcal{S}} \varphi \to \psi$,

where $\vdash_{\mathcal{S}}$ is the consequence relation of the logic $\mathcal{S}$, be it a deducibility relation in some proof system or the consequence associated with some kind of semantics for $\mathcal{S}$. The DDT (also called, informally, the "Deduction Theorem") is commonly regarded as one of the central metalogical properties of classical and intuitionistic logic; the study of several of its generalizations has played a major role in shaping the evolution of Algebraic Logic since the early eighties [2,6]. In this paper we investigate, in the context of Abstract Algebraic Logic, several restricted versions of this property according to the cardinality of the set Γ. It happens that the two implications contained in (DDT) have a qualitatively different metalogical import. The implication from right to left is equivalent to the Hilbert-style rule of Modus Ponens or "Detachment"

(MP) $\quad \varphi, \varphi \to \psi \vdash_{\mathcal{S}} \psi$,

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where no set of "side assumptions" is needed, thus it is not affected by our aimed restriction. Our investigation primarily refers to the implication from left to right, which is what should be called, strictly-speaking, the "Deduction Theorem"

$$\text{(DT)} \qquad \Gamma, \varphi \vdash_{\boldsymbol{S}} \psi \quad \text{implies} \quad \Gamma \vdash_{\boldsymbol{S}} \varphi \to \psi\,.$$

The (DDT) was taken by Tarski [19] as one of the axioms for his abstract presentation of the consequence operator of classical logic, and independently he and Herbrand [15] proved that a logic satisfies it if and only if it is an axiomatic strengthening of the so-called positive implicative logic (the implicative fragment of intuitionistic logic, denoted as $\mathcal{IPL}_{\to}$); as a consequence $\mathcal{IPL}_{\to}$ is characterized by (MP) and (DT). One of the questions we address in this paper is how much of the strength of (DT) is really needed in order to obtain $\mathcal{IPL}_{\to}$. One of the non-formal conclusions one can draw from our study is that, while (DT) can be considerably weakened for this purpose, (MP) cannot. By contrast, some logics have been considered in the literature that are defined by Gentzen calculi without (MP) but with limited versions of (DT), for instance the logic called "weak Došen's logic" in [4] has only (DT0), and Visser's "Basic Propositional Logic" [21] has only (DT1), to use the notation introduced below.

While (MP) is (equivalent to) a Hilbert-style rule, we prove that (DT) cannot be expressed by this kind of rules (Theorem 3.4). Thus, the appropriate framework for our investigation is that of Gentzen-style calculi. A finitary logic has the (DT) if and only if it satisfies the infinite set of Gentzen-style rules (one for each $n \geq 0$)

$$(\mathrm{DT}n) \qquad \frac{\varphi_1, \ldots, \varphi_n, \varphi \rhd \psi}{\varphi_1, \ldots, \varphi_n \rhd \varphi \to \psi}.$$

(We use the symbol "$\rhd$" as separator of the two sides of a sequent, instead of "$\vdash$" or "$\to$" or "$\Rightarrow$" as in other works, in order to avoid any confusion.)

We consider, for each $n \geq 1$, the structural Gentzen systems $\mathfrak{G}_n$ defined by each rule (DTn) plus (MP), as well as the sentential logics $\mathcal{G}_n$ they define. We show that for $n \geq 2$ we obtain new Gentzen-style presentations of $\mathcal{IPL}_{\to}$, while for $n = 1$ we obtain a new, weaker logic $\mathcal{G}_1$, hence we conclude that (DT2) is the weakest form of (DT) still able to generate $\mathcal{IPL}_{\to}$. When looking for a Hilbert-style presentation of $\mathcal{G}_1$ we obtain in a natural way a stronger companion logic $\mathcal{H}_1$, still weaker than $\mathcal{IPL}_{\to}$. Both incorporate the following strengthened forms of Modus Ponens, where additional assumptions are present inside the formulas themselves rather than as "side assumptions". Consider the rules

$$(\mathrm{MP}n) \qquad \varphi_1 \to \big(\ldots \to (\varphi_n \to \varphi)\ldots\big), \varphi_1 \to \big(\ldots \to \big(\varphi_n \to (\varphi \to \psi)\big)\ldots\big) \vdash \varphi_1 \to \big(\ldots \to (\varphi_n \to \psi)\ldots\big).$$

For $n = 2$ this rule is present (in two different ways) in $\mathcal{G}_1$ and $\mathcal{H}_1$, while for $n \geq 3$ it gives alternative Hilbert-style presentations of $\mathcal{IPL}_{\to}$ not needing the typical Fregean axiom. The cases of (DT0) and of (MP1) are not dealt with in this paper, as they are much weaker and do not allow for a unified treatment along with the other systems, as seen in [3, 13].

We characterize and classify the Gentzen systems and the two new sentential logics from the point of view of Abstract Algebraic Logic, and we determine their algebraic models, viewed as algebras endowed with a closure operation, that is, as generalized matrices (also called abstract logics in the literature). Since for $n \geq 2$ the Gentzen systems $\mathfrak{G}_n$ yield presentations of $\mathcal{IPL}_{\to}$, it is natural that as their algebraic models we find the class of Hilbert algebras, but we find them endowed with different families of implicative filters. As a consequence of the new Hilbert-style presentations of $\mathcal{IPL}_{\to}$, we also obtain new presentations of the variety of Hilbert algebras.

In the algebraic study of $\mathfrak{G}_1$, $\mathcal{H}_1$ and $\mathcal{G}_1$ we find the class of *quasi-Hilbert algebras*, a quasi-variety that is strictly larger than the variety of Hilbert algebras but is still contained in the quasi-variety of implicative algebras. As to the classification of the sentential logics in the protoalgebraic hierarchy, it is particularly interesting to find that $\mathcal{G}_1$ is a protoalgebraic logic which is neither equivalential nor weakly algebraizable, a category where very few and rather ad-hoc examples were hitherto known.

The seminal ideas about putting cardinality restrictions to the Deduction Theorem appear for a restricted case in Torrens' 1980 Dissertation [20] and in full generality, as well as the definition of quasi-Hilbert algebras and their first properties, in Pla and Verdú's 1980 paper [16]. In these works the restricted cases of (DT) are considered only as algebraic properties of the closure operator of filter generation in these algebras, and are not connected with what we now call Gentzen systems and the notion of model of a Gentzen system. These ideas were incorporated in the preliminary work contained in the 1991 Ph. D. Dissertation [13] of one of the authors (JLGL) and in the 2001 Master Thesis [3] of another of the authors (FB), both supervised by the third one (JMF), who in the meantime had worked (but not published) on these topics. Some of the results proved in this paper were announced in Chapter 5 of [10].

Some terminology and notation In this paper we deal exclusively with the algebraic similarity type (or sentential language) $\mathcal{L}$ having only one binary operation $\to$. The term algebra, or absolutely free algebra of this type, also called the algebra of sentential formulas, will be denoted by $\boldsymbol{Fm}$, and its universe by Fm. Variables, or atomic formulas, are denoted by $x, y, z, \ldots$, arbitrary formulas (terms) by lowercase greek letters (φ, ψ, etc.) and sets of formulas by uppercase ones (Γ, etc.). By a *sentential logic* $\boldsymbol{\mathcal{S}}$ we understand a consequence relation $\vdash_{\boldsymbol{\mathcal{S}}}$ between sets of formulas and formulas that is invariant under substitutions. All sentential logics defined by a Hilbert-style calculus or by a Gentzen calculus are finitary; however in the general theory we also admit non-finitary logics. The *interderivability relation of* $\boldsymbol{\mathcal{S}}$ is the binary relation $\varphi \dashv\vdash_{\boldsymbol{\mathcal{S}}} \psi$ that holds when both $\varphi \vdash_{\boldsymbol{\mathcal{S}}} \psi$ and $\psi \vdash_{\boldsymbol{\mathcal{S}}} \varphi$.

By a *sequent* in this paper we understand a finite, non-empty sequence of formulas which we denote by $\varphi_1, \ldots, \varphi_n \rhd \varphi$ (i. e, we consider only "single-conclusion" sequents). For the uniformity of some general expressions it will be convenient to extend the notation $\varphi_1, \ldots, \varphi_n$ to the case $n = 0$ by making it mean the empty sequence. A *Gentzen system* is to sequents what a sentential logic is to formulas: a consequence relation between sets of sequents and sequents that is invariant under substitutions; usually such relations are defined by Gentzen calculi, that is, by sets of Gentzen-style rules or rule schemes that are applied modulo substitutions. Notice that all Gentzen systems here considered will have all structural rules.

More details will be given in each section. For all unexplained notions and for a discussion of their significance we refer the reader to the survey paper [12] or to the more comprehensive book [7].

Structure of the paper The paper begins with Section 1 on *quasi-Hilbert algebras*, the quasi-variety that will appear in the modelling of the logical systems (either Gentzen systems or sentential logics) treated in the main body of the paper. By putting this material first we hope not to break the logical thread of the paper, which indeed constitutes its main motivation. In Section 2 the Gentzen systems are presented and their algebraic models are determined. Finally in Section 3 several sentential logics are considered, some by means of the Gentzen systems already studied, some by means of suitable Hilbert-style calculi. These logics are studied and classified according to the paradigms of Abstract Algebraic Logic. Some of them turn out to yield new, alternative presentations of positive implicative logic.

1 Quasi-Hilbert algebras

Our algebras will be of type $\mathcal{L} = \langle \to \rangle$, but we use the abbreviation $\top := x \to x$ and for some purposes treat it as a constant term; we explain why after Definition 1.1. Let $\boldsymbol{A} = \langle A, \to \rangle$ be an $\mathcal{L}$-algebra. We denote by $\mathrm{Co}\boldsymbol{A}$ its congruence lattice. If $\boldsymbol{A}, \boldsymbol{B}$ are two $\mathcal{L}$-algebras, then $\mathrm{Hom}(\boldsymbol{A}, \boldsymbol{B})$ is the set of all homomorphisms from $\boldsymbol{A}$ to $\boldsymbol{B}$. Elements of algebras are denoted by lowercase latin letters (a, b, etc.). In order to present some quasi-equations in a compact form it will be convenient to write $\varphi \approx \psi \approx \xi$ to denote $\varphi \approx \xi \,\&\, \psi \approx \xi$, for any $\varphi, \psi, \xi \in Fm$.

Definition 1.1 Let us consider the following $\mathcal{L}$-equations and $\mathcal{L}$-quasi-equations.

(i1) $x \to x \approx y \to y$.

(i2) $x \to y \approx y \to x \approx \top \Longrightarrow x \approx y$.

(trans) $x \to y \approx y \to z \approx \top \Longrightarrow x \to z \approx \top$.

(pre) $y \to \top \approx \top$.

(k) $x \to (y \to x) \approx \top$.

(mp2) $x_1 \to (x_2 \to x) \approx x_1 \to \big(x_2 \to (x \to y)\big) \approx \top \Longrightarrow x_1 \to (x_2 \to y) \approx \top$.

(fre) $\big(x \to (y \to z)\big) \to \big((x \to y) \to (x \to z)\big) \approx \top$.

Using them we define the following classes of algebras.

1. An *implicative algebra* is an $\mathcal{L}$-algebra $\boldsymbol{A} = \langle A, \rightarrow \rangle$ that satisfies the quasi-equations (i1), (i2), (trans) and (pre). This class is denoted by **IA**.

2. A *quasi-Hilbert algebra* is an $\mathcal{L}$-algebra $\boldsymbol{A} = \langle A, \rightarrow \rangle$ that satisfies the quasi-equations (i1), (i2), (k) and (mp2). This class is denoted by **QHA**.

3. A *Hilbert algebra* is an $\mathcal{L}$-algebra $\boldsymbol{A} = \langle A, \rightarrow \rangle$ that satisfies the quasi-equations (i1), (i2), (k) and (fre). This class is denoted by **HA**.

A note on terminology: (i1) and (i2) stand for *implicative*, (trans) for *transitivity*, (pre) for *prefixing*, and (fre) for *Fregean*. The others are either standard or will be easily understood later on.

By definition the three classes are quasi-varieties. The equation (i1), together with our notational convention, entails that any algebra $\boldsymbol{A}$ in any of these classes has an algebraic constant, which we denote by 1, such that $\top^{\boldsymbol{A}} = a \rightarrow a = 1$ for any $a \in A$. Hence we can view these algebras as $\langle \rightarrow, \top \rangle$-algebras of type (2,0), and we need only re-write equation (i1) as $x \rightarrow x \approx \top$. Then the class **IA** can be identified with the class of *implicative algebras* as defined by H. Rasiowa in [17]. As a matter of fact, these algebras can be defined as the algebras of type $\langle \rightarrow, \top \rangle$, with $\top^{\boldsymbol{A}} = 1$, where the relation $\leq$ defined as

(1) $$a \leq b \quad \text{iff} \quad a \rightarrow b = 1$$

is a partial ordering on A with maximum element 1. For this reason in this paper it will be convenient to treat the formal expression $x \leq y$ as an abbreviation of the equation $x \rightarrow y \approx \top$, that is, $x \rightarrow y \approx x \rightarrow x$. Similarly, the class **HA** is identified with the class of *Hilbert algebras* studied by Diego in [8, 9], called *positive implication algebras* by Rasiowa in [17]. The same process shows that our class **QHA** is the class of quasi-Hilbert algebras introduced in [16]. Diego showed that **HA** is actually a variety, while Rasiowa showed that **IA** is not. As for **QHA**, it is not known whether it is a variety or a proper quasi-variety; some results in this section can be relevant for an eventual solution of this open problem.

There is a way to associate a Hilbert algebra with every partially ordered set $\langle A, \leq \rangle$ with maximum $1 \in A$. One defines a binary operation $\rightarrow : A \times A \longrightarrow A$ as

(2) $$a \rightarrow b = \begin{cases} 1 & \text{if } a \leq b, \\ b & \text{otherwise.} \end{cases}$$

It is easy to check that the algebra $\boldsymbol{A} = \langle A, \rightarrow \rangle$ is a Hilbert algebra, which will be called *the algebra canonically associated with* $\leq$.

Now we are going to establish the relationships between the three classes of algebras just defined and to prove some elementary properties of quasi-Hilbert algebras we need here or later on. We begin by proving that quasi-Hilbert algebras are indeed implicative algebras.

Lemma 1.2 *Let* $\boldsymbol{A} = \langle A, \rightarrow \rangle \in$ **QHA** *and* $a, b, c \in A$. *Then:*

1. (pre) $a \rightarrow 1 = 1$.
2. (mp1) $a \rightarrow b = 1$ *and* $a \rightarrow (b \rightarrow c) = 1$ *imply* $a \rightarrow c = 1$.
3. (trans) $a \rightarrow b = 1$ *and* $b \rightarrow c = 1$ *imply* $a \rightarrow c = 1$.

Proof.
1. By (k) $a \rightarrow 1 = a \rightarrow (a \rightarrow a) = 1$.
2. From the assumptions $1 \rightarrow (a \rightarrow b) = 1 \rightarrow 1 = 1$ and $1 \rightarrow \big(a \rightarrow (b \rightarrow c)\big) = 1 \rightarrow 1 = 1$, hence by (mp2) $1 \rightarrow (a \rightarrow c) = 1$. Since from (pre) $(a \rightarrow c) \rightarrow 1 = 1$, (i2) yields $a \rightarrow c = 1$.
3. From the assumptions $a \rightarrow b = 1$ and $a \rightarrow (b \rightarrow c) = a \rightarrow 1 = 1$; then by (mp1) $a \rightarrow c = 1$. □

It follows that the relation defined in a quasi-Hilbert algebra as in (1) is an ordering relation with maximum 1. We go on making frequent use of this fact, and of (mp2) in an essential way.

Lemma 1.3 *Let* $\boldsymbol{A} = \langle A, \rightarrow \rangle \in$ **QHA** *and* $a, b, c, d \in A$. *Then:*

1. (subtract) $1 \rightarrow a = a$.
2. (isot) *If* $a \leq b$, *then* $b \rightarrow c \leq a \rightarrow c$ *and* $c \rightarrow a \leq c \rightarrow b$.
3. (cong) $a \rightarrow b = 1$ *and* $c \rightarrow d = 1$ *imply* $(b \rightarrow c) \rightarrow (a \rightarrow d) = 1$.

4. (contrac) $a \to (a \to b) = a \to b$.
5. $a \le (a \to b) \to b$.
6. (q-cp) $a \to (b \to c) = 1$ *if and only if* $b \to (a \to c) = 1$.
7. (q-fre1) $a \to (b \to c) = 1$ *if and only if* $(a \to b) \to (a \to c) = 1$.
8. (q-fre2) $(a \to b) \to \big((b \to a) \to a\big) = 1$ *if and only if* $(b \to a) \to \big((a \to b) \to b\big) = 1$.

Proof.

1. Since $(1 \to a) \to (1 \to a) = 1$ and by (pre) $(1 \to a) \to 1 = 1$, using (mp1) gives $(1 \to a) \to a = 1$. Now $a \to (1 \to a) = 1$ by (k), hence by (i2) $1 \to a = a$.

2. If $a \le b$, then $a \to b = 1$, so $(b \to c) \to (a \to b) = (b \to c) \to 1 = 1$ by (pre). Since $(b \to c) \to \big(a \to (b \to c)\big) = 1$ by (k), we use (mp2) and get $(b \to c) \to (a \to c) = 1$, that is, $b \to c \le a \to c$. $c \to a \le c \to b$ is proved similarly.

3. The assumptions amount to $a \le b$ and $c \le d$. Then applying both isotonies from 2. we get the result.

4. Since $\big(a \to (a \to b)\big) \to \big(a \to (a \to b)\big) = 1$ and by (pre) also $\big(a \to (a \to b)\big) \to (a \to a) = 1$ we can apply (mp2) and obtain $\big(a \to (a \to b)\big) \to (a \to b) = 1$. But $(a \to b) \to \big(a \to (a \to b)\big) = 1$ by (k), so by (i2) $\big(a \to (a \to b)\big) = a \to b$.

5. $a \to \big((a \to b) \to a\big) = 1$ by (k), and $a \to \big((a \to b) \to (a \to b)\big) = a \to 1 = 1$, so by (mp2) $a \to \big((a \to b) \to b\big) = 1$.

6. From $a \le b \to c$ and (isot) we obtain $(b \to c) \to c \le a \to c$ and by 5. $b \le (b \to c) \to c$ hence $b \le a \to c$. This shows the equivalence.

7. Assume $a \to (b \to c) = 1$. Then $(a \to b) \to \big(a \to (b \to c)\big) = 1$ by (pre). Since $(a \to b) \to (a \to b) = 1$, by (mp2) $(a \to b) \to (a \to c) = 1$. Now for the converse assume that $(a \to b) \to (a \to c) = 1$, that is, $a \to b \le a \to c$. Since $b \le a \to b$ by (k), $b \le a \to c$, that is, $b \to (a \to c) = 1$ which by (q-cp) gives $a \to (b \to c) = 1$.

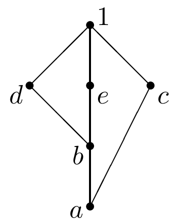
8. Applying (q-cp) to the assumption we obtain $(b \to a) \to \big((a \to b) \to a\big) = 1$, but by (pre) we have $(b \to a) \to \big((a \to b) \to (a \to b)\big) = 1$. Now (mp2) yields $(b \to a) \to \big((a \to b) \to b\big) = 1$. □

The property labelled (subtract) is usually called the *subtractive identity*. The two properties in (isot), called *isotonicity properties*, say that the operation $\to$ is monotonic in its second (right) argument and anti-monotonic in its first (left) argument. These are the typical properties linking implication and ordering in a large number of structures and, in another level of abstraction, in category theory. Of course each has an alternative presentation as a quasi-equation, using (1). The property called (cong) refers to *congruence*, as will be made explicit later on, and (contrac) refers to *contraction*. It is particularly interesting to notice that the last three properties are quasi-equations of the form $\alpha \approx \top \iff \beta \approx \top$ corresponding to equations $\alpha \approx \beta$ that hold in the variety of Hilbert algebras, in particular to its main characteristic identities *commutation of premises* $x \to (y \to z) \approx y \to (x \to z)$ and the *Fregean identity* $x \to (y \to z) \approx (x \to y) \to (x \to z)$. However, as we show at the end of Section 3.2, this phenomenon does not hold for all such identities.

Theorem 1.4 $\mathsf{HA} \subsetneq \mathsf{QHA} \subsetneq \mathsf{IA}$.

Proof. That every quasi-Hilbert algebra is an implicative algebra is shown in Lemma 1.2. Consider the $\mathcal{L}$-algebra with universe the real unit interval and the operation $a \to b = 1$ when $a \le b$ and $a \to b = 0$ otherwise. Since this is an ordered set with maximum, this definition ensures it is an implicative algebra, but it does not satisfy (k): $1/2 \to \big(1 \to 1/2\big) = 1/2 \to 0 = 0 \ne 1$. Hence this is not a quasi-Hilbert algebra. In order to prove that every Hilbert algebra is a quasi-Hilbert one, we need only show that (mp2) holds in HA. Assume that $\boldsymbol{A} \in \mathsf{HA}$ and that $a \to (b \to c) = 1$ and $a \to \big(b \to (c \to d)\big) = 1$ for some $a, b, c, d \in A$. From the second assumption $a \le b \to (c \to d) \le (b \to c) \to (b \to d)$ by (fre), but $(b \to c) \to (b \to d) \le a \to (b \to d)$ by (isot) from the first assumption. Hence $a \le a \to (b \to d)$, and by (contrac), which is known to hold in Hilbert algebras, this yields $a \to (b \to d) = 1$. This shows that $\mathsf{HA} \subseteq \mathsf{QHA}$. That the inclusion is proper is shown by the example below. □

Example 1.5 The $\mathcal{L}$-algebra $\boldsymbol{A}_1 = \langle A_1, \to \rangle$ is defined by the following operation:



$\to$	a	b	c	d	e	1
a	1	1	1	1	1	1
b	$\boxed{c}$	1	c	1	1	1
c	$\boxed{b}$	b	1	d	e	1
d	a	$\boxed{e}$	c	1	e	1
e	a	$\boxed{d}$	c	d	1	1
1	a	b	c	d	e	1

The diagram represents the ordering relation $\leq$ associated with $\boldsymbol{A}_1$. The boxed values in the table for $\to$ are those where the operation does not coincide with the canonical one associated with the ordering $\leq$.

It is easy to check that $\boldsymbol{A}_1 \in \mathsf{QHA}$ and $\boldsymbol{A}_1 \notin \mathsf{HA}$; for the second part use that

$$\big(d \to (c \to a)\big) \to \big((d \to c) \to (d \to a)\big) = d \neq 1.$$

This example will be used at several points in the paper. It happens to be the unique algebra with less than seven elements that is a quasi-Hilbert algebra but not a Hilbert algebra, and was found with a simple computer program.

Lemma 1.6 *Let $\boldsymbol{A} = \langle A, \to \rangle \in \mathsf{QHA}$, and let $\theta \in \mathrm{Co}\boldsymbol{A}$. Then:*

1. $\theta = \big\{ \langle a, b \rangle \in A^2 : \langle a \to b, 1 \rangle \in \theta$ *and* $\langle b \to a, 1 \rangle \in \theta \big\}$.
2. *If a, $b \in A$ with $\langle a \to b, 1 \rangle \in \theta$, then there is some $b' \in A$ such that $\langle b, b' \rangle \in \theta$, $b \leq b'$ and $a \leq b'$.*
3. *For every $a \in A$ the equivalence class a/θ is directed with respect to the ordering relation $\leq$.*

Proof.

1. The inclusion $\theta \subseteq \big\{ \langle a, b \rangle \in A^2 : \langle a \to b, 1 \rangle \in \theta$ and $\langle b \to a, 1 \rangle \in \theta \big\}$ is obvious, since for every $a \in A$ we have $a \to a = 1$. We now check the converse inclusion. Let a, $b \in A$ be such that $\langle a \to b, 1 \rangle \in \theta$ and $\langle b \to a, 1 \rangle \in \theta$; let us see that $\langle a, b \rangle \in \theta$. We split the proof in two cases:

1a. If $b \leq a$: From 1.3.5 we know that $b \leq a \leq (a \to b) \to b$. Then by (isot) and (contrac) we obtain that $(a \to b) \to b \leq (a \to b) \to a \leq (a \to b) \to \big((a \to b) \to b\big) = (a \to b) \to b$. Hence, $(a \to b) \to b = (a \to b) \to a$. From the fact that $\langle a \to b, 1 \rangle \in \theta$ we obtain by (subtract) that $\big\langle (a \to b) \to b, b \big\rangle \in \theta$ and that $\big\langle (a \to b) \to a, a \big\rangle \in \theta$. From the last two statements it follows that $\langle a, b \rangle \in \theta$.

1b. If a, b are arbitrary: By 1.3.5 we know that $a \leq (a \to b) \to b$. Then by (isot) and (contrac) we obtain that $(a \to b) \to a \leq (a \to b) \to \big((a \to b) \to b\big) = (a \to b) \to b$. On the other hand, $\langle a \to b, 1 \rangle \in \theta$ and $\langle b \to a, 1 \rangle \in \theta$, which gives us $\big\langle \big((a \to b) \to b\big) \to \big((a \to b) \to a\big), b \to a \big\rangle \in \theta$, that is, that $\big\langle \big((a \to b) \to b\big) \to \big((a \to b) \to a\big), 1 \big\rangle \in \theta$. We see that the elements $(a \to b) \to b$ and $(a \to b) \to a$ satisfy the assumption of the preceding case, and this implies that $\big\langle (a \to b) \to b, (a \to b) \to a \big\rangle \in \theta$. Now using that $\langle a \to b, 1 \rangle \in \theta$ and (subtract), we conclude that $\langle b, a \rangle \in \theta$.

2. Let a, $b \in A$ be such that $\langle a \to b, 1 \rangle \in \theta$. Define $b' = (a \to b) \to b$. Since $\langle a \to b, 1 \rangle \in \theta$ and $1 \to b = b$ we have $\langle b', b \rangle \in \theta$. From (k) we know that $b \leq b'$; and by 1.3.5 we know that $a \leq b'$.

3. Let b, $c \in A$ be such that $\langle a, b \rangle \in \theta$, $\langle a, c \rangle \in \theta$. We need to find some $d \in A$ with $\langle a, d \rangle \in \theta$ and $b \leq d$, $c \leq d$. From the assumptions it follows that $\langle b, c \rangle \in \theta$. Hence, $\langle b \to c, 1 \rangle \in \theta$. By part 2 there is some $d \in A$ with $\langle c, d \rangle \in \theta$, $b \leq d$ and $c \leq d$. Clearly $\langle a, d \rangle \in \theta$; as a consequence this d satisfies the needed properties. □

If $\boldsymbol{A}$ is finite, then part 3 of the lemma implies that every equivalence class has a maximum element, which gives a canonical representant of the class.

QHA is a *pointed class of algebras*, therefore from part 1 of the lemma we have:

Corollary 1.7 QHA *is a pointed congruence-regular class, that is, the congruences of its members are determined by their* 1*-class.* □

It is remarkable that we have not just *relative congruence regularity*, as is to be expected in a quasi-variety, but the full property (i. e., for all congruences and not just for those yielding a quasi-Hilbert algebra in the quotient). In view of the algebraizability results we will obtain in Section 3 it is also remarkable that the precise result of Lemma 1.6.1 also does hold for all congruences and not just for the relative ones. These facts may hint that perhaps **QHA** is in fact a variety. What we do know is the following:

Theorem 1.8 *Let $\boldsymbol{A} \in$ **QHA** and $\theta \in \mathrm{Co}\boldsymbol{A}$. Then $\boldsymbol{A}/\theta$ is an implicative algebra that satisfies the property* (isot).

Proof. Let us first show that the quotient is an implicative algebra. Since equations are preserved under quotients we know that $\boldsymbol{A}/\theta$ satisfies (i1) and (pre) and we need only check that $\boldsymbol{A}/\theta$ satisfies the quasi-equations (i2) and (trans). Lemma 1.6.1 easily implies that $\boldsymbol{A}/\theta$ satisfies (i2). We now see that (trans) is also satisfied. Take any $a, b, c \in A$ such that $\langle a \to b, 1\rangle \in \theta$ and $\langle b \to c, 1\rangle \in \theta$ and let us show that $\langle a \to c, 1\rangle \in \theta$. From the fact that $\langle a \to b, 1\rangle \in \theta$ and Lemma 1.6.2 we infer that there is some $b' \in A$ such that $\langle b, b'\rangle \in \theta$ and $a \leq b'$. Hence, $\langle b' \to c, 1\rangle \in \theta$; and using Lemma 1.6.2 again we obtain an element $c' \in A$ such that $\langle c, c'\rangle \in \theta$ and $b' \leq c'$. From $a \leq b' \leq c'$ we infer that $a \to c' = 1$. Since $\langle c, c'\rangle \in \theta$, from the previous fact we obtain that $\langle a \to c, 1\rangle = \langle a \to c, a \to c'\rangle \in \theta$. This completes the proof of (trans). Now to prove (isot), let $a, b, c \in A$ be such that $\langle a \to b, 1\rangle \in \theta$; let us show both that $\big\langle (c \to a) \to (c \to b), 1\big\rangle \in \theta$ and that $\big\langle (b \to c) \to (a \to c), 1\big\rangle \in \theta$. From the assumption and Lemma 1.6.2 we know there is some $b' \in A$ such that $\langle b, b'\rangle \in \theta$ and $a \leq b'$. Hence, $(c \to a) \to (c \to b') = 1$ and $(b' \to c) \to (a \to c) = 1$. From this and the fact that $\langle b, b'\rangle \in \theta$ we conclude both that $\big\langle (c \to a) \to (c \to b), 1\big\rangle \in \theta$ and that $\big\langle (b \to c) \to (a \to c), 1\big\rangle \in \theta$. □

Hilbert and quasi-Hilbert algebras play in this paper a twofold role: as models of certain sentential logics and as models of certain Gentzen systems. While for the first function the algebraic theory is enough, the second one is exerted through several inductive closure systems that can be found in any algebra of the similarity type:

Definition 1.9 Let $\boldsymbol{A} = \langle A, \to\rangle$ be an $\mathcal{L}$-algebra. A subset $F \subseteq A$ is *closed under Modus Ponens* (MP) when, for all $a, b \in A$, if $a \in F$ and $a \to b \in F$, then $b \in F$; and it is an *implicative filter* when it is closed under MP and for each $a \in A$, $a \to a \in F$. The family of all implicative filters of $\boldsymbol{A}$ will be denoted by $\mathcal{F}(\boldsymbol{A})$. Trivially, $\mathcal{F}(\boldsymbol{A})$ is an inductive closure system, and for each $X \subseteq A$, $\mathrm{F}(X)$ denotes the implicative filter generated by X, that is, the least implicative filter containing X. An implicative filter is said to be n*-generated*, for $n \in \omega$, when it is of the form $\mathrm{F}(X)$ for some $X \subseteq A$ with $|X| \leq n$; and it is *finitely generated* when it is n-generated for some $n \in \omega$.

Thus, implicative filters in implicative algebras are the subsets closed under MP and containing 1. Recall that in a partially ordered set $\langle A, \leq\rangle$ a subset $F \subseteq A$ is an *order filter* when $F \neq \emptyset$ and if $a \in F$ and $a \leq b$, then $b \in F$; if A has a maximum element 1, then this implies that $1 \in F$. If $\boldsymbol{A} \in$ **IA**, then trivially every implicative filter of $\boldsymbol{A}$ is an order filter, and $\{1\}$ is the least implicative filter of $\boldsymbol{A}$. If $\boldsymbol{A}$ is the Hilbert algebra canonically associated with an ordering relation $\leq$ with maximum, then the implicative filters of $\boldsymbol{A}$ coincide with the order filters of $\leq$. In quasi-Hilbert algebras the 2-generated implicative filters can be neatly described:

Lemma 1.10 *If $\boldsymbol{A} \in$ **QHA** and $a, b \in A$, then*

$$\mathrm{F}(a, b) = \{c \in A : a \to (b \to c) = 1\} \;\; \textit{and} \;\; \mathrm{F}(a) = \{c \in A : a \to c = 1\}.$$

Proof. Put $F = \{c \in A : a \to (b \to c) = 1\}$. By (k) and (pre) we see that $a, b, 1 \in F$. Moreover, F is closed under MP because of (mp2). Finally, if $F' \in \mathcal{F}(\boldsymbol{A})$ and $a, b \in F'$, then applying closure under MP twice we get that $c \in F'$ for every $c \in F$. This shows that F is the least implicative filter containing a and b. When $a = b$ we obtain the second expression using (contrac). □

In the next section we see that this property characterizes in some sense the quasi-Hilbert algebras, and that it is not possible to extend this way of generating implicative filters beyond two generators. By contrast, it is well known that in Hilbert algebras it extends to an arbitrary number of generators:

Lemma 1.11 *If $\boldsymbol{A} \in$ **HA** and $X \subseteq A$, then*

$$\mathrm{F}(X) = \{c \in A : a_1 \to \big(a_2 \to \cdots \to (a_n \to c)...\big) = 1 \;\; \textit{for some}\; a_1, \ldots, a_n \in X\}. \qquad \square$$

The following rather general fact about finitely generated closed sets will be used:

Lemma 1.12 *Let $\mathcal{C}$ and $\mathcal{C}'$ be two closure systems on the same set A, with associated closure operators* C *and* C'*, respectively, such that $\mathcal{C} \subseteq \mathcal{C}'$. Let $n \in \omega$ be such that $\mathcal{C}$ contains all n-generated members of $\mathcal{C}'$. Then for all $a_1, \ldots, a_n \in A$, $\mathrm{C}(a_1, \ldots, a_n) = \mathrm{C}'(a_1, \ldots, a_n)$.* □

2 The Gentzen systems and their models

The Gentzen systems we are going to consider are defined by calculi based on one or more of the following Gentzen-style rules, where $n \geq 0$ (recall that for $n = 0$ the notation $\varphi_1, \ldots, \varphi_n$ stands for the empty sequence).

(DTn) $$\frac{\varphi_1, \ldots, \varphi_n, \varphi \rhd \psi}{\varphi_1, \ldots, \varphi_n \rhd \varphi \to \psi},$$

(CONG) $$\frac{\varphi \rhd \psi \qquad \alpha \rhd \beta}{\psi \to \alpha \rhd \varphi \to \beta},$$

(weakMP2) $$\frac{\emptyset \rhd \varphi_1 \to (\varphi_2 \to \varphi) \qquad \emptyset \rhd \varphi_1 \to \big(\varphi_2 \to (\varphi \to \psi)\big)}{\emptyset \rhd \varphi_1 \to (\varphi_2 \to \psi)}.$$

Axioms of Gentzen calculi (also called *initial sequents*) are simply rules with an empty set of premises, that is, sequents. Hence, a Hilbert-style rule of the form $\varphi_1, \ldots, \varphi_n \vdash \psi$ can be considered as a Gentzen-style axiom if viewed as the sequent $\varphi_1, \ldots, \varphi_n \rhd \psi$. This applies even to a Hilbert-style axiom (i. e., a single formula φ), considered again as a rule without premises; the corresponding sequent would have an empty left side but can anyway be viewed as the Gentzen-style axiom $\emptyset \rhd \varphi$. We use the same name for each under the different views; hopefully the context will prevent any confusion. In this and the next section we consider the following Hilbert-style axioms and rules.

(I) $\varphi \to \varphi$,

(PRE) $\varphi \vdash \psi \to \varphi$,

(K) $\varphi \to (\psi \to \varphi)$,

(FRE) $\big(\varphi \to (\psi \to \xi)\big) \to \big((\varphi \to \psi) \to (\varphi \to \xi)\big)$,

(MP) $\varphi, \varphi \to \psi \vdash \psi$,

and, for each $n \in \omega$, $n \geq 1$,

(MPn) $$\varphi_1 \to \big(\ldots \to (\varphi_n \to \varphi)\ldots\big),\ \varphi_1 \to \big(\ldots \to \big(\varphi_n \to (\varphi \to \psi)\big)\ldots\big) \vdash \varphi_1 \to \big(\ldots \to (\varphi_n \to \psi)\ldots\big).$$

Note that actually (MP) corresponds to what could be named (MP0); modulo all the structural rules (which we will have) this rule can be equally presented as any one of the following Gentzen-style rules,

$$\frac{\Gamma \rhd \varphi \quad \Gamma \rhd \varphi \to \psi}{\Gamma \rhd \psi} \qquad \text{or} \qquad \frac{\Gamma \rhd \varphi \quad \Gamma, \psi \rhd \xi}{\Gamma, \varphi \to \psi \rhd \xi}.$$

In the sequel we will refer to any of them simply as (MP).

There is still another way of turning a Hilbert-style rule (R) $\varphi_1, \ldots, \varphi_n \vdash \psi$ into a Gentzen-style one, which we call the weak one: the rule

(weakR) $$\frac{\emptyset \rhd \varphi_1 \quad \ldots \quad \emptyset \rhd \varphi_n}{\emptyset \rhd \psi}.$$

For instance, the above Gentzen-style rule (weakMP2) has been obtained from the Hilbert-style rule (MP2) in this way.

Definition 2.1 For each $n \in \omega$ the Gentzen system $\mathfrak{G}_n$ is the one defined by all structural rules and the rules (MP) and (DTn). The Gentzen system $\mathfrak{G}_\omega$ is defined by all the structural rules, the rule (MP) and the rules (DTn) for all $n \in \omega$.

Let us write $\mathfrak{G} \leq \mathfrak{G}'$ to mean that the Gentzen system $\mathfrak{G}'$ is *stronger* than $\mathfrak{G}$, in the sense that every derivation in $\mathfrak{G}$ is also a derivation in $\mathfrak{G}'$. Notice that, since we have the structural rules of weakening and contraction, the rule (DT$n+1$) actually encompasses the rule (DTn); hence we have

(3) $$\mathfrak{G}_{\boldsymbol{0}} \leq \mathfrak{G}_{\boldsymbol{1}} \leq \cdots \leq \mathfrak{G}_{\boldsymbol{n}} \leq \mathfrak{G}_{\boldsymbol{n+1}} \leq \cdots \leq \mathfrak{G}_{\boldsymbol{\omega}}.$$

That all these Gentzen systems are different, that is, that all inequalities are strict, will be shown in Theorem 2.8 with the help of the algebraic models of these Gentzen systems. An exception will be the first inequality, because we are not going to study $\mathfrak{G}_{\boldsymbol{0}}$ in this paper: it is too weak to allow for a uniform treatment along with the rest of the family (anyway, it is true that it is different from $\mathfrak{G}_{\boldsymbol{1}}$). Observe that having all the rules (DTn) together can be condensed in the usual *rule scheme* for right-introduction of implication

$$\frac{\mathit{\Gamma}, \varphi \rhd \psi}{\mathit{\Gamma} \rhd \varphi \to \psi},$$

where $\mathit{\Gamma}$ is a meta-variable ranging over finite sequences of arbitrary length (including 0). Thus actually $\mathfrak{G}_{\boldsymbol{\omega}}$ is the well-known structural system defined by (MP) and (DT).

Lemma 2.2 *The Gentzen system $\mathfrak{G}_{\boldsymbol{1}}$, and hence all the $\mathfrak{G}_{\boldsymbol{n}}$ except possibly $\mathfrak{G}_{\boldsymbol{0}}$, has the following derivable sequents:* (I), (PRE), (K)*, and the derivable rule* (CONG)*.*

P r o o f. (I) results from applying (DT0) to the structural axiom $\varphi \rhd \varphi$. (PRE) is derived by (DT1) from the sequent $\varphi, \psi \rhd \varphi$, which is derived from the axiom by weakening. Then applying (DT0) to (PRE) we obtain (K). Finally (CONG) is derived in the following way.

$$\dfrac{\dfrac{\dfrac{\dfrac{\varphi \rhd \psi}{\varphi, \psi \to \alpha \rhd \psi}\,(\mathrm{w}) \qquad \dfrac{\dfrac{}{\psi \to \alpha \rhd \psi \to \alpha}\,(\mathrm{ax})}{\varphi, \psi \to \alpha \rhd \psi \to \alpha}\,(\mathrm{w})}{\varphi, \psi \to \alpha \rhd \alpha}\,(\mathrm{MP}) \qquad \alpha \rhd \beta}{\varphi, \psi \to \alpha \rhd \beta}\,(\mathrm{cut})}{\psi \to \alpha \rhd \varphi \to \beta}\,(\mathrm{DT1})$$

□

In general, algebraic models of structural Gentzen systems take the form of *generalized matrices*, structures $\langle \boldsymbol{A}, \mathrm{C} \rangle$, where $\boldsymbol{A}$ is an algebra of the relevant similarity type (here $\langle \to \rangle$) and $\mathrm{C} : \mathcal{P}(A) \longrightarrow \mathcal{P}(A)$ is a Þ*nitary closure operator* on A, the universe of $\boldsymbol{A}$. Such a structure is a *model of a rule*

(4) $$\frac{\{\, \mathit{\Gamma}_i \rhd \varphi_i : i < n \,\}}{\mathit{\Gamma} \rhd \varphi}$$

whenever, for each $h \in \mathrm{Hom}(\boldsymbol{Fm}, \boldsymbol{A})$, if for all $i < n$, $h(\varphi_i) \in \mathrm{C}\big(h(\mathit{\Gamma}_i)\big)$, then also $h(\varphi) \in \mathrm{C}\big(h(\mathit{\Gamma})\big)$. A generalized matrix is a *model of a Gentzen system* when it is a model of all its derivable rules; for this it is enough that it is a model of the rules of any calculus defining the system; note that by definition generalized matrices are always models of all the structural rules.

Being a model of a particular rule is usually more transparently expressed without using interpretations but arbitrary elements of the algebra. For instance, $\langle \boldsymbol{A}, \mathrm{C} \rangle$ is a model of (DT1) if and only if for all $a, b, c \in A$, if $c \in \mathrm{C}(a, b)$, then $b \to c \in \mathrm{C}(a)$.

There is a duality between finitary closure operators and *inductive closure systems* (i. e., families of subsets closed under arbitrary intersections and under unions of directed subfamilies). More precisely, if C is a finitary closure operator, then the family of its *closed sets* $\mathcal{C} = \{F \subseteq A : \mathrm{C}(F) = F\}$ is an inductive closure system, and conversely if $\mathcal{C}$ is an inductive closure system, then the mapping on $\mathcal{P}(A)$ defined, for $X \subseteq A$, by $\mathrm{C}(X) = \bigcap \{F \in \mathcal{C} : X \subseteq F\}$ is a finitary closure operator. These correspondences are inverse to one another. Then, generalized matrices can also be presented as structures $\langle \boldsymbol{A}, \mathcal{C} \rangle$, where $\mathcal{C} \subseteq \mathcal{P}(A)$ is an inductive closure system. Some properties of generalized matrices are better formulated in terms of closure operators and some in terms of closure systems; thus this duality of presentation and notation is very convenient and will not lead to any misunderstanding.

Notice that if $\langle \boldsymbol{A},\ \mathrm{C}\rangle$ is a model of a Gentzen-style rule that is in fact a Hilbert-style rule, then each closed set F of $\mathcal{C}$ is closed under the rule in the usual way, that is, the ordinary matrix $\langle \boldsymbol{A},\ F\rangle$ is a matrix model of the rule. For instance, comparing with Definition 1.9 we see that implicative filters are subsets closed under (I) and (MP), and the matrices they determine are the models of these rules.

The *Frege relation* of a closure operator C on a set A is the relation

$$\boldsymbol{\Lambda}(\mathrm{C}) = \big\{\langle a, b\rangle \in A^2 : \mathrm{C}(a) = \mathrm{C}(b)\big\}\,.$$

This is always an equivalence relation, but if A is the universe of an algebra, then $\boldsymbol{\Lambda}(\mathrm{C})$ need not be a congruence; the largest congruence of $\boldsymbol{A}$ contained in it is called the *Tarski congruence* of the generalized matrix $\langle \boldsymbol{A},\ \mathrm{C}\rangle$ and is denoted by $\widetilde{\boldsymbol{\Omega}}_{\boldsymbol{A}}(\mathrm{C})$, or by $\widetilde{\boldsymbol{\Omega}}_{\boldsymbol{A}}(\mathcal{C})$ if the closure system presentation is preferred. A generalized matrix is *reduced* when its Tarski congruence is the identity. The algebraic counterpart of a Gentzen system is taken to be the class of the algebraic reducts of all its reduced models. For more details on these notions see [10].

Lemma 2.3 *Let $\langle \boldsymbol{A}, \mathrm{C}\rangle$ be a model of* (CONG). *Then its Frege relation $\boldsymbol{\Lambda}(\mathrm{C})$ is a congruence of $\boldsymbol{A}$, hence $\boldsymbol{\Lambda}(\mathrm{C}) = \widetilde{\boldsymbol{\Omega}}_{\boldsymbol{A}}(\mathrm{C})$, and $\langle \boldsymbol{A}, \mathrm{C}\rangle$ is reduced if and only if it satisfies*

(5) *for all $a,\ b \in A$, if $\mathrm{C}(a) = \mathrm{C}(b)$, then $a = b$.*

Proof. Assume $\langle a, b\rangle,\ \langle c, d\rangle \in \boldsymbol{\Lambda}(\mathrm{C})$, that is, $\mathrm{C}(a) = \mathrm{C}(b)$ and $\mathrm{C}(c) = \mathrm{C}(d)$, or equivalently $a \in \mathrm{C}(b)$, $b \in \mathrm{C}(a)$, $c \in \mathrm{C}(d)$, and $d \in \mathrm{C}(c)$. Then by (CONG), these imply that both $a \to c \in \mathrm{C}(b \to d)$ and $b \to d \in \mathrm{C}(a \to b)$, that is, $\langle a \to c, b \to d\rangle \in \boldsymbol{\Lambda}(\mathrm{C})$. This shows that $\boldsymbol{\Lambda}(\mathrm{C})$ is a congruence and hence that $\boldsymbol{\Lambda}(\mathrm{C}) = \widetilde{\boldsymbol{\Omega}}_{\boldsymbol{A}}(\mathrm{C})$. A generalized matrix is reduced when $\widetilde{\boldsymbol{\Omega}}_{\boldsymbol{A}}(\mathrm{C})$ is the identity, hence in this case this amounts to saying that $\boldsymbol{\Lambda}(\mathrm{C})$ is the identity, and this is what (5) expresses. □

By Lemma 2.2, all models of all the Gentzen systems under consideration, except possibly of $\mathfrak{G}_{\mathbf{0}}$, have the properties of Lemma 2.3. This makes working with such models very smooth; this is why the case of $\mathfrak{G}_{\mathbf{0}}$ cannot be treated uniformly with the other cases, and is left out from this paper. Next we characterize the reduced models of all our Gentzen systems in terms of the corresponding closure systems. We begin with the case $n = 1$.

Theorem 2.4 *Let $\langle \boldsymbol{A}, \mathcal{C}\rangle$ be a generalized matrix. Then $\langle \boldsymbol{A}, \mathcal{C}\rangle$ is a reduced model of $\mathfrak{G}_{\mathbf{1}}$ if and only if $\boldsymbol{A}$ is a quasi-Hilbert algebra and $\mathcal{C}$ is a family of implicative filters containing all the 2-generated implicative filters of $\boldsymbol{A}$.*

Proof.

($\Rightarrow$). Let $\langle \boldsymbol{A}, \mathcal{C}\rangle$ be a reduced model of $\mathfrak{G}_{\mathbf{1}}$, with associated closure operator C. Then by assumption $\langle \boldsymbol{A}, \mathrm{C}\rangle$ is a model of (DT0), (DT1), (MP) and the derived rules of Lemma 2.2. Moreover it is reduced, so (5) holds by Lemma 2.3. Now by (I) $a \to a \in \mathrm{C}(\emptyset)$ for all $a \in A$. This implies that $\mathrm{C}(a \to a) = \mathrm{C}(b \to b) = \mathrm{C}(\emptyset)$ for all $a,\ b \in A$, and hence that $a \to a = b \to b$. Hence $\boldsymbol{A}$ satisfies (i1) and we can put $1 = a \to a$. Now $\mathrm{C}(\emptyset) = \{1\}$: If $a \in \mathrm{C}(\emptyset)$, then since $1 \in \mathrm{C}(\emptyset)$, also $\mathrm{C}(a) = \mathrm{C}(1) = \mathrm{C}(\emptyset)$ and so $a = 1$. In particular $1 \in F$ for all $F \in \mathcal{C}$. Being a model of (MP) amounts to saying that each $F \in \mathcal{C}$ is closed under MP in the sense of Definition 1.9, and thus is an implicative filter.

Now we show that $\boldsymbol{A} \in \mathsf{QHA}$. (i1) has already been proved. To prove (i2), assume $a \to b = b \to a = 1$. This means $a \to b,\ b \to a \in \mathrm{C}(\emptyset)$, and by (DT0) we obtain $a \in \mathrm{C}(b)$ and $b \in \mathrm{C}(a)$, that is, $\mathrm{C}(a) = \mathrm{C}(b)$ which implies $a = b$. (k) follows from (K). To prove (mp2) assume $a \to (b \to c) = 1$ and $a \to \big(b \to (c \to d)\big) = 1$. Since $\mathrm{C}(\emptyset) = \{1\}$, after using (MP) several times we obtain $d \in \mathrm{C}(a, b)$, and now by (DT1) $a \to (b \to d) \in \mathrm{C}(\emptyset)$, that is, $a \to (b \to d) = 1$. This completes the proof of (mp2) and hence that $\boldsymbol{A} \in \mathsf{QHA}$.

Finally we show that $\mathcal{C}$ contains all 2-generated implicative filters of $\boldsymbol{A}$, that is, that for all $a,\ b \in A$, $\mathrm{C}\big(\mathrm{F}(a, b)\big) \subseteq \mathrm{F}(a, b)$. Since we have already seen that $\mathcal{C} \subseteq \mathcal{F}(\boldsymbol{A})$, $\mathrm{F}(a, b) \subseteq \mathrm{C}(a, b)$, and therefore $\mathrm{C}\big(\mathrm{F}(a, b)\big) \subseteq \mathrm{C}(a, b)$. Now if $c \in \mathrm{C}\big(\mathrm{F}(a, b)\big)$, then $c \in \mathrm{C}(a, b)$ so by (DT1) and (DT0) $a \to (b \to c) \in \mathrm{C}(\emptyset)$, that is, $a \to (b \to c) = 1$. But by Lemma 1.10 this implies that $c \in \mathrm{F}(a, b)$. This completes the first half of the proof.

($\Leftarrow$). Let $\boldsymbol{A} \in \mathsf{QHA}$, and let $\mathcal{C} \subseteq \mathcal{F}(\boldsymbol{A})$ be an inductive closure system containing all 2-generated members of $\mathcal{F}(\boldsymbol{A})$. By assumption all $F \in \mathcal{C}$ are closed under MP and contain 1, hence $\langle \boldsymbol{A}, \mathrm{C}\rangle$ is a model of (MP). Note that by Lemma 1.12 $\mathrm{C}(a,b) = \mathrm{F}(a,b)$ for all $a, b \in A$. Then $1 \in \mathrm{C}(\emptyset) \subseteq \mathrm{C}(1) = \mathrm{F}(1) = \{1\}$ so $\mathrm{C}(\emptyset) = \{1\}$. Let us show that $\langle \boldsymbol{A}, \mathrm{C}\rangle$ is a model of (DT1): If $c \in \mathrm{C}(a,b) = \mathrm{F}(a,b)$, by Lemma 1.10 $a \to (b \to c) = 1$, that is, $a \to (b \to c) \in \mathrm{C}(\emptyset)$, and by (MP) $b \to c \in \mathrm{C}(a)$; this shows (DT1). Hence $\langle \boldsymbol{A}, \mathrm{C}\rangle$ is a model of $\mathfrak{G}_{\mathbf{1}}$ and hence of (CONG), thus in order to show that it is reduced, by Lemma 2.3 it is enough to show (5): If $\mathrm{C}(a) = \mathrm{C}(b)$, then $b \in \mathrm{C}(a)$ so by (DT0) $a \to b \in \mathrm{C}(\emptyset)$, that is, $a \to b = 1$, and similarly $b \to a = 1$. Since $\boldsymbol{A} \in \mathsf{QHA}$ we now use (i2) to obtain $a = b$. This shows that $\langle \boldsymbol{A}, \mathrm{C}\rangle$ is a reduced model of $\mathfrak{G}_{\mathbf{1}}$. □

Note that in particular for any $\boldsymbol{A} \in \mathsf{QHA}$, the generalized matrix $\big\langle \boldsymbol{A}, \mathcal{F}(\boldsymbol{A})\big\rangle$ is a reduced model of $\mathfrak{G}_{\mathbf{1}}$. However, unlike in other similar situations, this property does not characterize it uniquely, as the examples included in the proof of the last part of Theorem 3.5 show.

The preceding result describes in algebraic form the reduced models of the Gentzen calculus $\mathfrak{G}_{\mathbf{1}}$. As follows from standard results of the general theory, $\langle \boldsymbol{A}, \mathcal{C}\rangle$ is a model of $\mathfrak{G}_{\mathbf{1}}$ if and only if there is a strict surjective homomorphism from $\langle \boldsymbol{A}, \mathcal{C}\rangle$ onto a generalized matrix $\langle \boldsymbol{B}, \mathcal{D}\rangle$ where $\boldsymbol{B} \in \mathsf{QHA}$ and $\mathcal{D} \subseteq \mathcal{F}(\boldsymbol{B})$ and contains all 2-generated members of $\mathcal{F}(\boldsymbol{B})$. This does not add any essentially new information and we will not repeat characterizations of this kind.

Next we characterize the reduced models of the Gentzen systems $\mathfrak{G}_{\boldsymbol{n}}$ for $n > 1$. However, we do not obtain different classes of algebras for each different Gentzen system in the family (that they are indeed different will be shown soon): as the next result shows, if we have at least (DT2), then we obtain a Hilbert algebra, and the difference lies in the family of closed sets of the model.

Theorem 2.5 *Let $n \geq 2$. A generalized matrix $\langle \boldsymbol{A}, \mathcal{C}\rangle$ is a reduced model of the Gentzen system $\mathfrak{G}_{\boldsymbol{n}}$ if and only if $\boldsymbol{A}$ is a Hilbert algebra and $\mathcal{C}$ is a family of implicative filters of $\boldsymbol{A}$ containing all the $(n+1)$-generated filters of A.*

P r o o f.
($\Rightarrow$). Since $\mathfrak{G}_{\mathbf{1}} \leq \mathfrak{G}_{\boldsymbol{n}}$, we can continue the proof in the first part of Theorem 2.4. We know that $\boldsymbol{A} \in \mathsf{QHA}$ and $\mathcal{C} \subseteq \mathcal{F}(\boldsymbol{A})$, so to show that $\boldsymbol{A} \in \mathsf{HA}$ we need only show (fre): By (MP) $c \in \mathrm{C}\big(a \to (b \to c),\ a \to b,\ a\big)$, and by (DT2) and using that $\mathrm{C}(\emptyset) = \{1\}$ we obtain that $\big(a \to (b \to c)\big) \to \big((a \to b) \to (a \to c)\big) = 1$. It only remains to show that $\mathrm{F}(a_0, \dots, a_n) \in \mathcal{C}$, and we proceed as in the proof of Theorem 2.4: If $c \in \mathrm{C}\big(\mathrm{F}(a_0, \dots, a_n)\big)$, then also $c \in \mathrm{C}(a_0, \dots, a_n)$ and by using (DT0) – (DTn) and $\mathrm{C}(\emptyset) = \{1\}$ we get $a_0 \to \big(\cdots \to (a_n \to c)...\big) = 1$, which by Lemma 1.11 implies that $c \in \mathrm{F}(a_0, \dots, a_n)$.

($\Leftarrow$). Since $\mathsf{HA} \subseteq \mathsf{QHA}$ we can use Theorem 2.4 to conclude that $\langle \boldsymbol{A}, \mathcal{C}\rangle$ is a reduced model of $\mathfrak{G}_{\mathbf{1}}$. It remains to show that the assumption here implies that the generalized matrix is a model of (DTn): Assume that $c \in \mathrm{C}(a_0, \dots, a_n)$. By Lemma 1.12, $c \in \mathrm{F}(a_0, \dots, a_n)$ and since $\boldsymbol{A}$ is a Hilbert algebra we can use Lemma 1.11, (pre), (q-cp) and (contrac) and obtain that precisely $a_0 \to \big(\cdots \to (a_n \to c)...\big) = 1$. As in the proof of Theorem 2.4 we know that $\mathrm{C}(\emptyset) = \{1\}$, so actually $a_0 \to \big(\cdots \to (a_n \to c)...\big) \in \mathrm{C}(\emptyset)$, and then by (MP) n times we conclude that $a_n \to c \in \mathrm{C}(a_0, \dots a_{n-1})$. □

If we put together the previous result for all values of n (or, by (3), for an infinite number of them), we will obtain families of implicative filters containing all the finitely-generated ones; but only the family of all of them can satisfy this, because it has to be inductive. Thus we have:

Theorem 2.6 *A generalized matrix $\langle \boldsymbol{A}, \mathcal{C}\rangle$ is a reduced model of the Gentzen system $\mathfrak{G}_{\boldsymbol{\omega}}$ if and only if $\boldsymbol{A} \in \mathsf{HA}$ and $\mathcal{C}$ is the family $\mathcal{F}(\boldsymbol{A})$ of all implicative filters of $\boldsymbol{A}$.* □

These results give what one can consider, in some sense, a proof-theoretic characterization of the relevant classes of algebras:

Corollary 2.7 *Let $\boldsymbol{A}$ be an $\mathcal{L}$-algebra and let $n \geq 2$. Then:*

1. *$\boldsymbol{A} \in \mathsf{QHA}$ if and only if there is an inductive closure system $\mathcal{C}$ on A such that the generalized matrix $\langle \boldsymbol{A}, \mathcal{C}\rangle$ is a reduced model of the Gentzen system $\mathfrak{G}_{\mathbf{1}}$.*

2. *$\boldsymbol{A} \in \mathsf{HA}$ if and only if there is an inductive closure system $\mathcal{C}$ on A such that the generalized matrix $\langle \boldsymbol{A}, \mathcal{C}\rangle$ is a reduced model of the Gentzen system $\mathfrak{G}_{\boldsymbol{n}}$.*

3. *$\boldsymbol{A} \in \mathsf{HA}$ if and only if there is an inductive closure system $\mathcal{C}$ on A such that the generalized matrix $\langle \boldsymbol{A}, \mathcal{C}\rangle$ is a reduced model of the Gentzen system $\mathfrak{G}_{\boldsymbol{\omega}}$.* □

Another consequence of the previous theorems, in Abstract Algebraic Logic terms, is that these algebras are the canonical algebraic counterparts of these Gentzen systems:

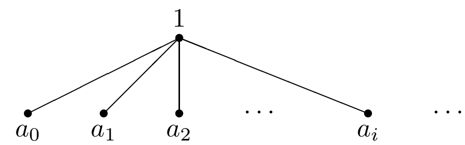
$$\mathsf{Alg}\,\mathfrak{G}_{\boldsymbol{1}} = \mathsf{QHA}, \qquad \mathsf{Alg}\,\mathfrak{G}_{\boldsymbol{n}} = \mathsf{Alg}\,\mathfrak{G}_{\boldsymbol{\omega}} = \mathsf{HA} \quad (n \geq 2).$$

With the characterization of the reduced models of each of the Gentzen systems it is now easy to show that they are all different:

Theorem 2.8 *The Gentzen systems in Definition* 1.2 *are all different, and thus*

$$\mathfrak{G}_{\boldsymbol{1}} < \mathfrak{G}_{\boldsymbol{2}} < \cdots < \mathfrak{G}_{\boldsymbol{n}} < \mathfrak{G}_{\boldsymbol{n+1}} < \cdots < \mathfrak{G}_{\boldsymbol{\omega}}\,.$$

Proof. It is enough to show that the Gentzen systems have different models. All counterexamples can be found on a single algebra, a denumerable Hilbert algebra indeed: Let $A_\omega = \{1\} \cup \{a_i : i \in \omega\}$ be an ordered set, where the a_i's form an antichain, and 1 is the maximum:



The canonical Hilbert algebra associated with this order is given by the operation $\to$ defined as in (2), that is, $a_i \to a_i = 1 \to 1 = a_i \to 1 = 1$, $1 \to a_i = a_i$, and $a_i \to a_j = a_j$ if $i \neq j$. Thus this makes $\boldsymbol{A}_\omega = \langle A_\omega, \to \rangle$ a Hilbert algebra (and a fortiori a quasi-Hilbert algebra). The implicative filters coincide with the order filters, which in this case are all subsets containing 1. For each $n \geq 1$, denote by $\mathcal{F}_n$ the family of all the n-generated filters: they are all subsets containing 1 and at most n of the a_i's. Each of these families is clearly closed under intersections, hence if we add the total set A_ω to it, we obtain a closure system, which is trivially closed under unions of directed subfamilies (observe that such a subfamily must necessarily be finite and have a maximum, because of the bound in the number of elements of the filters, and hence this maximum will be the union of the family). Obviously for each $n \geq 1$, $\mathcal{F}_{n+1}$ does not contain all the $(n+2)$-generated filters, hence Theorems 2.4 and 2.5 tell us that the generalized matrix $\langle \boldsymbol{A}_\omega, \mathcal{F}_{n+1} \cup \{A_\omega\}\rangle$ is a reduced model of $\mathfrak{G}_{\boldsymbol{n}}$, but is not a model of $\mathfrak{G}_{\boldsymbol{n+1}}$. This also implies that neither of these generalized matrices is a model of $\mathfrak{G}_{\boldsymbol{\omega}}$. □

Thus we have given an algebraic proof of what is essentially a proof-theoretic fact:

Corollary 2.9 *For each* $n \geq 1$*, the rule* (DT$n+1$) *is not derivable in the Gentzen system* $\mathfrak{G}_{\boldsymbol{n}}$*, that is, with the rules* (MP) *and* (DTn)*, together with all structural rules.* □

However, as we show in the next section, for $n \geq 2$ the rule (DTn) will be admissible, while it will not for $n = 1$; see Corollaries 3.2 and 3.13. Actually, we will see that all these Gentzen systems, except $\mathfrak{G}_{\boldsymbol{1}}$, have the same derivable sequents and thus define the same sentential logic, the implicative fragment of intuitionistic propositional logic.

3 The sentential logics

In this section we analyse several sentential logics related to the Gentzen systems considered so far: Some are related to the establishment of restrictions in the set of side formulas in the Gentzen-style rule (DT), and some to the introduction of additional assumptions in the formulas involved in the rule (MP), which means making it actually stronger. Let us begin with those which are but alternative presentations of positive implicative logic.

3.1 Connections with positive implicative logic

We denote by $\mathcal{IPL}_{\to}$ the $\langle\to\rangle$-fragment of intuitionistic propositional logic, i. e., Hilbert's positive implicative logic. As is well-known, this sentential logic can be defined by the Hilbert-style calculus with axioms (K) and (FRE) and the rule (MP), and it can also be defined through the Gentzen system we have called $\mathfrak{G}_{\boldsymbol{\omega}}$. More precisely, if $\boldsymbol{\mathcal{S}}$ is a sentential logic and $\mathfrak{G}$ is a Gentzen system, we say that $\mathfrak{G}$ *defines* $\boldsymbol{\mathcal{S}}$ or that it is *adequate* for

$\boldsymbol{\mathcal{S}}$ when $\varphi_1, \dots, \varphi_n \vdash_{\boldsymbol{\mathcal{S}}} \varphi$ if and only if the sequent $\varphi_1, \dots, \varphi_n \rhd \varphi$ is a derivable sequent of $\mathfrak{G}$. We say that a sentential logic $\boldsymbol{\mathcal{S}}$ *satisfies a Gentzen-style rule* (4) when for all substitutions σ, if $\sigma[\Gamma_i] \vdash_{\boldsymbol{\mathcal{S}}} \sigma(\varphi_i)$ for all $i < n$, then $\sigma[\Gamma] \vdash_{\boldsymbol{\mathcal{S}}} \sigma(\varphi)$; in practice however, particular rules are formulated with arbitrary formulas, that is, as rule schemes with meta-variables, and then it is not necessary to deal with substitutions.

Theorem 3.1 *The Gentzen system* $\mathfrak{G}_{\boldsymbol{\omega}}$ *and all the Gentzen systems* $\mathfrak{G}_{\boldsymbol{n}}$ *for* $n \geq 2$ *are adequate for the sentential logic* $\boldsymbol{\mathcal{IPL}}_{\to}$.

P r o o f. That $\mathfrak{G}_{\boldsymbol{\omega}}$ is adequate for $\boldsymbol{\mathcal{IPL}}_{\to}$ is well-known. By (3), it will be enough to show that $\mathfrak{G}_{\boldsymbol{2}}$ is adequate for $\boldsymbol{\mathcal{IPL}}_{\to}$. If $\varphi_1, \dots, \varphi_n \rhd \varphi$ is derivable in $\mathfrak{G}_{\boldsymbol{2}}$, then it is so in $\mathfrak{G}_{\boldsymbol{\omega}}$, and hence $\varphi_1, \dots, \varphi_n \vdash_{\boldsymbol{\mathcal{IPL}}_{\to}} \varphi$. To prove the converse, it is enough to show that all Hilbert-style axioms and rules of a presentation of $\boldsymbol{\mathcal{IPL}}_{\to}$ correspond to derivable sequents of $\mathfrak{G}_{\boldsymbol{2}}$. (MP) does so by definition, and (K) does so by Lemma 2.2, so only the case of (FRE) has to be shown. In the proof below we let Δ stand for the sequence $\varphi,\ \varphi \to \psi,\ \varphi \to (\psi \to \xi)$.

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\varphi,\ \varphi\to\psi \rhd \psi}\,(\mathrm{MP})}{\Delta \rhd \psi}\,(\mathrm{w}) \qquad \dfrac{\dfrac{\dfrac{}{\varphi\rhd\varphi}\,(\mathrm{ax})}{\Delta\rhd\varphi}\,(\mathrm{w}) \quad \dfrac{\dfrac{}{\varphi\to(\psi\to\xi)\rhd\varphi\to(\psi\to\xi)}\,(\mathrm{ax})}{\Delta\rhd\varphi\to(\psi\to\xi)}\,(\mathrm{w})}{\Delta\rhd\psi\to\xi}\,(\mathrm{MP})}{\Delta\rhd\xi}\,(\mathrm{MP})}{\varphi\to\psi,\ \varphi\to(\psi\to\xi)\rhd\varphi\to\xi}\,(\mathrm{DT2})}{\varphi\to(\psi\to\xi)\rhd(\varphi\to\psi)\to(\varphi\to\xi)}\,(\mathrm{DT1})}{\emptyset\rhd\big(\varphi\to(\psi\to\xi)\big)\to\big((\varphi\to\psi)\to(\varphi\to\xi)\big)}\,(\mathrm{DT0})
$$

□

This means that all these Gentzen systems share the same derivable sequents, and that they correspond to the entailments of $\boldsymbol{\mathcal{IPL}}_{\to}$. Since this logic satisfies (DT) and all (DTm), from this and Corollary 2.9 we have:

Corollary 3.2 *Let* $n \geq 2$ *and* $m > n$. *Then the rules* (DT) *and* (DTm) *are admissible but not derivable in the Gentzen system* $\mathfrak{G}_{\boldsymbol{n}}$. □

We have found an infinite number of alternative Gentzen-style presentations of positive implicative logic $\boldsymbol{\mathcal{IPL}}_{\to}$, which are proof-theoretically different (they have different derivable rules) but have the same derivable sequents (and hence the same admissible rules). Regarding the logic, this tells us that it is not necessary to have full (DT) as a rule to obtain $\boldsymbol{\mathcal{IPL}}_{\to}$: it is enough to have (DT2), in the presence of (MP) and all structural rules. By contrast, as we prove below, (DT1) is not enough.

The preceding theorem yields a new characterization of the positive implicative logic. By a well-known result of Herbrand and Tarski, see [22, Corollary 2.4.3], this logic is the weakest logic satisfying the Deduction-Detachment Theorem. Now we also have:

Corollary 3.3 $\boldsymbol{\mathcal{IPL}}_{\to}$ *is the weakest sentential logic satisfying* (MP) *and* (DT2).

P r o o f. $\boldsymbol{\mathcal{IPL}}_{\to}$ satisfies (MP), and (DT2) because it satisfies the stronger (DT). Now if another sentential logic $\boldsymbol{\mathcal{S}}$ satisfies them, this means that the set of sequents $\{\varphi_1, \dots, \varphi_n \rhd \varphi : \varphi_1, \dots, \varphi_n \vdash_{\boldsymbol{\mathcal{S}}} \varphi\}$ is closed under these rules. Hence it contains all derivable sequents of $\mathfrak{G}_{\boldsymbol{2}}$, among which are those corresponding to $\boldsymbol{\mathcal{IPL}}_{\to}$ by Theorem 3.1. Thus this logic is stronger than $\boldsymbol{\mathcal{IPL}}_{\to}$. □

Models of Gentzen systems allow also to compare Gentzen-style rules with Hilbert-style ones. For this it is enough to identify a Hilbert-style rule $\varphi_1, \dots, \varphi_n \vdash \varphi$ with the Gentzen-style rule $\dfrac{\emptyset}{\varphi_1, \dots, \varphi_n \rhd \varphi}$. Thus it makes sense to say that a generalized matrix $\langle \boldsymbol{A}, \mathrm{C} \rangle$ is a *generalized model of a sentential logic* $\boldsymbol{\mathcal{S}}$ when it is a model of all Gentzen-style rules of the form $\dfrac{\emptyset}{\Gamma \rhd \varphi}$ which are such that $\Gamma \vdash_{\boldsymbol{\mathcal{S}}} \varphi$; that is, when if $\Gamma \vdash_{\boldsymbol{\mathcal{S}}} \varphi$, then $h(\varphi) \in \mathrm{C}\big(h[\Gamma]\big)$ for all $h \in \mathrm{Hom}(\boldsymbol{Fm}, \boldsymbol{A})$. As a consequence, each $F \in \mathcal{C}$ is an $\boldsymbol{\mathcal{S}}$-filter, that is, the matrix $\langle \boldsymbol{A}, F \rangle$ is an (ordinary) model of $\boldsymbol{\mathcal{S}}$. It is easy to show that the converse also holds: any generalized matrix all whose closed sets are $\boldsymbol{\mathcal{S}}$-filters, is a generalized model of $\boldsymbol{\mathcal{S}}$.

Then again we find an algebraic proof of another fact of proof-theoretic interest:

Theorem 3.4 *The Gentzen-style rule* (DT) *cannot be replaced by any set of Hilbert-style rules in the usual structural Gentzen-style presentation of* $\mathcal{IPL}_{\rightarrow}$.

Proof. The mentioned presentation, consisting of (MP) and (DT), is that of the Gentzen system $\mathfrak{G}_{\boldsymbol{\omega}}$, whose reduced models have been determined in Theorem 2.6. We have already said that (MP) is actually (equivalent to) a Hilbert-style rule. If (DT) could be replaced by a set of Hilbert-style rules then we would have a system all whose non-structural rules are Hilbert-style rules. Then the models of this system would coincide with the generalized models of the sentential logic defined by the system, that is, of $\mathcal{IPL}_{\rightarrow}$. In particular every generalized model of $\mathcal{IPL}_{\rightarrow}$ would also be a model of $\mathfrak{G}_{\boldsymbol{\omega}}$, that is, of (DT); but it is easy to build a counterexample to this. Take the Hilbert algebra $\boldsymbol{A}_\omega$ described in the proof of Theorem 2.8 and the generalized matrix given by the family of implicative filters $\mathcal{C} = \big\{\{1\}, \{1, a_0\}, A_\omega\big\}$. This family is closed under intersections and under unions of directed subfamilies, hence this is a generalized matrix. All its members are implicative filters, and $\boldsymbol{A}_\omega$ is a Hilbert algebra, thus $\langle \boldsymbol{A}_\omega, \mathcal{C}\rangle$ is a generalized model of $\mathcal{IPL}_{\rightarrow}$, but it is not a model of (DT): $a_0 \in \mathrm{C}(a_1) = A_\omega$, but $a_1 \rightarrow a_0 = a_0 \notin \mathrm{C}(\emptyset) = \{1\}$. □

(This fact can also be derived from the proof-theoretical characterization of all logics with (DT) due to Herbrand and Tarski and mentioned before.)

The relation between $\mathfrak{G}_{\boldsymbol{\omega}}$ and $\mathcal{IPL}_{\rightarrow}$ is, from the abstract algebraic logic point of view, much stronger than that between the $\mathfrak{G}_{\boldsymbol{n}}$ and $\mathcal{IPL}_{\rightarrow}$ for $n \geq 2$; moreover, the latter are intrinsically weaker in two ways. All this is shown in Theorem 3.5, but we first recall some terminology. Here the *algebraizability of Gentzen systems* is understood in the sense of [18], which is an extension of Blok and Pigozzi's well-known notion for sentential logics [1]. In [14] a very general notion of equivalence between many-sided Gentzen systems is developed; as particular cases one finds the notion of an algebraizable (single-conclusion) Gentzen system and the notion of equivalence between such a Gentzen system and a sentential logic, which we use in the proof of Theorem 3.5. *Full adequacy* concerns the double role of generalized matrices as models of Gentzen systems and as generalized models of sentential logics (the matrices being the ordinary models). For an algebra $\boldsymbol{A}$ we denote by $\mathcal{F}i_{\boldsymbol{\mathcal{S}}}\boldsymbol{A}$ the family of all $\boldsymbol{\mathcal{S}}$-filters over $\boldsymbol{A}$; then $\big\langle \boldsymbol{A}, \mathcal{F}i_{\boldsymbol{\mathcal{S}}}\boldsymbol{A}\big\rangle$ is the largest generalized model of $\boldsymbol{\mathcal{S}}$ on $\boldsymbol{A}$. Of special importance in the general theory are the so-called *full generalized models*, which are the inverse images of those of the form $\big\langle \boldsymbol{A}, \mathcal{F}i_{\boldsymbol{\mathcal{S}}}\boldsymbol{A}\big\rangle$ under strict homomorphisms. If $\boldsymbol{\mathcal{S}}$ is a sentential logic with theorems, a Gentzen system $\mathfrak{G}$ is said to be *fully adequate for* $\boldsymbol{\mathcal{S}}$ when the models of $\mathfrak{G}$ coincide with the full generalized models of $\boldsymbol{\mathcal{S}}$; in particular then $\mathfrak{G}$ is adequate for $\boldsymbol{\mathcal{S}}$ in the sense specified above, but their relationships are much stronger in several respects. See [10] and [12, Section 5] for more details.

Theorem 3.5

1. *The Gentzen system* $\mathfrak{G}_{\boldsymbol{\omega}}$ *is algebraizable and its equivalent algebraic semantics is the class* **HA**.

2. *The Gentzen system* $\mathfrak{G}_{\boldsymbol{\omega}}$ *is fully adequate for the sentential logic* $\mathcal{IPL}_{\rightarrow}$.

3. *For each* $n \geq 1$ *the Gentzen system* $\mathfrak{G}_{\boldsymbol{n}}$ *is not algebraizable and is not fully adequate for any sentential logic.*

Proof.

1. In [1, pp. 25–26] it is shown that the sentential logic $\mathcal{IPL}_{\rightarrow}$ is algebraizable and that its equivalent algebraic semantics is the class **HA**. Therefore it will be enough to show that $\mathfrak{G}_{\boldsymbol{\omega}}$ is equivalent to $\mathcal{IPL}_{\rightarrow}$ by means of mutually inverse translations. Then this equivalence can be composed with the known equivalence between the consequence of $\mathcal{IPL}_{\rightarrow}$ and the relative equational consequence of **HA** and give an equivalence between the consequence of $\mathfrak{G}_{\boldsymbol{\omega}}$ and the same relative equational consequence, and thus $\mathfrak{G}_{\boldsymbol{\omega}}$ will be algebraizable with respect to **HA**. The translations between $\mathfrak{G}_{\boldsymbol{\omega}}$ and $\mathcal{IPL}_{\rightarrow}$ are:

$$\boldsymbol{\varrho}(\varphi_1, \dots, \varphi_n \rhd \varphi) = \varphi_1 \rightarrow \big(\cdots \rightarrow (\varphi_n \rightarrow \varphi)...\big), \quad \boldsymbol{\varrho}(\emptyset \rhd \varphi) = \varphi, \quad \boldsymbol{\tau}(\varphi) = \emptyset \rhd \varphi\,.$$

One way to show that $\mathfrak{G}_{\boldsymbol{\omega}}$ is $(\boldsymbol{\varrho}, \boldsymbol{\tau})$-equivalent to $\mathcal{IPL}_{\rightarrow}$ is to prove the following three things:

1a. If $\Gamma \vdash_{\mathcal{IPL}_{\rightarrow}} \varphi$, then the sequent $\boldsymbol{\tau}(\varphi)$ is derivable from the sequents of the set $\boldsymbol{\tau}[\Gamma]$ in $\mathfrak{G}_{\boldsymbol{\omega}}$: By Theorem 3.1 the assumption implies that the sequent $\Gamma \rhd \varphi$ is derivable in $\mathfrak{G}_{\boldsymbol{\omega}}$, and from this and a finite number of cuts we obtain the required derivation.

1b. If the sequent $\Gamma \rhd \varphi$ is derivable in $\mathfrak{G}_{\omega}$ from the sequents $\{\Gamma_i \rhd \varphi_i : i < n\}$, then

$$\big\{\varrho[\Gamma_i \rhd \varphi_i] : i < n\big\} \vdash_{\mathcal{IPL}_{\to}} \varrho(\Gamma \rhd \varphi).$$

For this it is enough to see that the ϱ-translations of all rules of $\mathfrak{G}_{\omega}$ are rules of $\mathcal{IPL}_{\to}$. (MP) as a Gentzen-style rule is the sequent $\varphi, \varphi \to \psi \rhd \psi$, so $\varrho(\mathrm{MP}) = \varphi \to \big((\varphi \to \psi) \to \psi\big)$ which is a theorem of $\mathcal{IPL}_{\to}$. The ϱ-translations of all (DTn) yield rules of $\mathcal{IPL}_{\to}$ simply because the premise and the conclusion have the same ϱ-translation. Also the structural rules have to be considered. The translation of the axiom is (I), a theorem of $\mathcal{IPL}_{\to}$, and that of the weakening rule is a rule of $\mathcal{IPL}_{\to}$ by (PRE). The translation of the contraction rule is also a rule of $\mathcal{IPL}_{\to}$, precisely by contraction. The interesting case is the cut rule: in a particular application this rule takes the form

$$\frac{\varphi_1, \dots, \varphi_n \rhd \varphi \qquad \varphi_1, \dots, \varphi_n, \varphi \rhd \psi}{\varphi_1, \dots, \varphi_n \rhd \psi}$$

for some finite n. We realize that its ϱ-translation, if $n \geq 1$, is exactly the rule called (MPn) at the beginning of Section 2; this rule is a derived rule of $\mathcal{IPL}_{\to}$ thanks to (DT) and (MP). If $n = 0$, then the translation is just (MP), again a rule of $\mathcal{IPL}_{\to}$.

1c. For each formula φ, $\varphi \dashv\vdash_{\mathcal{IPL}_{\to}} \varrho\big(\tau(\varphi)\big)$, which is trivial because $\varrho\big(\tau(\varphi)\big) = \varphi$.

2. It is well-known that the $\mathcal{IPL}_{\to}$-filters on a Hilbert algebra coincide with the implicative filters. As a consequence, by [10, Theorem 3.8], the reduced full generalized models of $\mathcal{IPL}_{\to}$ are the generalized matrices of the form $\big\langle \boldsymbol{A}, \mathcal{F}(\boldsymbol{A})\big\rangle$ with $\boldsymbol{A} \in \mathsf{HA}$. Now if we compare with Theorem 2.6 we realize that they coincide with the reduced models of the Gentzen system $\mathfrak{G}_{\omega}$. Now full generalized models of a sentential logic are the strict inverse images of the reduced ones, and the same happens to models of a Gentzen system. Hence the full generalized models of $\mathcal{IPL}_{\to}$ coincide with the models of $\mathfrak{G}_{\omega}$, that is, $\mathfrak{G}_{\omega}$ is fully adequate for $\mathcal{IPL}_{\to}$.

3. Consider the Hilbert algebra $\boldsymbol{A}_{\omega}$ constructed in the proof of Theorem 2.8 and the generalized matrices $\langle \boldsymbol{A}_{\omega}, \mathcal{F}_{n+1} \cup \{A_{\omega}\}\rangle$ and $\langle \boldsymbol{A}_{\omega}, \mathcal{F}_{n+2} \cup \{A_{\omega}\}\rangle$. As we have seen, they are different, but by Theorem 2.5 they are both models of $\mathfrak{G}_{n}$ (for $n \geq 2$) and they are both reduced, that is, their Tarski congruence is the identity. This shows that the Tarski operator $\widetilde{\boldsymbol{\Omega}}_{\boldsymbol{A}_{\omega}}$ is not one-to-one when considered as an operator on the set of models of $\mathfrak{G}_{n}$ on $\boldsymbol{A}_{\omega}$. By [14, Theorem 4.7], this implies the Gentzen system cannot be algebraizable, with respect to any quasi-variety of algebras (in [14] the models of $\mathfrak{G}_{n}$ over a fixed algebra are called the $\mathfrak{G}_{n}$-filters). Moreover, the same fact also prevents $\mathfrak{G}_{n}$ from being fully adequate for any sentential logic, by [10, Theorem 2.30]. The same argument also works for $n = 1$ but using Theorem 2.4 instead of Theorem 2.3. □

A Gentzen system defined by a different calculus satisfying 1. and 2. of this theorem is given by the (very general) result of [10, Theorem 4.45]. Since the fully adequate system, when it exists, is unique, we conclude that the two systems coincide. While the proof in [10] is based on very general considerations, the proof above illustrates the role of the Hilbert-style rules (MPn) in a Gentzen system with (DT).

3.2 Two new logics

The second focus of our research is on strengthened forms of Modus Ponens, incorporating additional assumptions, as a replacement for the Fregean axiom (FRE) in the axiomatic definition of positive implicative logic. These forms are the rules called (MPn) in Section 2.

Definition 3.6 For each $n \in \omega$, $n \geq 1$, we denote by $\mathcal{H}_n$ the sentential logic defined by the Hilbert-style calculus with axiom (K) and rules (MP) and (MP$n+1$).

The ordering relation $\leq$ between sentential logics we are going to use is similar to that between Gentzen systems: If $\mathcal{S}$ and $\mathcal{S}'$ are two sentential logics (over the same language) we say that $\mathcal{S}$ is *weaker* than $\mathcal{S}'$, and that $\mathcal{S}'$ is *stronger* than $\mathcal{S}$, when as a consequence relation $\vdash_{\mathcal{S}}$ is included in $\vdash_{\mathcal{S}'}$, that is, when $\Gamma \vdash_{\mathcal{S}} \varphi$ implies $\Gamma \vdash_{\mathcal{S}'} \varphi$; we denote this by $\mathcal{S} \leq \mathcal{S}'$.

Lemma 3.7 *For each* $n \geq 1$, $\mathcal{H}_n \leq \mathcal{H}_{n+1}$ *and* $\mathcal{H}_n \leq \mathcal{IPL}_{\to}$.

Proof. That all these logics are weaker than $\mathcal{IPL}_{\to}$ is seen because all rules (MP$n+1$) are derived rules of $\mathcal{IPL}_{\to}$, which is a straightforward consequence of (DT) and (MP). To show the first part we see that (MP$n+1$)

can be derived from (MP$n+2$) together with (K) and (MP):

1.	$\varphi_1 \to \big(\cdots \to \big(\varphi_n \to (\varphi_{n+1} \to \varphi)\big)...\big) = \alpha$	Assumption
2.	$\varphi_1 \to \big(\cdots \to \big(\varphi_n \to (\varphi_{n+1} \to (\varphi \to \psi))\big)...\big) = \beta$	Assumption
3.	$\alpha \to \big((\varphi \to (\varphi \to \varphi)) \to \alpha\big)$	(K)
4.	$\big(\varphi \to (\varphi \to \varphi)\big) \to \alpha$	(MP) $1, 3$
5.	$\beta \to \big((\varphi \to (\varphi \to \varphi)) \to \beta\big)$	(K)
6.	$\big(\varphi \to (\varphi \to \varphi)\big) \to \beta$	(MP) $2, 5$
7.	$\big(\varphi \to (\varphi \to \varphi)\big) \to \big(\varphi_1 \to \big(\cdots \to \big(\varphi_n \to (\varphi_{n+1} \to \psi)\big)...\big)\big)$	(MP$n+2$) $4, 6$
8.	$\varphi \to (\varphi \to \varphi)$	(K)
9.	$\varphi_1 \to \big(\cdots \to \big(\varphi_n \to (\varphi_{n+1} \to \psi)\big)...\big)$	(MP) $7, 8$ □

Lemma 3.8 *The sentential logic $\mathcal{H}_1$ satisÞes the following axioms and rules:*

(MP1)	$\varphi_1 \to \varphi,\ \varphi_1 \to (\varphi \to \psi) \vdash \varphi_1 \to \psi,$
(I)	$\varphi \to \varphi,$
(PRE)	$\varphi \vdash \psi \to \varphi,$
TRANS)	$\varphi \to \psi,\ \psi \to \xi \vdash \varphi \to \xi,$
(CONGRU)	$\varphi \to \psi,\ \alpha \to \beta \vdash (\psi \to \alpha) \to (\varphi \to \beta).$

P r o o f . (MP1) follows from (MP2) as in the proof of Lemma 3.7; actually, the same proof as given there also holds for $n = 0$. Let us prove the other four:

(I) :

1.	$\varphi \to (\varphi \to \varphi)$	(K)
2.	$\varphi \to \big((\varphi \to \varphi) \to \varphi\big)$	(K)
3.	$\varphi \to \varphi$	(MP1) $1, 2$

(PRE) :

1.	φ	Assumption
2.	$\varphi \to (\psi \to \varphi)$	(K)
3.	$\psi \to \varphi$	(MP) $1, 2$

(TRANS) :

1.	$\varphi \to \psi$	Assumption
2.	$\psi \to \xi$	Assumption
3.	$\varphi \to (\psi \to \xi)$	(PRE) 2
4.	$\varphi \to \xi$	(MP1) $1, 3$

(CONGRU) :

1.	$\varphi \to \psi$	Assumption
2.	$\alpha \to \beta$	Assumption
3.	$\varphi \to (\alpha \to \beta)$	(PRE) 2
4.	$(\psi \to \alpha) \to \big(\varphi \to (\alpha \to \beta)\big)$	(PRE) 3
5.	$(\psi \to \alpha) \to (\varphi \to \psi)$	(PRE) 1
6.	$(\psi \to \alpha) \to \big(\varphi \to (\psi \to \alpha)\big)$	(K)
7.	$(\psi \to \alpha) \to (\varphi \to \alpha)$	(MP2) $5, 6$
8.	$(\psi \to \alpha) \to (\varphi \to \beta)$	(MP2) $4, 7$ □

Theorem 3.9 *For all $n \geq 2$, $\mathcal{H}_n = \mathcal{IPL}_\to$.*

P r o o f . Given Lemma 3.7 it is enough to show that $\mathcal{IPL}_\to \leq \mathcal{H}_2$, and for this we have just to show that (FRE) is a theorem of $\mathcal{H}_2$. Since $\mathcal{H}_1 \leq \mathcal{H}_2$ we can use that the latter also satisfies (I) and (PRE). Then:

1.	$(\varphi \to \psi) \to (\varphi \to \psi)$	(I)
2.	$\big(\varphi \to (\psi \to \xi)\big) \to \big((\varphi \to \psi) \to (\varphi \to \psi)\big)$	(PRE) 1
3.	$\big(\varphi \to (\psi \to \xi)\big) \to \big((\varphi \to \psi) \to \big(\varphi \to (\psi \to \xi)\big)\big)$	(K)
4.	$\big(\varphi \to (\psi \to \xi)\big) \to \big((\varphi \to \psi) \to (\varphi \to \xi)\big)$	(MP3) $2, 3$ □

Therefore we have obtained an infinite number of alternative Hilbert-style presentations of $\mathcal{IPL}_{\to}$ where the Fregean axiom (FRE) is replaced by any of the rules (MPn) for $n \geq 3$. Notice that each of these yields a new *quasi-equational presentation of the class of Hilbert algebras*, because this class is the equivalent algebraic semantics of $\mathcal{IPL}_{\to}$, an algebraizable logic; then [1, Theorem 2.17] tells us how to convert every (Hilbert-style) axiomatization of the logic into a (quasi-equational) presentation of its equivalent algebraic semantics.

By contrast, (MP2) and (MP1) define strictly weaker sentential logics. Here we treat the case of (MP2), which is connected with the logic defined by the rules (MP) and (DT1) as Gentzen-style rules. (MP1) is too weak, as is (DT0), and the logics related to these rules will be treated elsewhere.

Definition 3.10 The sentential logic $\mathcal{G}_1$ is the logic defined by the Gentzen system $\mathfrak{G}_1$.

Thus, the derivability relation of $\mathcal{G}_1$ satisfies all the Gentzen-style rules of $\mathfrak{G}_1$, of which we highlight (DT1). The proof-theoretical relations between $\mathcal{G}_1$ and $\mathcal{H}_1$ are described in the next two results; in part 2 of the first one we also obtain a kind of "poor man" Hilbert-style presentation of $\mathcal{G}_1$.

Theorem 3.11

1. $\mathcal{G}_1$ *is the sentential logic deÞned by the Gentzen system deÞned by the rules* (K), (MP) *and* (weakMP2).

2. $\mathcal{G}_1$ *is the sentential logic deÞned by the Hilbert-style calculus having as axioms all theorems of* $\mathcal{H}_1$ *and the rule* (MP).

3. *The logics* $\mathcal{G}_1$ *and* $\mathcal{H}_1$ *have the same theorems.*

4. *The sentential logic* $\mathcal{H}_1$ *is the inferential strengthening of* $\mathcal{G}_1$ *by the rule* (MP2).

P r o o f. Let us denote by $\mathcal{S}$ the sentential logic defined by the Gentzen system with rules (K), (MP) and (weakMP2). Given the definition of $\mathcal{H}_1$, the set of its theorems is closed under (weakMP2) as well. Using this, it is easy to see that $\mathcal{S}$ coincides with the sentential logic defined by the Hilbert-style calculus having as axioms all theorems of $\mathcal{H}_1$ and the rule (MP), and that $\mathcal{S}$ and $\mathcal{H}_1$ have the same theorems. Thus, in order to show 1. and 2. it is enough to show that $\mathcal{S} = \mathcal{G}_1$. Then 3. and 4. follow immediately.

First we show show that $\mathcal{S} \leq \mathcal{G}_1$ by checking that the rules of the Gentzen system defining $\mathcal{S}$ are derivable in $\mathfrak{G}_1$: By definition this is true for (MP), and (K) has been proved in Lemma 2.2. (weakMP2) is shown as follows.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\emptyset \rhd \varphi_1 \to (\varphi_2 \to \varphi)}{\varphi_1 \rhd \varphi_2 \to \varphi}\,(\mathrm{DT0})}{\varphi_1, \varphi_2 \rhd \varphi}\,(\mathrm{DT1}) \qquad \dfrac{\dfrac{\emptyset \rhd \varphi_1 \to \big(\varphi_2 \to (\varphi \to \psi)\big)}{\varphi_1 \rhd \varphi_2 \to (\varphi \to \psi)}\,(\mathrm{DT0})}{\varphi_1, \varphi_2 \rhd \varphi \to \psi}\,(\mathrm{DT1})}{\varphi_1, \varphi_2 \rhd \psi}\,(\mathrm{MP})}{\varphi_1 \rhd \varphi_2 \to \psi}\,(\mathrm{DT1})}{\emptyset \rhd \varphi_1 \to (\varphi_2 \to \psi)}\,(\mathrm{DT0})$$

In order to show that $\mathcal{S} \geq \mathcal{G}_1$ we first check that if $\langle \alpha_0, \dots, \alpha_{n-1} \rangle$ is a proof in the Hilbert-style presentation of $\mathcal{S}$ from $\{\varphi, \psi\}$, then $\emptyset \vdash_{\mathcal{H}_1} \varphi \to (\psi \to \alpha_i)$ for every $i < n$. We proceed by induction and consider several cases:

1) If α_i is an axiom of $\mathcal{S}$, then $\emptyset \vdash_{\mathcal{H}_1} \alpha_i$; by (K) and (MP) we obtain that $\emptyset \vdash_{\mathcal{H}_1} \psi \to \alpha_i$. And by a similar reasoning we obtain that $\emptyset \vdash_{\mathcal{H}_1} \varphi \to (\psi \to \alpha_i)$.

2) If $\alpha_i \in \{\varphi, \psi\}$, then we just use that $\emptyset \vdash_{\mathcal{H}_1} \varphi \to (\psi \to \varphi)$ or that $\emptyset \vdash_{\mathcal{H}_1} \varphi \to (\psi \to \psi)$.

3) If α_i is obtained by applying (MP) to α_j and $\alpha_j \to \alpha_i$, by the inductive hypothesis $\emptyset \vdash_{\mathcal{H}_1} \varphi \to (\psi \to \alpha_j)$ and $\emptyset \vdash_{\mathcal{H}_1} \varphi \to \big(\psi \to (\alpha_j \to \alpha_i)\big)$; from this and (MP2) we conclude that $\emptyset \vdash_{\mathcal{H}_1} \varphi \to (\psi \to \alpha_i)$.

Now we can show that $\mathcal{S}$ satisfies (DT1): If $\varphi, \psi \vdash_{\mathcal{S}} \xi$, then by what we have just shown $\emptyset \vdash_{\mathcal{H}_1} \varphi \to (\psi \to \xi)$, and hence $\emptyset \vdash_{\mathcal{S}} \varphi \to (\psi \to \xi)$. Now using (MP) we obtain that $\varphi \vdash_{\mathcal{S}} \psi \to \xi$. Therefore $\mathcal{S} \geq \mathcal{G}_1$, which ends the proof. □

Thus $\mathcal{H}_1$ is a kind of *strong version* of $\mathcal{G}_1$: The theories of $\mathcal{H}_1$ are exactly the theories of $\mathcal{G}_1$ that are closed under the "strong form" of rule (MP2), whose "weak form" (weakMP2) is satisfied by both.

Theorem 3.12 $\mathcal{G}_1 < \mathcal{H}_1 < \mathcal{IPL}_{\to}$.

Proof. In Theorem 3.11 we have proved that $\mathcal{G}_1 \leq \mathcal{H}_1$, and from Lemma 3.7 and Theorem 3.9 it follows that $\mathcal{H}_1 \leq \mathcal{IPL}_{\to}$. To see that these inequalities are strict we are going to use simple models of these logics on the algebra $\boldsymbol{A}_1$ considered in Example 1.5. This is a quasi-Hilbert algebra, and the quasi-identities (k) and (mp2) are part of the definition of this quasi-variety. Moreover 1 is the maximum of the order, which satisfies (1). This tells us that the set $\{1\}$ is closed under the rules (K), (MP) and (MP2), and hence the matrix $\langle \boldsymbol{A}_1\,\{1\}\rangle$ is a model of $\mathcal{H}_1$. However it is not a model of $\mathcal{IPL}_{\to}$ because it is not a model of (FRE), as witnessed by the same example used in 1.5 to see that $\boldsymbol{A}_1$ is not a Hilbert algebra. This establishes that $\mathcal{H}_1 < \mathcal{IPL}_{\to}$. Now consider the set $\{1, c, e\}$. It is easy to check that this is an implicative filter of $\boldsymbol{A}_1$. Since by Theorem 2.4 the generalized matrix constituted by all its implicative filters is a model of the Gentzen system $\mathfrak{G}_1$, it is also a generalized model of the logic defined by the system, i. e., $\mathcal{G}_1$. As a consequence, the matrix $\langle \boldsymbol{A}_1,\, \{1, c, e\}\rangle$ is a model of $\mathcal{G}_1$. But it is not a model of $\mathcal{H}_1$ because it is not a model of (MP2): $1 \to (d \to (b \to a)) = c$ and $1 \to (d \to b) = e$, while $1 \to (d \to a) = a \notin \{1, c, e\}$. This establishes that $\mathcal{G}_1 < \mathcal{H}_1$. □

Corollary 3.13 *The rule* (DT2) *is not admissible in the Gentzen system* $\mathfrak{G}_1$.

Proof. Admissibility of (DT2) in $\mathfrak{G}_1$ amounts to $\mathcal{G}_1$ satisfying (DT2). But since we have just proven that $\mathcal{G}_1$ is strictly weaker than $\mathcal{IPL}_{\to}$, this would contradict Corollary 3.3 asserting that $\mathcal{IPL}_{\to}$ is the weakest logic satisfying (MP) and (DT2) simultaneously. □

Thus, we have two distinct logics, of implicative character, but weaker than positive implicative logic. We end the paper by classifying these new logics according to several standard criteria of Abstract Algebraic Logic and determining their algebraic counterparts. Concerning classification, we consider two criteria: one is their position in the so-called *protoalgebraic hierarchy* [7, 12], where the main subclasses are the equivalential logics and the algebraizable logics (which come in several flavours: weakly, plain, finitely, strongly, regularly, etc.), and the other is the kind of *congruence properties* [10] it has, which results in the classes of selfextensional logics and of Fregean logics. As we will see, their algebraic behaviour is very different, and where one has good properties the other has not, and conversely. Recall that $\mathcal{IPL}_{\to}$ is one of the best-behaved logics in these respects: It is finitely, regularly and strongly algebraizable, with HA as its equivalent algebraic semantics, and is not just selfextensional, it is even Fregean.

Theorem 3.14 *The logics* $\mathcal{G}_1$ *and* $\mathcal{H}_1$ *are both protoalgebraic.*

Proof. We can apply the characterization of protoalgebraicity found in [7, Theorem 1.1.3]: A sentential logic $\mathcal{S}$ is protoalgebraic if and only if there exists a (possibly infinite and possibly empty) set $I(x, y)$ of formulas in two variables such that $\vdash_{\mathcal{S}} I(x, x)$ and $x,\ I(x, y) \vdash_{\mathcal{S}} y$. Since both logics satisfy (I) and (MP), the one-formula set $\{x \to y\}$ meets the requirement. □

Theorem 3.15

1. $\mathcal{H}_1$ *is an implicative logic in the sense of Rasiowa.*
2. $\mathcal{H}_1$ *is finitely, regularly algebraizable, with defining equations* $E(x) = \{x \approx \top\}$ *and with equivalence formulas* $\Delta(x, y) = \{x \to y,\ y \to x\}$.
3. *The equivalent algebraic semantics of* $\mathcal{H}_1$ *is the class* $\mathsf{Alg}\ \mathcal{H}_1 = \mathsf{QHA}$ *of quasi-Hilbert algebras.*
4. *The reduced models of* $\mathcal{H}_1$ *are all the matrices of the form* $\langle \boldsymbol{A}, \{1\}\rangle$ *with* $\boldsymbol{A} \in \mathsf{QHA}$.
5. *The reduced full generalized models of* $\mathcal{H}_1$ *are all the generalized matrices of the form*

$$\big\langle \boldsymbol{A},\ \big\{F \in \mathcal{F}(\boldsymbol{A}) : F \textit{ is closed under } \text{(MP2)}\big\}\big\rangle$$

with $\boldsymbol{A} \in \mathsf{QHA}$.

Proof. 1. The commonly called implicative logics are those called "standard systems of implicative extensional propositional calculi" by Rasiowa in [17], and are defined by a set of syntactical conditions, all of which have already been proved for $\mathcal{H}_1$: (I), (PRE), (TRANS), (MP), (CONGRU), this last one in a weaker form than that shown in Lemma 3.8: $\varphi \to \psi,\ \psi \to \varphi,\ \alpha \to \beta,\ \beta \to \alpha \vdash (\psi \to \alpha) \to (\varphi \to \beta)$. 2. is a consequence of 1. in the general theory, as shown in [1, Example 5.2]. By [1, Theorem 2.17], the equivalent algebraic semantics $\mathbf{Alg}\,\mathcal{H}_1$ is the quasi-variety axiomatized by the equations and quasi-equations obtained from the axioms and rules, respectively, of any Hilbert-style presentation of the logic after "translating" them with the help of the defining equation, plus two additional conditions: $E\big(\Delta(x,x)\big)$ and $E\big(\Delta(x,y)\big) \Rightarrow x \approx y$. In our case this last one gives (i2). By a process similar to that in [1], one can check that (MP) plus the first additional condition amount to equation (i1). Then after having (i1), axiom (K) gives equation (k) and rule (MP2) gives the quasi-equation (mp2). Hence we get exactly the quasi-equational base of the quasi-variety $\mathbf{QHA}$. This shows 3. Now 4. and 5. follow also from the general theory, given the previous results and the axiomatization of $\mathcal{H}_1$. □

Theorem 3.16 *$\mathcal{H}_1$ does not satisfy* (DT0)*, hence a fortiori it does not satisfy* (DT1)*, and it is not selfextensional.*

Proof. From (q-cp) in Lemma 1.3 and Theorem 3.15 it is clear that $x \to (y \to z) \vdash_{\mathcal{H}_1} y \to (x \to z)$. But by contrast $\emptyset \nvdash_{\mathcal{H}_1} \big(x \to (y \to z)\big) \to \big(y \to (x \to z)\big)$ because $\big(x \to (y \to z)\big) \to \big(y \to (x \to z)\big) \approx \top$ does not hold in $\mathbf{QHA}$, as witnessed by Example 1.5, where $\big(d \to (c \to a)\big) \to \big(c \to (d \to a)\big) = d \neq 1$. Now, again by (q-cp), $x \to (y \to z) \dashv\vdash_{\mathcal{H}_1} y \to (x \to z)$. Let us now check that $t \to \big(x \to (y \to z)\big) \nvdash_{\mathcal{H}_1} t \to \big(y \to (x \to z)\big)$. Take the algebra $\boldsymbol{A}_1$ from Example 1.5 again. There $e \to \big(d \to (c \to a)\big) = 1$ while $e \to \big(c \to (d \to a)\big) = d \neq 1$, therefore $t \to \big(x \to (y \to z)\big) \nvdash_{\mathcal{H}_1} t \to \big(y \to (x \to z)\big)$. This implies that the interderivability relation $\dashv\vdash_{\mathcal{H}_1}$ is not a congruence of the formula algebra: $\mathcal{H}_1$ is not selfextensional. □

We see that $\mathcal{H}_1$ belongs to one of the inner classes of the protoalgebraic hierarchy (which means it has the best properties regarding its algebraic behaviour), while it does not satisfy even the weakest form of the Deduction Theorem and is not selfextensional. By contrast, the logic $\mathcal{G}_1$ does indeed satisfy (DT1) by definition, and we are going to see that it is selfextensional, but does not belong to any of the special classes of the said hierarchy. In the next result we use the notation $\mathbb{V}$ to denote the operator of forming the variety (equational class) generated by a class of algebras.

Theorem 3.17

1. *$\mathcal{G}_1$ is not weakly algebraizable, hence a fortiori it is not algebraizable, and is not equivalential.*
2. *$\mathcal{G}_1$ is selfextensional, but it is not Fregean.*
3. *The algebraic counterpart of $\mathcal{G}_1$ is* $\mathbf{Alg}\,\mathcal{G}_1 = \mathbb{V}(\mathbf{QHA})$.
4. *The reduced full generalized models of $\mathcal{G}_1$ are those of the form $\big\langle \boldsymbol{A}, \mathcal{F}(\boldsymbol{A})\big\rangle$ for all $\boldsymbol{A} \in \mathbb{V}(\mathbf{QHA})$.*

Proof.

1. Consider the algebra $\boldsymbol{A}_1$ of Example 1.5. Since $\boldsymbol{A}_1 \in \mathbf{QHA}$, by Theorem 2.4 the family of all its implicative filters determines a model of $\mathfrak{G}_1$ and hence a generalized model of $\mathcal{G}_1$. Since the sets $\{1\}$ and $\{1, c\}$ are implicative filters, the matrices $\big\langle \boldsymbol{A}_1, \{1\}\big\rangle$ and $\big\langle \boldsymbol{A}_1, \{1, c\}\big\rangle$ are models of $\mathcal{G}_1$. By inspection of the table of $\to$ in $\boldsymbol{A}_1$ it is easy to see that this algebra has only three congruences: the two trivial ones (the identity and the total relations) and the congruence with blocks $\{a, c\}$ and $\{1, b, d, e\}$. Therefore $\boldsymbol{\Omega}_{\boldsymbol{A}_1}\big(\{1\}\big) = \boldsymbol{\Omega}_{\boldsymbol{A}_1}\big(\{1, c\}\big) = \mathrm{Id}_{A_1}$. Thus both matrices are reduced, and the Leibniz operator is not one-to-one on the models of $\mathcal{G}_1$ on the same algebra. By definition this implies it is not weakly algebraizable. Now let $\boldsymbol{B}$ be the subalgebra of $\boldsymbol{A}_1$ with universe $\{1, c\}$. Clearly $\big\langle \boldsymbol{B}, \{1, c\}\big\rangle$ is a submatrix of $\big\langle \boldsymbol{A}_1, \{1, c\}\big\rangle$ and $\boldsymbol{\Omega}_{\boldsymbol{B}}\big(\{1, c\}\big) = B \times B$, so $\big\langle \boldsymbol{B}, \{1, c\}\big\rangle$ is not reduced. Thus, the class of reduced matrices of $\mathcal{G}_1$ is not closed under the operation of taking submatrices. By [5, Theorem 3.2.1] (see also [12, Theorem 3.15(2)]), this implies that $\mathcal{G}_1$ is not equivalential.

2. By Lemma 2.2, the Gentzen-style rule (CONG) is a derived rule of the Gentzen system $\mathfrak{G}_1$, therefore the sentential logic $\mathcal{G}_1$ has the congruence property, that is, it is selfextensional. Now $\mathcal{G}_1$ is protoalgebraic and has theorems. By [10, Theorem 3.18], a protoalgebraic Fregean logic with theorems must be algebraizable. Since we have just seen that $\mathcal{G}_1$ is not even weakly algebraizable, we conclude that it is not Fregean.

3. Recall that, by [10, Proposition 3.2], since $\mathcal{G}_1$ is protoalgebraic, its algebraic counterpart $\mathbf{Alg}\,\mathcal{G}_1$ coincides with the class of the algebraic reducts of its reduced matrices. We first check that $\mathbf{Alg}\,\mathcal{G}_1 \subseteq \mathbb{V}(\mathbf{QHA})$. Since $\mathcal{G}_1$ is selfextensional, its interderivability relation $\dashv\vdash_{\mathcal{G}_1}$ is a fully invariant congruence of the formula algebra and by [10, Proposition 2.26], the variety generated by the class $\mathbf{Alg}\,\mathcal{G}_1$ is the same as the variety generated by the

Lindenbaum-Tarski quotient $\boldsymbol{Fm}/\dashv\vdash_{\mathcal{G}_1}$. But here this relation can be neatly characterized through the following chain of equivalences:

$$\begin{array}{llll} \varphi \dashv\vdash_{\mathcal{G}_1} \psi & \text{iff} & \emptyset \vdash_{\mathcal{G}_1} \varphi \to \psi \text{ and } \emptyset \vdash_{\mathcal{G}_1} \psi \to \varphi & \text{by (DT0)} \\ & \text{iff} & \emptyset \vdash_{\mathcal{H}_1} \varphi \to \psi \text{ and } \emptyset \vdash_{\mathcal{H}_1} \psi \to \varphi & \text{by Theorem 3.11(3)} \\ & \text{iff} & \emptyset \vDash_{\mathsf{QHA}} \varphi \approx \psi & \text{by Theorem 3.15(2,3).} \end{array}$$

By the full invariance of $\dashv\vdash_{\mathcal{G}_1}$, this implies that $\mathbb{V}(\mathsf{Alg}\,\mathcal{G}_1) = \mathbb{V}(\boldsymbol{Fm}/\dashv\vdash_{\mathcal{G}_1}) = \mathbb{V}(\mathsf{QHA})$. Therefore we have $\mathsf{Alg}\,\mathcal{G}_1 \subseteq \mathbb{V}(\mathsf{Alg}\,\mathcal{G}_1) = \mathbb{V}(\mathsf{QHA})$.

Now let us see that $\mathsf{Alg}\,\mathcal{G}_1 \supseteq \mathbb{V}(\mathsf{QHA})$. Since QHA is a quasi-variety, it is enough to check that if $\boldsymbol{A} \in \mathsf{QHA}$ and $\theta \in \mathrm{Co}\boldsymbol{A}$, then $\boldsymbol{A}/\theta \in \mathsf{Alg}\,\mathcal{G}_1$. Now $1/\theta \subseteq A$ and it is easy to check that $1/\theta$ is an implicative filter of $\boldsymbol{A}$. Hence by Theorem 2.4 (through the same reasoning as in the first part of 1.), $\langle \boldsymbol{A}, 1/\theta\rangle$ is a matrix model of $\mathcal{G}_1$, and thus its reduction gives a reduced matrix, therefore if we take the algebraic reduct, we have that $\boldsymbol{A}/\boldsymbol{\Omega}_{\boldsymbol{A}}(1/\theta) \in \mathsf{Alg}\,\mathcal{G}_1$. So we have just to check that $\boldsymbol{\Omega}_{\boldsymbol{A}}(1/\theta) = \theta$. It is clear that θ is compatible with $1/\theta$, therefore $\theta \subseteq \boldsymbol{\Omega}_{\boldsymbol{A}}(1/\theta)$. In particular we obtain that $1/\theta \subseteq 1/\boldsymbol{\Omega}_{\boldsymbol{A}}(1/\theta)$. But from the fact that $\boldsymbol{\Omega}_{\boldsymbol{A}}(1/\theta)$ is compatible with $1/\theta$ we obtain the reverse inclusion, and thus $1/\theta = 1/\boldsymbol{\Omega}_{\boldsymbol{A}}(1/\theta)$. By the congruence-regularity of QHA seen in Corollary 1.7 we conclude that $\theta = \boldsymbol{\Omega}_{\boldsymbol{A}}(1/\theta)$.

4. By the general theory [10], the reduced full generalized models of $\mathcal{G}_1$ have the form $\big\langle \boldsymbol{A}, \mathcal{F}i_{\mathcal{G}_1}(\boldsymbol{A})\big\rangle$ with $\boldsymbol{A} \in \mathsf{Alg}\,\mathcal{G}_1$, where $\mathcal{F}i_{\mathcal{G}_1}(\boldsymbol{A})$ is the family of all $\mathcal{G}_1$-filters over $\boldsymbol{A}$. Thus we have to show that if $\boldsymbol{A} \in \mathbb{V}(\mathsf{QHA})$, then $\mathcal{F}i_{\mathcal{G}_1}(\boldsymbol{A}) = \mathcal{F}(\boldsymbol{A})$. The inclusion from left to right is a consequence of the fact that (I) and (MP) are rules of $\mathcal{G}_1$. If $\boldsymbol{A} \in \mathsf{QHA}$, then the inclusion from right to left is also clear, because $\big\langle \boldsymbol{A}, \mathcal{F}(\boldsymbol{A})\big\rangle$ is a generalized model of $\mathcal{G}_1$ because, by Theorem 2.4, it is a model of $\mathfrak{G}_1$.

Finally assume that $\boldsymbol{A} = \boldsymbol{B}/\theta$ for some $\boldsymbol{B} \in \mathsf{QHA}$ and some $\theta \in \mathrm{Co}\boldsymbol{B}$, and that $F \in \mathcal{F}(\boldsymbol{A})$; we have to check that $F \in \mathcal{F}i_{\mathcal{G}_1}(\boldsymbol{A})$. The projection $\pi : \big\langle \boldsymbol{B}, \pi^{-1}[F]\big\rangle \longrightarrow \big\langle \boldsymbol{A}, F\big\rangle$ is a strict surjective homomorphism. Since $F \in \mathcal{F}(\boldsymbol{A})$, $\big\langle \boldsymbol{A}, F\big\rangle$ is a model of the Hilbert-style calculus with axiom (I) and rule (MP). Hence, $\big\langle \boldsymbol{B}, \pi^{-1}[F]\big\rangle$ is also a model of the same calculus, that is, $\pi^{-1}[F] \in \mathcal{F}(\boldsymbol{B})$. Since $\boldsymbol{B} \in \mathsf{QHA}$, we have that $\pi^{-1}[F] \in \mathcal{F}(\boldsymbol{B}) = \mathcal{F}i_{\mathcal{G}_1}(\boldsymbol{B})$ as previously seen. Thus, $\big\langle \boldsymbol{B}, \pi^{-1}[F]\big\rangle$ is a model of $\mathcal{G}_1$. From this, [10, Proposition 1.19] allows us to conclude that $\langle \boldsymbol{A}, F\rangle$ is also a model of $\mathcal{G}_1$, that is, $F \in \mathcal{F}i_{\mathcal{G}_1}(\boldsymbol{A})$. □

Since QHA is a quasi-variety, $\mathsf{Alg}\,\mathcal{G}_1 = \mathbb{V}(\mathsf{QHA}) = \{\boldsymbol{A}/\theta : \boldsymbol{A} \in \mathsf{QHA} \text{ and } \theta \in \mathrm{Co}\boldsymbol{A}\}$. Since $\mathsf{QHA} \subseteq \mathsf{IA}$ which is a proper quasi-variety, in principle the class $\mathsf{Alg}\,\mathcal{G}_1$ might extend outside the class of implicative algebras. But this does not happen here, because by Theorem 1.8 all these quotients are indeed implicative algebras (which moreover satisfy the isotonicity quasi-identities).

Now, Theorem 3.17.4 describes the reduced full generalized models of $\mathcal{G}_1$. By the general completeness theorem of any logic with respect to the class of all its reduced full generalized models, we have that for all $\Gamma \cup \{\varphi\} \subseteq Fm$,

$$\Gamma \vdash_{\mathcal{G}_1} \varphi \ \text{ iff } \ h(\varphi) \in \mathrm{F}\big(h[\Gamma]\big) \text{ for every } \boldsymbol{A} \in \mathbb{V}(\mathsf{QHA}) \text{ and every } h \in \mathrm{Hom}(\boldsymbol{Fm}, \boldsymbol{A}).$$

It is interesting to discover that for this kind of completeness one can take just the algebras in QHA instead of the whole generated variety:

Theorem 3.18 *For all $\Gamma \subseteq Fm$ and every $\varphi \in Fm$,*

$$\Gamma \vdash_{\mathcal{G}_1} \ \textit{iff} \ h(\varphi) \in \mathrm{F}\big(h[\Gamma]\big) \textit{ for every } \boldsymbol{A} \in \mathsf{QHA} \textit{ and every } h \in \mathrm{Hom}(\boldsymbol{Fm}, \boldsymbol{A}). \tag{6}$$

Proof. The theorem asserts the identity of two logics. The one described in the right-hand side of (6) is the logic defined by the class of generalized matrices $\mathsf{K} = \big\{\langle \boldsymbol{A}, \mathcal{F}(\boldsymbol{A})\rangle : \boldsymbol{A} \in \mathsf{QHA}\big\}$. Let us first show that this logic is finitary. We can "split" each generalized matrix in K into its constituent matrices (one for each of the closed sets of the closure system) and obtain the class $\mathsf{K}' = \big\{\langle \boldsymbol{A}, F\rangle : \boldsymbol{A} \in \mathsf{QHA} \text{ and } F \in \mathcal{F}(\boldsymbol{A})\big\}$. The logic defined by this class is the same as the one defined by K. But K' is clearly closed under ultraproducts, because being a quasi-Hilbert algebra is quasi-equational, and being an implicative filter is to be a model of (I) and (MP). Therefore by a well-known theorem (see e. g. [7, Corollary 0.4.6]) the sentential logic defined by the class is finitary. Since $\mathcal{G}_1$ is also finitary, we need only prove (6) for a finite set $\Gamma = \{\varphi_1, \ldots, \varphi_n\}$. By definition

$\varGamma \vdash_{\mathcal{G}_1} \varphi$ if and only if the sequent $\varphi_1, \ldots, \varphi_n \rhd \varphi$ is derivable in $\mathfrak{G}_1$, and by the completeness of every Gentzen system with respect to the class of all its reduced models, this amounts to saying that $h(\varphi) \in \mathrm{C}\big(h[\varGamma]\big)$ for every reduced model $\langle \boldsymbol{A}, \mathcal{C}\rangle$ of $\mathfrak{G}_1$ and every $h \in \mathrm{Hom}(\boldsymbol{Fm}, \boldsymbol{A})$. Theorem 2.4 characterizes all such reduced models, but there can be several ones on the same algebra. However, as we are taking all of them, it is enough to take on each algebra the one with the least closure operator, that is, with the largest closure system, which is $\mathcal{F}(\boldsymbol{A})$. Thus the above amounts to saying that $h(\varphi) \in \mathrm{F}\big(h[\varGamma]\big)$ for every $\boldsymbol{A} \in \mathsf{QHA}$ and every $h \in \mathrm{Hom}(\boldsymbol{Fm}, \boldsymbol{A})$, as was to be proved. □

We end the paper by refuting a conjecture mentioned just before Theorem 1.4: Is it true that for any equation $\alpha \approx \beta$ that holds in the variety **HA**, the two quasi-equations summarized in the expression

$$\alpha \approx \top \iff \beta \approx \top \tag{7}$$

do hold in the quasi-variety **QHA** ? It is remarkable (see Lemma 1.6) that this holds true for some of the most characteristic identities of Hilbert algebras (hence the name "quasi-Hilbert algebras"), but in general the answer is negative: the counterexample in the second part of the proof of Theorem 3.16 shows that the quasi-equation

$$t \to \big(x \to (y \to z)\big) \approx \top \Longrightarrow t \to \big(y \to (x \to z)\big) \approx \top$$

does not hold in **QHA**. But even without considering a particular counterexample, a discussion in the context of the theory of algebraizability already tells us that this conjecture cannot hold, and moreover it helps in understanding *why*. Notice that the equation $\alpha \approx \top$ is the translation of a formula α involved in the algebraizability of $\mathcal{H}_1$ with respect to the class **QHA** (Theorem 3.15). According to this theory [1, 7, 12], the quasi-equation (7) holds in **QHA** if and only if $\alpha \dashv\vdash_{\mathcal{H}_1} \beta$. On the other hand, $\mathcal{IPL}_{\to}$ is also algebraizable with the same translations with respect to **HA**, and the equation $\alpha \approx \beta$ holds in **HA** if and only if $\vdash_{\mathcal{IPL}_{\to}} \alpha \to \beta$ and $\vdash_{\mathcal{IPL}_{\to}} \beta \to \alpha$, but since $\mathcal{IPL}_{\to}$ satisfies (DT), this amounts to $\alpha \dashv\vdash_{\mathcal{IPL}_{\to}} \beta$. Thus the truth of the conjecture would imply that the two logics have the same interderivability relation, and this is certainly not the case, as $\mathcal{IPL}_{\to}$ is selfextensional while $\mathcal{H}_1$ is not.

4 Conclusions and open problems

We have examined the effects of putting restrictions on the cardinality of the set of side assumptions in the Deduction Theorem, viewed as a Gentzen-style rule, and of adding additional assumptions to Modus Ponens (inside its formulas), viewed as a Hilbert-style rule. As a result, a denumerable collection of new Gentzen systems and two new sentential logics have been isolated. These logics are weaker than the positive implicative logic. We have determined their algebraic models and the relationships between them, and have classified them according to several standard criteria of abstract algebraic logic. In passing we have found new, alternative presentations of positive implicative logic and have characterized it in terms of the restricted Deduction Theorem: it is the weakest logic satisfying Modus Ponens and the Deduction Theorem with at most 2 side formulas.

This work has lead to the class **QHA** of quasi-Hilbert algebras, a quasi-variety of implicative algebras larger than the variety of Hilbert algebras. Its algebraic properties reflect those of the corresponding logics and Gentzen systems. The main open question about it is whether this class is a proper quasi-variety, i. e., whether it is not a variety. Using Theorem 1.8 one can show that **QHA** is a variety if and only if for every $\boldsymbol{A} \in \mathsf{QHA}$ and every $\theta \in \mathrm{Co}\boldsymbol{A}$, the algebra $\boldsymbol{A}/\theta$ satisfies (mp2). Corollary 2.7 gives an equivalent "logical" formulation of the problem: **QHA** is a variety if and only if for every $\boldsymbol{A} \in \mathsf{QHA}$ and every $\theta \in \mathrm{Co}\boldsymbol{A}$, the generalized matrix $\big\langle \boldsymbol{A}/\theta, \mathcal{F}(\boldsymbol{A}/\theta)\big\rangle$ is a model of the rule (DT1). An interesting problem from quite a different context is to investigate whether the logic $\mathcal{H}_1$ is the "strong version" of $\mathcal{G}_1$ in the sense of [11], which in this case amounts to asking whether a $\mathcal{G}_1$-filter on an algebra in $\mathbb{V}(\mathsf{QHA})$ is Leibniz if and only if it is closed under the rule (MP2). As a problem, this turns out to be equivalent to the first one: using non-trivial results from [11] it is possible to show that a positive answer here implies that **QHA** is a variety, and conversely.

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