Reasoning about coherent conditional probability in the logic FCP($\boldsymbol{Ł\Pi}$)

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Abstract. Very recently, a (fuzzy modal) logic to reason about coherent conditional probability, in the sense of de Finetti, has been introduced by the authors. Under this approach, a conditional probability $\mu(\cdot \mid \cdot)$ is taken as a primitive notion that applies over conditional events of the form "φ *given* ψ", $\varphi|\psi$ for short, where ψ is not the impossible event. The logic, called FCP($Ł\Pi$), exploits an idea already used by Hájek and colleagues to define a logic for (unconditional) probability in the frame of fuzzy logics. Namely, we take the probability of the conditional event "$\varphi|\psi$" as the truth-value of the (fuzzy) modal proposition $P(\varphi \mid \psi)$, read as "$\varphi|\psi$ is probable". The logic FCP($Ł\Pi$), which is built up over the many-valued logic $Ł\Pi\frac{1}{2}$ (a logic which combines the well-known Lukasiewicz and Product fuzzy logics), is shown to be complete for modal theories with respect to the class of probabilistic Kripke structures induced by coherent conditional probabilities. Indeed, checking coherence of a (generalized) probability assessment to an arbitrary family of conditional events becomes tantamount to checking consistency of a suitable defined theory over the logic FCP($Ł\Pi$). In this paper we review and provide further results for the logic FCP($Ł\Pi$). In particular, we extend the previous completeness result when we allow the presence of non-modal formulas in the theories, which are used to describe logical relationships among events. This increases the knowledge modelling power of FCP($Ł\Pi$). Moreover we also show compactness results for our logic. Finally, FCP($Ł\Pi$) is shown to be a powerful tool for knowledge representation. Indeed, following ideas already investigated in the related literature, we show how FCP($Ł\Pi$) allows for the definition of suitable notions of default rules which enjoy the core properties of nonmonotonic reasoning characterizing system **P** and **R**.

1 Introduction: conditional probability and fuzzy logic

Probability theory is certainly the most well-known and deeply investigated formalism between those that aim at modelling reasoning under uncertainty. Such a research has had a remarkable influence also in the field of logic. Indeed, many logics which allow reasoning about probability have been proposed, some of them rather early. We may cite [2, 9, 10, 13, 14, 20, 23, 28–34] as some of the most relevant references. Besides, it is worth mentioning the recent book [21] by Halpern, where a deep investigation of uncertainty (not only probability) representations and uncertainty logics is presented. In general, all the above logical formalisms present some kind of probabilistic operators but all of them, with the exception of [14], are based on the classical two valued-logic.

An alternative treatment, originally proposed in [18] and further elaborated in [17] and in [16], allows the axiomatization of uncertainty logics in the framework of fuzzy logic. The basic idea is to consider, for each classical (two-valued) proposition φ, a (fuzzy) modal proposition $P\varphi$ which reads "φ is probable" and taking as truth-degree of $P\varphi$ the probability of φ. Then one can define theories about the $P\varphi$'s over a particular fuzzy logic including, as axioms, formulas corresponding to the basic postulates of probability theory. The advantage of such an approach is that algebraic operations needed to compute with probabilities (or with any other uncertainty model) are embedded in the connectives of the many-valued logical framework, resulting in clear and elegant formalizations.

In reasoning with probability, a crucial issue concerns the notion of *conditional probability*. Traditionally, given a probability measure μ on an algebra of possible worlds W, if the agent observes that the actual world is in $A \subseteq W$, then the updated probability measure $\mu(\cdot \mid A)$, called conditional probability, is defined as $\mu(B \mid A) = \mu(B \cap A)/\mu(A)$, provided that $\mu(A) > 0$. If $\mu(A) = 0$ the conditional probability remains then undefined. This yields both philosophical and logical problems. For instance, in [16] where the logic FP($SL\Pi$) is presented, conditional probability statements are handled by formulas $P(\varphi \mid \psi)$ which denote an abbreviation for $P\psi \rightarrow_\Pi P(\varphi \wedge \psi)$. Such a definition exploits the properties of Product logic implication $\rightarrow_\Pi$, whose truth function behaves like a truncated division:

$$e(\Phi \rightarrow_\Pi \Psi) = \begin{cases} 1, & \text{if } e(\Phi) \leq e(\Psi) \\ e(\Psi)/e(\Phi), & \text{otherwise.} \end{cases}$$

However, with such a logical modelling, whenever the probability of the conditioning event χ is 0, $P(\varphi \mid \chi)$ takes as truth-value 1. Therefore, this yields problems when dealing with zero probabilities.

Two well-known proposals which aim at solving this problem consist in either adopting a non-standard probability approach (where events are measured on the hyper-real interval [0, 1] rather than on to the usual real interval), or in taking conditional probability as a primitive notion. In the first case, the assignment of zero probability is only allowed to impossible events, while other events can take on an infinitesimal probability. This clearly permits to avoid situations in which the conditioning event has null probability. The second approach (that goes back to de Finetti, Rényi and Popper among others) considers conditional probability and conditional events as basic notions, not derived from the notion of unconditional probability, and provides adequate axioms. Coletti and Scozzafava's book [6] includes a rich elaboration of different issues of reasoning with *coherent* conditional probability, i.e. conditional probability in de Finetti's sense. We take from there the following definition.

Definition 1 ([6]). *Let $\mathcal{G}$ be a Boolean algebra and let $\mathcal{B} \subseteq \mathcal{G}$ be closed with respect to finite unions (additive set). Let $\mathcal{B}^0 = \mathcal{B}\backslash\{\emptyset\}$. A* conditional probability *on the set $\mathcal{G}\times\mathcal{B}^0$ of conditional events, denoted as $E|H$, is a function $\mu : \mathcal{G}\times\mathcal{B}^0 \to [0, 1]$ satisfying the following axioms:*

(i) $\mu(H \mid H) = 1$*, for all* $H \in \mathcal{B}^0$
(ii) $\mu(\cdot \mid H)$ *is a (finitely additive) probability on* $\mathcal{G}$ *for any given* $H \in \mathcal{B}^0$
(iii) $\mu(E\cap A \mid H) = \mu(E \mid H)\cdot\mu(A \mid E\cap H)$*, for all* $A \in \mathcal{G}$ *and* $E, H, E\cap H \in \mathcal{B}^0$*.*

Two different logical treatments based respectively on the above two solutions have been recently proposed, both using a fuzzy logic approach as the one mentioned above. In fact, they both define probability logics over the fuzzy logic $L\Pi\frac{1}{2}$, which combines the well-known Lukasiewicz and Product fuzzy logics. In [11] Flaminio and Montagna introduce the logic FP($SL\Pi$) whose models include non-standard probabilities. On the other hand, we defined in [27] the logic FCP($L\Pi$) in whose models conditional probability is a primitive notion. In this logic a modal operator P which directly applies to conditional events of the form $\varphi|\chi$ is introduced. Unconditional probability, then, arises as non-primitive whenever the conditioning event is a (classical) tautology. The obvious reading of a statement like $P(\varphi \mid \chi)$ is "the conditional event "φ given χ" is probable". Similarly to the case mentioned above, the truth-value of $P(\varphi \mid \chi)$ will be given by the conditional probability $\mu(\varphi \mid \chi)$. In [27], FCP($L\Pi$) is shown to be complete with respect to a class of Kripke structures suitably equipped with a conditional probability. Moreover, it is also shown that checking the *coherence* of an assessment to a family of conditional events, in the sense of de Finetti, Coletti and Scozzafava[1], is tantamount to checking consistency of a suitable defined theory in FCP($L\Pi$).

In the first part of this paper (Sections 2, 3 and 5) we review the main definitions and results concerning FCP($L\Pi$), while in the second part we extend some of those previous results, and also present some new results and perspectives. Indeed, in [27], strong completeness was proved with respect to modal theories, i.e. theories only including modal (probabilistic) formulas. Although they are the most interesting kind of formulas, this clearly restricted the type of deductions allowed. In Section 4, we provide strong completeness for general theories as well, i.e. theories including both modal and non-modal formulas. Moreover, in Section 6 we generalize a result obtained by Flaminio [12] on the compactness of our logic for coherent assessments, and we discuss a different approach to obtain similar compactness results. To conclude, in Section 7 we show that FCP($L\Pi$) is a powerful tool from the knowledge representation point of view. Indeed, many complex statements, both quantitaive and qualitative, concerning

[1] Roughly speaking, an assessment to an arbitrary family of conditional events is called coherent when it can be extended to a whole conditional probability [6].

conditional probabilities can be represented, as well as suitable notions of default rules which capture the core properties of nonmonotonic reasoning carved in system P and in some extension.

2 Preliminaries: the $Ł\Pi\frac{1}{2}$ logic

The language of the $Ł\Pi$ logic is built in the usual way from a countable set of propositional variables, three binary connectives $\to_L$ (Łukasiewicz implication), $\odot$ (Product conjunction) and $\to_\Pi$ (Product implication), and the truth constant $\bar{0}$. A truth-evaluation is a mapping e that assigns to every propositional variable a real number from the unit interval [0, 1] and extends to all formulas as follows:

$$e(\bar{0}) = 0, \qquad e(\varphi \to_L \psi) = \min(1 - e(\varphi) + e(\psi), 1),$$
$$e(\varphi \odot \psi) = e(\varphi) \cdot e(\psi), \quad e(\varphi \to_\Pi \psi) = \begin{cases} 1, & \text{if } e(\varphi) \le e(\psi) \\ e(\psi)/e(\varphi), & \text{otherwise} \end{cases}.$$

The truth constant $\overline{1}$ is defined as $\varphi \to_L \varphi$. In this way we have $e(\overline{1}) = 1$ for any truth-evaluation e. Moreover, many other connectives can be defined from those introduced above:

$$\begin{array}{ll}
\neg_L\varphi \text{ is } \varphi \to_L \bar{0}, & \neg_\Pi\varphi \text{ is } \varphi \to_\Pi \bar{0}, \\
\varphi \wedge \psi \text{ is } \varphi \& (\varphi \to_L \psi), & \varphi \vee \psi \text{ is } \neg_L(\neg_L\varphi \wedge \neg_L\psi), \\
\varphi \oplus \psi \text{ is } \neg_L\varphi \to_L \psi, & \varphi \& \psi \text{ is } \neg_L(\neg_L\varphi \oplus \neg_L\psi), \\
\varphi \ominus \psi \text{ is } \varphi \& \neg_L\psi, & \varphi \equiv \psi \text{ is } (\varphi \to_L \psi)\&(\psi \to_L \varphi), \\
\Delta\varphi \text{ is } \neg_\Pi\neg_L\varphi, & \nabla\varphi \text{ is } \neg_\Pi\neg_\Pi\varphi,
\end{array}$$

with the following interpretations:

$$\begin{array}{ll}
e(\neg_L\varphi) = 1 - e(\varphi), & e(\neg_\Pi\varphi) = \begin{cases} 1, \text{ if } e(\varphi) = 0 \\ 0, \text{ otherwise} \end{cases}, \\
e(\varphi \wedge \psi) = \min(e(\varphi), e(\psi)), & e(\varphi \vee \psi) = \max(e(\varphi), e(\psi)), \\
e(\varphi \oplus \psi) = \min(1, e(\varphi) + e(\psi)), & e(\varphi \& \psi) = \max(0, e(\varphi) + e(\psi) - 1), \\
e(\varphi \ominus \psi) = \max(0, e(\varphi) - e(\psi)), & e(\varphi \equiv \psi) = 1 - |e(\varphi) - e(\psi)|, \\
e(\Delta\varphi) = \begin{cases} 1, \text{ if } e(\varphi) = 1 \\ 0, \text{ otherwise} \end{cases}, & e(\nabla\varphi) = \begin{cases} 1, \text{ if } e(\varphi) > 0 \\ 0, \text{ otherwise} \end{cases}.
\end{array}$$

The logic $Ł\Pi$ is defined Hilbert-style as the logical system whose axioms and rules are the following[2]:

(i) Axioms of Łukasiewicz Logic:
 (Ł1) $\varphi \to_L (\psi \to_L \varphi)$
 (Ł2) $(\varphi \to_L \psi) \to_L ((\psi \to_L \chi) \to_L (\varphi \to_L \chi))$
 (Ł3) $(\neg_L\varphi \to_L \neg_L\psi) \to_L (\psi \to_L \varphi)$
 (Ł4) $((\varphi \to_L \psi) \to_L \psi) \to_L ((\psi \to_L \varphi) \to_L \varphi)$
(ii) Axioms of Product Logic[3]:

[2] This definition, proposed in [4], is actually a simplified version of the original definition of $Ł\Pi$ given in [7].

[3] Actually Product logic axioms also include axiom A7 $[\bar{0} \to_\Pi \varphi]$ which is redundant in $Ł\Pi$.

(A1) $(\varphi \rightarrow_\Pi \psi) \rightarrow_\Pi ((\psi \rightarrow_\Pi \chi) \rightarrow_\Pi (\varphi \rightarrow_\Pi \chi))$
(A2) $(\varphi \odot \psi) \rightarrow_\Pi \varphi$
(A3) $(\varphi \odot \psi) \rightarrow_\Pi (\psi \odot \varphi)$
(A4) $(\varphi \odot (\varphi \rightarrow_\Pi \psi) \rightarrow_\Pi (\psi \odot (\psi \rightarrow_\Pi \varphi))$
(A5a) $(\varphi \rightarrow_\Pi (\psi \rightarrow_\Pi \chi)) \rightarrow_\Pi ((\varphi \odot \psi) \rightarrow_\Pi \chi)$
(A5b) $((\varphi \odot \psi) \rightarrow_\Pi \chi) \rightarrow_\Pi (\varphi \rightarrow_\Pi (\psi \rightarrow_\Pi \chi))$
(A6) $((\varphi \rightarrow_\Pi \psi) \rightarrow_\Pi \chi) \rightarrow_\Pi (((\psi \rightarrow_\Pi \varphi) \rightarrow_\Pi \chi) \rightarrow_\Pi \chi)$
(Π1) $\neg_\Pi \neg_\Pi \chi \rightarrow_\Pi (((\varphi \odot \chi) \rightarrow_\Pi (\psi \odot \chi)) \rightarrow_\Pi (\varphi \rightarrow_\Pi \psi))$
(Π2) $\varphi \wedge \neg_\Pi \varphi \rightarrow_\Pi \bar{0}$

(iii) The following additional axioms relating Łukasiewicz and Product logic connectives:
($\neg$) $\neg_\Pi \varphi \rightarrow_L \neg_L \varphi$
(Δ) $\Delta(\varphi \rightarrow_L \psi) \equiv \Delta(\varphi \rightarrow_\Pi \psi)$
($L\Pi$) $\varphi \odot (\psi \ominus \chi) \equiv (\varphi \odot \psi) \ominus (\varphi \odot \chi)$

(iv) Deduction rules of $L\Pi$ are modus ponens for $\rightarrow_L$ (modus ponens for $\rightarrow_\Pi$ is derivable), and necessitation for Δ: from φ derive $\Delta\varphi$.

The logic $L\Pi\frac{1}{2}$ is the logic obtained from $L\Pi$ by expanding the language with a propositional variable $\overline{\frac{1}{2}}$ and adding the axiom:

($L\Pi\frac{1}{2}$) $\overline{\frac{1}{2}} \equiv \neg_L \overline{\frac{1}{2}}$

Obviously, a truth-evaluation e for $L\Pi$ is easily extended to an evaluation for $L\Pi\frac{1}{2}$ by further requiring $e(\overline{\frac{1}{2}}) = \frac{1}{2}$.

From the above axiom systems, the notion of proof from a theory (a set of formulas) in both logics, denoted $\vdash_{L\Pi}$ and $\vdash_{L\Pi 1/2}$ respectively, is defined as usual. Strong completeness of both logics for finite theories with respect to the given semantics has been proved in [7]. In what follows we will restrict ourselves to the logic $L\Pi\frac{1}{2}$.

Theorem 1. *For any finite set of formulas T and any formula φ of $L\Pi\frac{1}{2}$, we have $T \vdash_{L\Pi 1/2} \varphi$ iff $e(\varphi) = 1$ for any truth-evaluation e which is a model*[4] *of T.*

As it is also shown in [7], for each rational $r \in [0,1]$ a formula $\overline{r}$ is definable in $L\Pi\frac{1}{2}$ from the truth constant $\overline{\frac{1}{2}}$ and the connectives, so that $e(\overline{r}) = r$ for each evaluation e. Therefore, in the language of $L\Pi\frac{1}{2}$ we have a truth constant for each rational in [0, 1], and due to completeness of $L\Pi\frac{1}{2}$, the following book-keeping axioms for rational truth constants are provable:

($RL\Pi$1) $\neg_L \overline{r} \equiv \overline{1-r}$, ($RL\Pi$2) $\overline{r} \rightarrow_L \overline{s} \equiv \overline{\min(1, 1-r+s)}$,
($RL\Pi$3) $\overline{r} \odot \overline{s} \equiv \overline{r \cdot s}$, ($RL\Pi$4) $\overline{r} \rightarrow_\Pi \overline{s} \equiv \overline{r \Rightarrow_P s}$,

where $r \Rightarrow_P s = 1$ if $r \leq s$, $r \Rightarrow_P s = s/r$ otherwise.

[4] We say that an evaluation e is a *model* of a theory T whenever $e(\psi) = 1$ for each $\psi \in T$.

3 A logic of conditional probability

In this section we describe the fuzzy modal logic FCP($Ł\Pi$) —FCP for Fuzzy Conditional Probability—, built up over the many-valued logic $Ł\Pi\frac{1}{2}$ described in the previous section. The language of FCP($Ł\Pi$) is defined in two steps:

Non-modal formulas: they are built from a set V of propositional variables $\{p_1, p_2, \ldots p_n, \ldots\}$ using the classical binary connectives $\wedge$ and $\neg$. Other connectives like $\vee$, $\rightarrow$ and $\leftrightarrow$ are defined from $\wedge$ and $\neg$ in the usual way. Non-modal formulas (we will also refer to them as Boolean propositions) will be denoted by lower case Greek letters φ, ψ, etc. The set of non-modal formulas will be denoted by $\mathcal{L}$.

Modal formulas: they are built from elementary modal formulas of the form $P(\varphi \mid \chi)$, where φ and χ are non-modal formulas, using the connectives of $Ł\Pi$ ($\rightarrow_L$, $\odot$, $\rightarrow_\Pi$) and the truth constants $\overline{r}$, for each rational $r \in [0, 1]$. We shall denote them by upper case Greek letters Φ, Ψ, etc. Notice that we do not allow nested modalities.

Definition 2. *The axioms of the logic FCP($Ł\Pi$) are the following:*

(i) Axioms of Classical propositional Logic for non-modal formulas
(ii) Axioms of $Ł\Pi\frac{1}{2}$ for modal formulas
(iii) Probabilistic modal axioms:

(FCP1) $P(\varphi \rightarrow \psi \mid \chi) \rightarrow_L (P(\varphi \mid \chi) \rightarrow_L P(\psi \mid \chi))$
(FCP2) $P(\neg\varphi \mid \chi) \equiv \neg_L P(\varphi \mid \chi)$
(FCP3) $P(\varphi \vee \psi \mid \chi) \equiv ((P(\varphi \mid \chi) \rightarrow_L P(\varphi \wedge \psi \mid \chi)) \rightarrow_L P(\psi \mid \chi)$
(FCP4) $P(\varphi \wedge \psi \mid \chi) \equiv P(\psi \mid \varphi \wedge \chi) \odot P(\varphi \mid \chi)$
(FCP5) $P(\chi \mid \chi)$

Deduction rules of FCP($Ł\Pi$) are those of $Ł\Pi$ (i.e. modus ponens *and* necessitation *for Δ), plus:*

(iv) necessitation *for P: from φ derive $P(\varphi \mid \chi)$*
(v) substitution of equivalents *for the conditioning event: from $\chi \leftrightarrow \chi'$, derive $P(\varphi \mid \chi) \equiv P(\varphi \mid \chi')$*

The notion of proof is defined as usual. We will denote that in FCP($Ł\Pi$) a formula Φ follows from a theory (set of formulas) T by $T \vdash_{FCP} \Phi$. The only remark is that the rule of necessitation for $P(\cdot \mid \chi)$ can only be applied to Boolean theorems.

The semantics for FCP($Ł\Pi$) is given by *conditional probability Kripke structures* $K = \langle W, \mathcal{U}, e, \mu\rangle$, where:

- W is a non-empty set of possible worlds.
- $e : V \times W \rightarrow \{0, 1\}$ provides for each world a *Boolean* (two-valued) evaluation of the propositional variables, that is, $e(p, w) \in \{0, 1\}$ for each propositional variable $p \in V$ and each world $w \in W$. A truth-evaluation $e(\cdot, w)$ is extended to Boolean propositions as usual. For a Boolean formula φ, we will write $[\varphi]_W = \{w \in W \mid e(\varphi, w) = 1\}$.

- $\mu : \mathcal{U} \times \mathcal{U}^0 \to [0,1]$ is a conditional probability over a Boolean algebra $\mathcal{U}$ of subsets of W[5] where $\mathcal{U}^0 = \mathcal{U} \setminus \{\emptyset\}$, and such that $([\varphi]_W, [\chi]_W)$ is μ-measurable for any non-modal φ and χ (with $[\chi]_W \neq \emptyset$).
- $e(\cdot, w)$ is extended to elementary modal formulas by defining

$$e(P(\varphi \mid \chi), w) = \mu([\varphi]_W \mid [\chi]_W)^{6},$$

and to arbitrary modal formulas according to $Ł\Pi\frac{1}{2}$ semantics, that is:

$$\begin{aligned} e(\overline{r}, w) &= r, \\ e(\Phi \to_L \Psi, w) &= \min(1 - e(\Phi, w) + e(\Psi, w), 1), \\ e(\Phi \odot \Psi, w) &= e(\Phi, w) \cdot e(\Psi, w), \\ e(\Phi \to_\Pi \Psi, w) &= \begin{cases} 1, & \text{if } e(\Phi, w) \leq e(\Psi, w) \\ e(\Psi, w)/e(\Phi, w), & \text{otherwise} \end{cases} . \end{aligned}$$

Notice that if Φ is a modal formula the truth-evaluations $e(\Phi, w)$ depend only on the conditional probability measure μ and not on the particular world w.

The truth-degree of a formula Φ in a conditional probability Kripke structure $K = \langle W, \mathcal{U}, e, \mu \rangle$, written $\|\Phi\|^K$, is defined as

$$\|\Phi\|^K = \inf_{w \in W} e(\Phi, w).$$

When $\|\Phi\|^K = 1$ we will say that Φ is valid in K or that K is a model for Φ, and it will be also written $K \models \Phi$. Let T be a set of formulas. Then we say that K is a model of T if $K \models \Phi$ for all $\Phi \in T$. Now let $\mathcal{M}$ be a class of conditional probability Kripke structures. Then we define the truth-degree $\|\Phi\|_T^{\mathcal{M}}$ of a formula in a theory T relative to the class $\mathcal{M}$ as

$$\|\Phi\|_T^{\mathcal{M}} = \inf\{\|\Phi\|^K \mid K \in \mathcal{M},\ K \text{ being a model of } T\} .$$

The notion of logical entailment relative to the class $\mathcal{M}$, written $\models_{\mathcal{M}}$, is then defined as follows:

$$T \models_{\mathcal{M}} \Phi \text{ iff } \|\Phi\|_T^{\mathcal{M}} = 1 .$$

That is, Φ logically follows from a set of formulas T if every structure of $\mathcal{M}$ which is a model of T also is a model of Φ. If $\mathcal{M}$ denotes the whole class of conditional probability Kripke structures we shall write $T \models_{FCP} \Phi$ and $\|\Phi\|_T^{FCP}$.

It is easy to check that axioms FCP1-FCP5 are valid formulas in the class of all conditional probability Kripke structures. Moreover, the inference rule of substitution of equivalents preserves truth in a model, while the necessitation rule for P preserves validity in a model. Therefore we have the following soundness result.

Lemma 1. (Soundness) *The logic FCP(ŁΠ) is sound with respect to the class of conditional probability Kripke structures.*

[5] Notice that in our definition the factors of the Cartesian product are the same Boolean algebra. This is clearly a special case of what stated in Definition 1.

[6] When $[\chi]_W = \emptyset$, we define $e(P(\varphi \mid \chi), w) = 1$.

For any $\varphi, \psi \in \mathcal{L}$, define $\varphi \sim \psi$ iff $\vdash \varphi \leftrightarrow \psi$ in classical logic. The relation $\sim$ is an equivalence relation in the crisp language $\mathcal{L}$ and $[\varphi]$ will denote the equivalence class of φ, containing the propositions provably equivalent to φ. Obviously, the quotient set $\mathcal{L}/_\sim$ forms a Boolean algebra which is isomorphic to a subalgebra $\mathbf{B}(\Omega)$ of the power set of the set Ω of Boolean interpretations of the crisp language $\mathcal{L}$[7]. For each $\varphi \in \mathcal{L}$, we shall identify the equivalence class $[\varphi]$ with the set $\{\omega \in \Omega \mid \omega(\varphi) = 1\} \in \mathbf{B}(\Omega)$ of interpretations that make φ true. We shall denote by $\mathcal{CP}(\mathcal{L})$ the set of conditional probabilities over $\mathcal{L}/_{\sim_{FCP}} \times (\mathcal{L}/_{\sim_{FCP}} \setminus [\bot])$ or equivalently on $\mathbf{B}(\Omega) \times \mathbf{B}(\Omega)^0$.

Notice that each conditional probability $\mu \in \mathcal{CP}(\mathcal{L})$ induces a conditional probability Kripke structure $\langle \Omega, \mathbf{B}(\Omega), e_\mu, \mu\rangle$ where $e_\mu(p, \omega) = \omega(p) \in \{0, 1\}$ for each $\omega \in \Omega$ and each propositional variable p. We shall denote by $\mathcal{CPS}$ the class of Kripke structures induced by conditional probabilities $\mu \in \mathcal{CP}(\mathcal{L})$, i.e. $\mathcal{CPS} = \{\langle \Omega, \mathbf{B}(\Omega), e_\mu, \mu\rangle \mid \mu \in \mathcal{CP}(\mathcal{L})\}$. Abusing the language, we will say that a conditional probability $\mu \in \mathcal{CP}(\mathcal{L})$ is a *model* of a modal theory T whenever the induced Kripke structure $\Omega_\mu = \langle \Omega, \mathbf{B}(\Omega), e_\mu, \mu\rangle$ is a model of T. Besides, we shall often write $\mu(\varphi \mid \chi)$ actually meaning $\mu([\varphi] \mid [\chi])$. Actually, for many purposes we can restrict ourselves to the class of conditional probability Kripke structures $\mathcal{CPS}$. Indeed, the following lemma can be proved.

Lemma 2. *For each conditional probability Kripke structure $K = \langle W, \mathcal{U}, e, \mu\rangle$ there is a conditional probability $\mu^* : \mathbf{B}(\Omega) \times \mathbf{B}(\Omega)^0 \to [0, 1]$ such that $\|P(\varphi \mid \chi)\|^K = \mu^*(\varphi \mid \chi)$ for all $\varphi, \chi \in \mathcal{L}$ such that $[\chi] \neq \emptyset$. Therefore, it also holds that $\|\Phi\|_T = \|\Phi\|_T^{\mathcal{CPS}}$ for any modal formula Φ and any modal theory T.*

Next theorem states that FCP($Ł\Pi$) is strongly complete for finite modal theories with respect to the intended probabilistic semantics.

Theorem 2. (Strong finite probabilistic completeness of FCP($Ł\Pi$)) *Let T be a finite modal theory over FCP($Ł\Pi$) and Φ a modal formula. Then $T \vdash_{FCP} \Phi$ iff $e_\mu(\Phi) = 1$ for each conditional probability model μ of T.*

Proof. (Sketch)[8] Soundeness is clear. To prove completeness, following [16], the basic idea consists in transforming modal theories over FCP($Ł\Pi$) into theories over $Ł\Pi\frac{1}{2}$ and then take advantage of completeness of $Ł\Pi\frac{1}{2}$.

Define a theory, called $\mathcal{F}$, as follows:
(i) take as propositional variables of the theory variables of the form $f_{\varphi|\chi}$, where φ and χ are classical propositions from $\mathcal{L}$.
(ii) take as axioms of the theory the following ones, for each φ, ψ and χ:
($\mathcal{F}1$) $f_{\varphi|\chi}$, for φ being a classical tautology,
($\mathcal{F}2$) $f_{\varphi|\chi} \equiv f_{\varphi|\chi'}$, for any χ, χ' such that $\chi \leftrightarrow \chi'$ is a tautology,
($\mathcal{F}3$) $f_{\varphi\to\psi|\chi} \to_L (f_{\varphi|\chi} \to_L f_{\psi|\chi})$,

[7] Actually, $\mathbf{B}(\Omega) = \{\{\omega \in \Omega \mid \omega(\varphi) = 1\} \mid \varphi \in \mathcal{L}\}$. Needless to say, if the language has only finitely many propositional variables then the algebra $\mathbf{B}(\Omega)$ is just the whole power set of Ω, otherwise it is a strict subalgebra.
[8] For a full proof, please consult [27].

($\mathcal{F}4$) $f_{\neg\varphi|\chi} \equiv \neg_L f_{\varphi|\chi}$,
($\mathcal{F}5$) $f_{\varphi\vee\psi|\chi} \equiv [(f_{\varphi|\chi} \rightarrow_L f_{\varphi\wedge\psi|\chi}) \rightarrow_L f_{\psi|\chi}]$,
($\mathcal{F}6$) $f_{\varphi\wedge\psi|\chi} \equiv f_{\psi|\varphi\wedge\chi} \odot f_{\varphi|\chi}$,
($\mathcal{F}7$) $f_{\varphi|\varphi}$.

Then define the mapping $*$ from modal formulas to $L\Pi\frac{1}{2}$-formulas as follows: $(P(\varphi \mid \chi))^* = f_{\varphi|\chi}$, $\overline{r}^* = \overline{r}$, $(\Phi \circ \Psi)^* = \Phi^* \circ \Psi^*$, for $\circ \in \{\rightarrow_L, \odot, \rightarrow_\Pi\}$. Let us denote by T^* the set of all formulas translated from T. First, by the construction of $\mathcal{F}$, one can check that for any Φ,

$$T \vdash_{FCP} \Phi \text{ iff } T^* \cup \mathcal{F} \vdash_{L\Pi\frac{1}{2}} \Phi^*. \quad (1)$$

Next, one proves that the semantical analogue of (1) also holds, that is,

$$T \models_{FCP} \Phi \text{ iff } T^* \cup \mathcal{F} \models_{L\Pi\frac{1}{2}} \Phi^*. \quad (2)$$

¿From (1) and (2), to prove the theorem it remains to show that

$$T^* \cup \mathcal{F} \vdash_{L\Pi\frac{1}{2}} \Phi^* \text{ iff } T^* \cup \mathcal{F} \models_{L\Pi\frac{1}{2}} \Phi^*.$$

Note that $L\Pi\frac{1}{2}$ is strongly complete but only for finite theories. We have that the initial modal theory T is finite, so is T^*. However $\mathcal{F}$ contains infinitely many instances of axioms $\mathcal{F}1-\mathcal{F}7$. Nonetheless one can prove that such infinitely many instances can be replaced by only finitely many instances, by using propositional normal forms. This completes the proof of theorem.

As an easy consequence of this completeness theorem, we have the following appealing facts for two kinds of deductions that are usually of interest: if T is a finite modal theory over FCP($L\Pi$) and φ and χ are non-modal formulas, with $[\chi] \neq \emptyset$, then
(i) $T \vdash_{FCP} \overline{r} \rightarrow P(\varphi \mid \chi)$ iff $\mu(\varphi \mid \chi) \geq r$ for each cond. prob. μ model of T.
(ii) $T \vdash_{FCP} P(\varphi \mid \chi) \rightarrow \overline{r}$ iff $\mu(\varphi \mid \chi) \leq r$ for each cond. prob. μ model of T.

4 Extended completeness for FCP($L\Pi$)

The completeness result for FCP($L\Pi$) shown in Theorem 2 only considers (finite) modal theories, that is, theories involving only probabilistic formulas. However it is also worth considering theories also including non-modal formulas since they can allow us to take into account logical representations of the relationships between events. For instance, if φ and ψ represent incompatible events, we may want to include in our probabilistic theory the non-modal formula $\neg(\varphi \wedge \psi)$, or if the event ψ is *included* in φ then we may need to include the formula $\psi \rightarrow \varphi$. To prove completeness for these extended theories we need first to prove an extension of Lemma 2.

In what follows, let D be a propositional (non-modal) theory. Then, let $\Omega_D = \{w \in \Omega \mid w(\varphi) = 1, \forall\varphi \in D\}$ be the set of classical interpretations which are model of D. Finally let us denote by $\mathcal{M}_D$ the class of probabilistic Krikpe structures which are models of D, by $\mathcal{CP}(D)$ the class of conditional probabilities defined on $\mathbf{B}(\Omega_D) \times \mathbf{B}(\Omega_D)^0$, and by $\mathcal{CPS}(D)$ the class of probabilistic Krikpe models $\{(\Omega_D, \mathbf{B}(\Omega_D), e_\mu, \mu) \mid \mu \in \mathcal{CP}(D)\}$. Obviously, $\mathcal{CPS}(D) \subset \mathcal{M}_D$.

Lemma 3. *For each propositional theory D and conditional probability Kripke structure $K = \langle W, \mathcal{U}, e, \mu\rangle \in \mathcal{M}_D$, there is a conditional probability $\mu^* \in \mathcal{CP}(D)$ such that $\|P(\varphi \mid \chi)\|^K = \mu^*(\varphi \mid \chi)$ for all $\varphi, \chi \in \mathcal{L}$. Therefore, it also holds that $\|\Phi\|_T^{\mathcal{M}_D} = \|\Phi\|_T^{\mathcal{CPS}(D)}$ for any modal formula Φ and any modal theory T.*

With this preliminary result, one can show that the probabilistic completeness of FCP($Ł\Pi$) with respect to finite modal theories shown in Theorem 2 can be extended in the following terms. The proof is an straightforward extension of the one of that theorem and it is omitted.

Theorem 3. (Extended finite probabilistic completeness of FCP($Ł\Pi$)) *Let T be a finite modal theory over FCP($Ł\Pi$), D a propositional theory and Φ a modal formula. Then $T \cup D \vdash_{FCP} \Phi$ iff $e_\mu(\Phi) = 1$ for each conditional probability $\mu \in \mathcal{CP}(D)$ model of T.*

Analogously to the previous case of modal theories, the following particular cases are of interest. If T is a finite (modal) conditional theory over FCP($Ł\Pi$), D is a propositional (non-modal) theory, and φ and χ are non-modal formulas, then we have:

(i) $T \cup D \vdash \bar{r} \to P(\varphi \mid \chi)$ iff $\mu(\varphi \mid \chi) \geq r$, for each conditional probability $\mu \in \mathcal{CP}(D)$ model of T.

(ii) $T \cup D \vdash P(\varphi \mid \chi) \to \bar{r}$ iff $\mu(\varphi \mid \chi) \geq r$, for each conditional probability $\mu \in \mathcal{CP}(D)$ model of T.

5 Consistency and coherent assessments

One of the most important features of the conditional probability approach developed e.g. in [6] is based on the possibility of reasoning only from partial conditional probability assessments to an arbitrary family of conditional events (without requiring in principle any specific algebraic structure). However, it must be checked whether such assessments minimally agree with the rules of conditional probability. This minimal requiriment is that such an assessment can be extended at least to a proper conditional probability, and it is called *coherence*.

Definition 3 ([6]). *A probabilistic assessment $\{Pr(\varphi_i \mid \chi_i) = \alpha_i\}_{i=1,n}$ over a set of conditional events $\varphi_i \mid \chi_i$ (with χ_i not being a contradiction) is* coherent *if there is a conditional probability μ, in the sense of Definition 1, such that $Pr(\varphi_i \mid \chi_i) = \mu(\varphi_i \mid \chi_i)$ for all $i = 1, \ldots, n$.*

This notion of *coherence* can be alternatively found in the literature in a different form, like in [3], in terms of a betting scheme. An important result by Coletti and Scozzafava is the characterization of the coherence of an assessment in terms of the existence of a suitable nested familiy of probabilities (see e.g. [6] for details).

What we can easily show here is that the notion of coherence of a probabilistic assessment to a set of conditional events is tantamount to the consistency of a suitable defined theory over FCP($Ł\Pi$).

Theorem 4. *Let $\kappa = \{Pr(\varphi_i \mid \chi_i) = \alpha_i : i = 1, \ldots, n\}$ be a rational probabilistic assessment. Then κ is coherent iff the theory $T_\kappa = \{P(\varphi_i \mid \chi_i) \equiv \overline{\alpha}_i : i = 1, \ldots, n\}$ is consistent in FCP(ŁΠ), i.e. $T_\kappa \nvdash_{FCP} \bar{0}$.*

A very similar result is proved by Flaminio and Montagna in [11] in the framework of their logic FP(*SLΠ*). As a corollary, we point out that we can also deal with the coherence of the so-called *generalized* conditional probability assessments [15]. They basically correspond to interval-valued assessments.

Theorem 5. *Let $\kappa^* = \{Pr(\varphi_i \mid \chi_i) \in [\alpha_i, \beta_i] : i = 1, \ldots, n\}$ be a rational generalized probabilistic assessment. Then κ is coherent iff the theory $T_{\kappa^*} = \{\overline{\alpha}_i \to_L P(\varphi_i \mid \chi_i), P(\varphi_i \mid \chi_i) \to_L \overline{\beta}_i : i = 1, \ldots, n\}$ is consistent in FCP(ŁΠ), i.e. iff $T_{\kappa^*} \nvdash_{FCP} \bar{0}$.*

6 Compactness of coherent assessments

Very recently, Flaminio has shown [12] compactness of coherent assessments to conditional events, both under Flaminio and Montagna's probabilistic logic FP(*SLΠ*) and under our logic FCP(*ŁΠ*). In particular, for FCP(*ŁΠ*), they prove the following result.

Theorem 6 (Compactness of consistency, [12]). *Let us consider a modal theory $T = \{P(\varphi_i \mid \psi_i) \equiv \overline{\alpha_i}\}_{i \in I}$ over FCP(ŁΠ). Then T is consistent iff every finite subtheory of T is so.*

In other words, this means the following. Let $\kappa = \{Pr(\varphi_i \mid \psi_i) = \alpha_i\}_{i \in I}$ be a rational assessment of conditional probability over a class of conditional events $\mathcal{C} = \{\text{``}\varphi_i \mid \psi_i\text{''}\}_{i \in I}$. For every finite subset $\mathcal{J}$ of $\mathcal{C}$, let $\kappa_{\upharpoonright \mathcal{J}}$ denote the restriction of κ to $\mathcal{J}$. Then κ is coherent iff for every finite $\mathcal{J} \subseteq \mathcal{C}$, $\kappa_{\upharpoonright \mathcal{J}}$ is coherent.

The proof is based on the well-known theorem of Łos on the ultraproduct model construction. It is shown that, although the ultraproduct of conditional probabilistic Kripke structures is not closed w.r.t. this class of structures, the resulting non-standard conditional probability is such that its standard part defines a conditional probability that makes the job.

Actually, it is not difficult to check that Flaminio's proof also works for more general kinds of theories involving $Ł\Pi\frac{1}{2}$ connectives with continuous truth-functions, i.e., Łukasiewicz connectives and Product conjunction. In fact, if μ^* is a non-standard probability, the standard part of the corresponding truth-evaluation e_{μ^*} over the *continuous* fragment of $Ł\Pi\frac{1}{2}$ (that is, the $\to_\Pi$-free fragment) commutes with the standard part. In other words, if St denotes standard part, then $St(e_{\mu^*}(\Phi \circ \Psi)) = St(e_{\mu^*}(\Phi)) \circ St(e_{\mu^*}(\Psi))$, for $\circ \in \{\&, \to_L, \odot\}$. Thus we have the following more general probabilistic compactness result.

Theorem 7 (compactness). *Let T be a modal theory over FCP(ŁΠ) whose formulas only involve (at most) truth-constants and the $\&, \to_L, \odot$ connectives. Then T is consistent iff every finite subtheory of T is so.*

A parallel result for Flaminio and Montagna's logic FP($SL\Pi$) also holds.

Notice that, in particular, compactness for the so-called *generalized coherence* [15] also holds. This corresponds to state compactness of coherence for interval-valued conditional probability assessments of the kind $\kappa = \{\beta_i \le Pr(\varphi_i \mid \psi_i) \le \alpha_i\}_{i\in I}$, which amounts to compactness for the consistency of the theory $T = \{\overline{\beta_i} \rightarrow_L P(\varphi_i \mid \psi_i), P(\varphi_i \mid \psi_i) \rightarrow_L \overline{\alpha_i}\}_{i\in I}$.

These compactness results directly refer to the probabilistic logics FCP($L\Pi$) and FP($SL\Pi$), but without mentioning a possible similar result for the base logic $L\Pi\frac{1}{2}$. Compactness results for different fuzzy logics are known. In particular Cintula and Navara show compactness for several fuzzy logics in [5], where they comment that the same proof they provide for the compactness of Lukasiewicz logic (originally proved by Butnariu, Klement and Zafrany) also works for other fuzzy logics with connectives interpreted by continuous functions. This is the case again of the *continuous* fragment of $L\Pi\frac{1}{2}$. On the other hand, the completeness proof of FCP($L\Pi$) shows that one can translate a modal theory over FCP($L\Pi$) to a theory over $L\Pi\frac{1}{2}$. Hence, combining these two facts, we also obtain compactness results for modal (probabilistic) theories over FCP($L\Pi$) which do not involve the product implication connective $\rightarrow_\Pi$.

7 Applications to knowledge representation

It is worth pointing out that the logic FCP($L\Pi$) is actually very powerful from a knowledge representation point of view. Indeed, It allows to express several kinds of statements about conditional probability, from purely comparative staments like "the conditional event $\varphi \mid \chi$ is at least as probable than the conditional event $\psi \mid \delta$" as

$$P(\psi \mid \delta) \rightarrow_L P(\varphi \mid \chi),$$

or numerical probability statements like

- "the probability of $\varphi \mid \chi$ is 0.8" as $P(\varphi \mid \chi) \equiv \overline{0.8}$,
- '"the probability of $\varphi \mid \chi$ is at least 0.8" as $\overline{0.8} \rightarrow_L P(\varphi \mid \chi)$,
- "the probability of $\varphi \mid \chi$ is at most 0.8" as $P(\varphi \mid \chi) \rightarrow_L \overline{0.8}$,
- "$\varphi \mid \chi$ has positive probability" as $\neg_\Pi \neg_\Pi P(\varphi \mid \chi)$,

or even staments about *independence*, like "φ and ψ are independent given χ" as

$$P(\varphi \mid \chi \wedge \psi) \equiv P(\varphi \mid \chi).$$

Another interesting issue is the possibility of modelling default reasoning by means of conditional events and probabilities. This has been largely explored in the literature. Actually, from a semantical point of view, the logical framework that FCP($L\Pi$) offers is very close to the so-called *model-theoretic probabilistic logic* in the sense of Biazzo et al's approach [3], and the links established there to probabilistic reasoning under coherence and default reasoning[9]. Actually, FCP($L\Pi$) can provide a (syntactical) deductive system for such a rich

[9] See also [30] for another recent probabilistic logic approach to model defaults.

framework. Here, as an example, and following previous works (e.g. [3, 15, 26, 30]) we show how one can easily define over FCP($Ł\Pi$) a notion of default rule and default entailment using the deduction machinery of FCP($Ł\Pi$). Actually we will show two notions, one corresponding to the well-known System P, and the other to the Rational consequence relation.

We interpret a default $\chi \rightsquigarrow \varphi$ as a conditional object $\varphi|\chi$ with probability 1. Therefore such a default rule is modelled in FCP($Ł\Pi$) just by the formula $P(\varphi \mid \chi)$. Given a theory T over FCP($Ł\Pi$) we can define when T entails or validates a default, and hence define a consequence relation $\rightsquigarrow_T$ on non-modal formulas:

$$\chi \rightsquigarrow_T \varphi \text{ iff } T \vdash_{FCP} P(\varphi \mid \chi).$$

that is, iff for any Kripke probabilistic structure $K = \langle W, \mathcal{U}, e, \mu\rangle$ model of T we have $\mu([\varphi]_W \mid [\chi]_W) = 1$ (assuming $[\chi]_W \neq \emptyset$). Recall that $\rightsquigarrow_T$ can be a non-monotonic consequence relation due to the possibility of coherent conditional probabilities of assigning zero probabilities to the conditioning events. As it is well-known $\rightsquigarrow_T$ is a preferential relation (see e.g. [25]), i.e. it satisfies the following well-known properties characterizing system **P**:

1. **Reflexivity:** $\varphi \rightsquigarrow_T \varphi$
2. **Left logical equivalence:** if $\vdash \varphi \leftrightarrow \psi$ and $\varphi \rightsquigarrow_T \chi$ then $\psi \rightsquigarrow_T \chi$
3. **Right weakening:** if $\vdash \varphi \rightarrow \psi$ and $\chi \rightsquigarrow_T \varphi$ then $\chi \rightsquigarrow_T \psi$.
4. **And:** if $\varphi \rightsquigarrow_T \psi$ and $\varphi \rightsquigarrow_T \chi$ then $\varphi \rightsquigarrow_T \psi \wedge \chi$.
5. **Cautious Monotonicity:** if $\varphi \rightsquigarrow_T \psi$ and $\varphi \rightsquigarrow_T \chi$ then $\varphi \wedge \psi \rightsquigarrow_T \chi$.
6. **Or:** if $\varphi \rightsquigarrow_T \psi$ and $\chi \rightsquigarrow_T \psi$ then $\varphi \vee \chi \rightsquigarrow_T \psi$.

These properties are in turn due to the fact that the next derivabilities hold in FCP($Ł\Pi$):

1. $\vdash_{FCP} P(\varphi \mid \varphi)$
2. $\varphi \leftrightarrow \psi \vdash_{FCP} P(\chi \mid \varphi) \rightarrow_L P(\chi \mid \psi)$
3. $\varphi \rightarrow \psi \vdash_{FCP} P(\varphi \mid \chi) \rightarrow_L P(\psi \mid \chi)$
4. $\vdash_{FCP} P(\psi \mid \varphi) \& P(\chi \mid \varphi) \rightarrow_L P(\psi \wedge \chi \mid \varphi)$
5. $\vdash_{FCP} P(\psi \mid \varphi) \& P(\chi \mid \varphi) \rightarrow_L P(\chi \mid \varphi \wedge \psi)$
6. $\{P(\chi \mid \varphi), P(\chi \mid \psi)\} \vdash_{FCP} P(\chi \mid \varphi \vee \psi)$

Following this approach we can model entailment from a default theory (or conditional base) as follows. A conditional base $KB = \{\varphi_i \rightsquigarrow \chi_i\}_{i \in I}$ is taken to correspond to a theory $T_{KB} = \{P(\varphi_i \mid \chi_i)\}_{i \in I}$ over FCP($Ł\Pi$). Notice that in this coherent probability framework, $\chi \rightsquigarrow \varphi$ is not equivalent to $\chi \rightarrow \varphi$, the latter corresponding to a *strict* rule.

Definition 4 (Default entailment). *A default $\delta \rightsquigarrow \psi$ follows from a default theory $KB = \{\varphi_i \rightsquigarrow \chi_i\}$, writen $KB \vdash^* \delta \rightsquigarrow \psi$, if $T_{KB} \vdash_{FCP} P(\psi \mid \delta)$*

It is clear then that the above described six properties defining system **P**, taken as inference rules on defaults, are sound with respect this default entailment relation $\vdash^*$.

Following Lehmann and Magidor's ideas [25], we can also define *rational* consequence relations with coherent probabilistis semantics (actually, in [25] they use non-standard probabilistic models). So as to do this we need to fix a single probabilistic Kripke structure $K = \langle W, \mathcal{U}, \mu, e \rangle$ and then define the following consequence relation $\rightsquigarrow_K$ on propositional (non-modal) formulas:

$$\varphi \rightsquigarrow_K \psi \text{ iff } K \models P(\psi \mid \varphi),$$

or equivalently, iff $\mu([\psi]_W \mid [\varphi]_W) = 1$ (assuming $[\varphi]_M \neq \emptyset$). This consequence relation can be easily shown to be also a preferential relation, but moreover it can be shown to satisfy the further rational property:

7. **Rational Monotonicity:** if $\varphi \rightsquigarrow_K \psi$ and $\varphi \not\rightsquigarrow_K \neg\chi$ then $\varphi \wedge \chi \rightsquigarrow_K \psi$,

where the notation $\varphi \not\rightsquigarrow_K \psi$ means that the pair (φ, ψ) is not in the consequence relation $\rightsquigarrow_K$, i.e. that $K \not\models P(\psi \mid \varphi)$, i.e. that $\mu(\psi \mid \varphi) < 1$. This is a consequence of the validity of the following derivation in FCP($Ł\Pi$): $\{P(\psi \mid \varphi), \neg\Delta P(\neg\chi \mid \varphi)\} \vdash_{FCP} P(\psi \mid \varphi \wedge \chi)$.

8 Conclusions

In this paper we have investigated several aspects of the fuzzy modal logic FCP($Ł\Pi$) which allows reasoning about coherent conditional probability in the sense of de Finetti. To conclude, we would like to point out some open problems which deserve further investigations.

First, it remains to be studied whether we could use logics weaker than $Ł\Pi\frac{1}{2}$. Indeed, we could define, in the same way as we have done, a conditional probability logic over $Ł\Pi$ (i.e. without rational truth-constants in the language), yielding a kind of *qualitative* probability logic where we could reason for instance about comparative and conditional probability independence statements. Notice that a notion of φ being probable when φ is more probable than $\neg\varphi$, as considered in [22], could also still be defined in such a logic as the formula $\nabla(P(\varphi \mid \top) \ominus P(\neg\varphi \mid \top))$.

Second, we plan to study in more detail the links among FCP($Ł\Pi$) and different kinds of probabilistic nonmonotonic consequence relations as those defined in [26]. Also posible links to the very recent work by Arló-Costa and Parikh [1] on conditional probability and deafeasible reasoning deserve attention.

Finally, it would be worth exploring the definition in the framework of FCP($Ł\Pi$) of logics to reason about conditional objects in other uncertainty theories, like possibility theory or plausibility measures.

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