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Equational Characterization of the Subvarieties of BL Generated by t-norm Algebras

Abstract. In this paper we show that the subvarieties of **BL**, the variety of BL-algebras, generated by single BL-chains on [0, 1], determined by continuous t-norms, are finitely axiomatizable. An algorithm to check the subsethood relation between these subvarieties is provided, as well as another procedure to effectively find the equations of each subvariety. From a logical point of view, the latter corresponds to find the axiomatization of every residuated many-valued calculus defined by a continuous t-norm and its residuum. Actually, the paper proves the results for a more general class than t-norm BL-chains, the so-called regular BL-chains.

Keywords: BL-algebras, t-norm algebras, varieties, axiomatization

1. Introduction

Original Zadeh's functional definition for the intersection and union of fuzzy sets [31] used min and max respectively to combine the membership degrees (belonging to the unit real interval [0, 1]). But very soon these operations were generalized by t-norms and t-conorms respectively (see, for example [3]), and today they are widely used in Fuzzy Set theory. According to Zadeh [32], fuzzy logic in narrow sense is a logical system, which is an extension of many-valued logic, aiming at formalizing approximate reasoning. Building on this idea, scholars like Hájek (see the introductory chapter in [21]), Novák et al. [28], Gottwald [20] and others consider the core of fuzzy logic as a family of residuated many-valued logical calculi with truth values on the real unit interval [0, 1], and with min, max, a t-norm $*$ and its residuum $\rightarrow$ as basic truth functions interpreting lattice meet and joint connectives (additive "and" and "or"), a strong conjunction (multiplicative "and") and the implication, respectively.

Chronologically, the first studied residuated many-valued logics were Lukasiewicz's infinite-valued logic and Gödel's many-valued logic (also known as Dummet's logic), defined many years before the introduction of

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fuzzy logic. Both logics were proved to be complete with respect to the many-valued calculi defined on the real unit interval [0, 1] equipped with the usual order and with the so-called Lukasiewicz t-norm ($x*y = \max(x+y-1, 0)$) and Gödel t-norm ($x*y = \min(x, y)$) and their residua, respectively (see [29, 16]). Later, the so-called Product logic was introduced in [23] and proved to be complete in [0, 1] with respect to the product t-norm ($x * y = x \cdot y$) and its residuum. The corresponding algebraic varieties of these three logics are the variety of MV-algebras[1] (defined by Chang and noted **Ł**), the variety of Gödel algebras[2] (noted **G**) and the variety of Product algebras (noted **Π**). In [21] Hájek introduced the so-called Basic Fuzzy Logic, BL for short, as a common fragment of the above three logics, together with the corresponding variety of BL-algebras[3], which we will denotnote by **BL**. But his idea in mind was to define a logic to cope with the 1-tautologies common to all many-valued calculi in [0, 1] defined by a continous t-norm and its residuum, i.e. that BL would be the logic of continuous t-norms. This claim was proved very soon [22, 10].

It is well known that there exists a one-to-one correspondence between subvarieties of BL-algebras and axiomatic extensions of the BL logic, through a natural translation between algebraic equations and logical axioms. Therefore, when we study a family of subvarieties of **BL** and their equational characterization, we also obtain, by a simple translation, the corresponding axiomatic extensions of BL. It is worth observing that axiomatic extensions of BL are algebraizable in the sense of Blok and Pigozzi [8].

Some subvarieties of **BL** are already well-known. Subvarieties of **Ł** were fully described by Komori [27] and equationally defined by Di Nola and Lettieri in [15]. The lattice of subvarieties of **G** form a chain of subvarieties, $Boole = G_2 \subset G_3 \subset \ldots \subset G_n \subset \ldots.. \subset G$, corresponding to the ones generated by finite Gödel chains and upper bounded by the full variety **G** generated by any infinite Gödel chain. The equational definition of them as axiomatic extensions of BL are also well known. In [11], Cignoli and Torrens proved that the only subvarieties of Π are Boole and the full variety **Π** of product algebras itself.

Another subvariety studied in [17] is the so-called **SBL**, the subvariety of pseudo-complemented BL-algebras. The corresponding logic, also noted SBL, was proved to be the logic of the pseudo-complemented contin-

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[1] Definitionally equivalent to Wajsberg algebras [19].

[2] Equivalent to Linear Heyting algebras.

[3] BL-algebras are bounded residuated lattices satisfying the pre-linearity and divisibility conditions.

uous t-norms, i.e. t-norms whose associated negation $\neg x = x \to 0$ verifies $\min(x, \neg x) = 0$, or equivalently $\neg$ is Gödel negation.

Moreover in [14] the subvarieties generated by single-component chains (families of chains belonging to **Ł**, **G** and **Π**) are fully described and equationally defined.

But so far, the most general study of subvarieties of **BL** is to be found in the paper by Aglianó and Montagna [1]. A basic result of this paper is the decomposition of BL-chains as ordinal sums of Wajsberg hoops (algebraic structures already studied by Ferreirim in [18]). Other basic results are the characterization of the *generic* BL-chains (chains generating the full variety **BL**) and the equational definition of the subvarieties generated by BL-chains that are finite ordinal sums of Wajsberg hoops.

Finally, observe that not all axiomatic extensions of BL are *standard* complete, i.e. the corresponding subvarieties are not generated by the BL-chains of the variety over the real unit interval [0,1]. Even there are subvarieties of **BL** that contain no BL-chains over [0, 1] at all: take, for example, any subvariety generated by a finite Lukasiewicz or Gödel chain, or by the Chang MV-algebra. The logics defined by axiomatic extensions which are standard complete will be called *t-norm based* residuated fuzzy logics.

The goal of the present paper is the study and equational characterization of all subvarieties of **BL** generated by single standard BL-chains, i.e. BL-chains on [0, 1] induced by continuous t-norms. The main results of the paper can be summarized as follows:

- Each continuous t-norm has a canonical form defining a BL-chain which generates the same subvariety.
- There is a bijection between canonical BL-chain and subvarieties of **BL** generated by continuous t-norms.
- There exists an algorithm (we give one in the paper) to decide whether one of such varieties is subvariety of another one.
- The set of subvarieties of **BL** generated by standard BL-chains, is countable (this result was already proved in [25]) and each such variety is finitely axiomatizable. Moreover there exists an algorithm to find the equations.

Related to the latter item, we notice that in [25] the author proves that for every t-norm algebra, the set of formulas which are valid in it is Co-NP complete, thus improving a result in [4] stating that BL is Co-NP complete. The main difference between [4] and [25] is that in [25] one has to take account of the peculiarities of the t-norm but the author shows that all the information we need about the t-norm can be coded by a finite string. In this paper this will be proved in a stronger way, namely, we prove that for any t-norm BL-algebra there is a (unique) canonical t-norm BL-algebra which generates the same variety, and which can be represented as a finite ordinal sum of some finitely-many basic components.

From a logical point of view, the paper provides, for each continuous t-norm, an effective method (algorithm) to find the axiomatic extension of BL defining the logic of the given continuous t-norm. In other words, it provides the axiomatic definition of the residuated fuzzy logic which is standard complete with respect the BL-chain over [0,1] defined by that continuous t-norm. However, observe that these logics do not cover all the t-norm based residuated fuzzy logics, since there also exist those logics corresponding to subvarieties generated by arbitrary *families of t-norms* (with more than a single t-norm). To find all their axiomatic definitions is so far an open problem. Some initial results and suggestions are given in the conclusion section.

Finally let us mention that, in fact, the results we have got are also valid, with a few technical modifications, for a more general class than t-norm BL-chains, the so-called regular BL-chains, that strictly contain the standard ones (for example finite Gödel-chains are regular BL-chains while they are not standard). Therefore the paper actually studies the equational characterization of the subvarieties of **BL** generated by regular BL-chains.

The structure of the paper is as follows. Next section (preliminaries) contains the basic definitions we will need in the paper. The main result of Section 3 is the proof that the inclusion between varieties generated by regular BL-chains A depends only on the set $Fin(A)$, a set of finite ordinal sums of Wajsberg hoops satisfying some special conditions. In Section 4 we define the so-called canonical representation of a regular BL-chain and we provide an embedding algorithm. Section 5 is devoted to the proof that a variety generated by a regular BL-chain is finitely axiomatizable and that the described algorithm effectively finds the equations. Finally there is a last section with some conclusions and open problems.

2. Preliminaries

Triangular norms (t-norms for short) were defined by Schweizer and Sklar in the setting of Probabilistic Metric spaces [30] as operations on [0,1] satisfying commutativity, associativity, monotonicity and having 1 as neutral element. Basic t-norms are Minimum, Product and Lukasiewicz ($x * y = \max(x + y - 1, 0)$) and well known properties of t-norms are (see e.g. [26]):

(T1) A t-norm $*$ has a residuum $\to$ (residuated implication in fuzzy logical terms), defined as $x \to y = \max\{z \mid z * x \leq y\}$, if and only if the t-norm $*$ is left continuous.

(T2) If a t-norm $*$ has a residuum $\to$, then $x \leq y$ iff $x \to y = 1$ and thus $\max(x \to y, y \to x) = 1$ (pre-linearity property).

(T3) A t-norm $*$ is continuous if and only if $x * (x \to y) = \min(x, y)$ (divisibility property).

(T4) (From Mostert and Shields theorem) The set of idempotent elements of a continuous t-norm is a closed set with respect to the usual topology on [0,1]. Their complement is the union of a finite or countable family of open intervals (a, b). The restrictions of the t-norm on the closed intervals $[a, b]$ are operations isomorphic either to Lukasiewicz or product t-norms.

Given a continuous t-norm $*$, if $\to$ denotes its residuum, then the algebra $\langle [0,1], *, \to, 0, 1 \rangle$ will be denoted by $[0,1]_*$.

In fact (T4) says that any t-norm algebra $[0,1]_*$ decomposes into an ordinal sum of components where each component is either isomorphic to $[0,1]_\wedge$, where $\wedge$ denotes minimum (also known as Gödel t-norm), or isomorphic to $[0,1]_\odot$, where $\odot$ denotes product, or isomorphic to $[0,1]_\otimes$, where $\otimes$ denotes Lukasiewicz t-norm. In the sequel the algebra $[0,1]_\wedge$ will be denoted by $\mathcal{G}$ (for Gödel), the algebra $[0,1]_\odot$ will be denoted Π (for product) and the algebra $[0,1]_\otimes$ will be denoted $\mathcal{L}$ (for Lukasiewicz).

A BL-algebra is an algebraic structure $\mathbf{L} = (L, \vee, \wedge, *, \to, 0, 1)$ such that the following conditions are satisfied:

(BL1) $(L, \vee, \wedge, 0, 1)$ is a lattice with maximum 1 and minimum 0,

(BL2) $(L, *, 1)$ is a commutative semigroup with unit 1,

(BL3) $*$ and $\to$ form and adjoint pair, i.e.,
$z \leq (x \to y)$ iff $x * z \leq y$ for all x, y, z,

(BL4) $x \wedge y = x * (x \to y)$,

(BL5) $(x \to y) \vee (y \to x) = 1$.

Condition (BL3) is the residuation property, condition (BL4) is known as the *divisibility* condition and is equivalent to continuity on t-norms (see T3 above), while condition (BL5) is known as *pre-linearity* (related to T2). From (T2), and remembering that BL is the logic of continuous t-norms, the set of BL-algebras on the lattice $([0,1], \max, \min, 0, 1)$ coincides with the set of algebraic structures $([0,1], \max, \min, *, \rightarrow, 0, 1)$ where $*$ is a continuous t-norm and $\rightarrow$ is its residuated implication. Notice that all the conditions used in the definition of BL-algebras are equations except the adjointness condition (BL3), but Hájek himself proved in [21, Lemma 2.3.10] that it can be equivalently expressed as a set of equations, and thus BL-algebras form a variety, noted **BL** from now on.

Well-known subvarieties of **BL** are the variety **Ł** of MV-algebras, the variety **G** of Gödel algebras[4] and the variety **Π** of Product algebras[5]. Hájek also proved in [21] that **Ł** can be defined from the equations of **BL** by adding the equation

$$\neg\neg x = x, \tag{L}$$

G can be defined from the equations of **BL** by adding the equation

$$(x \rightarrow x * x) = 1, \tag{G}$$

and finally **Π** can be defined from the equations of **BL** adding the equations

$$x \wedge \neg x = 0 \tag{Π1}$$
$$\neg\neg z \rightarrow (((x * z) \rightarrow (y \rightarrow z)) \rightarrow (x \rightarrow y)) = 1. \tag{Π2}$$

Moreover Hájek proved ([21, Lemma 2.3.16]) that each BL-algebra is a subdirect product of a set of linearly ordered BL-algebras. This property, that is a consequence of the above pre-linearity condition (BL5), says that each subvariety is characterized by the family of linearly ordered BL-algebras it contains. Thus studying the subvarieties of BL-algebras amounts to studying varieties generated by families of linearly ordered BL-algebras, called BL-chains from now on. Thus it is very important for our goal to know the structure of the BL-chains.

The structure of BL-chains has been studied in [22] and in [10], where a complete representation of BL-chains was given. The notion of ordinal sum of BL-chains plays a key role in that representation and it is defined as follows.

Definition 2.1. (Cf. [22]) Let $\mathbf{C_i} = (C_i, \wedge_i, \vee_i, *_i, \rightarrow_i, 0_i, 1_i)$, $i = 1, 2$, be two BL-chains such that $1_1 = 0_2$ and $C_1 \setminus \{1_1\} \cap C_2 \setminus \{0_2\} = \emptyset$. The ordinal sum of $\mathbf{C_1}$ and $\mathbf{C_2}$, written $\mathbf{C_1} \oplus \mathbf{C_2}$ is a new BL-chain $(C_1 \cup C_2, \wedge, \vee, *, \rightarrow, 0_1, 1_2)$ whose operations $\wedge, \vee, *$ coincide with those of $\mathbf{C_i}$ when applied on

[4] See for example [21]. Gödel algebras are also known as *linear Heyting* algebras in the algebraic logic context, as for example in [5]. Actually, using our notation, they correspond to *pre-linear* Heyting algebras, i.e. Heyting algebras satisfying (BL5).

[5] See [23] and [11].

pairs of elements of C_i, $i = 1, 2$, and in the rest of pairs they are defined as follows, for all $x \in C_1$ and $y \in C_2$:

1. $x \wedge y = y \wedge x = x * y = y * x = x$,
2. $x \vee y = y \vee x = y$.

Finally, $\to$ is defined by:

$$x \to y = \begin{cases} 1_2, & \text{if } x \leq y \\ x \to_i y, & \text{if } x > y \text{ and } x, y \in C_i \\ y, & \text{if } x > y \text{ and } x \in C_2, y \in C_1. \end{cases}$$

The definition can be extended in the obvious way for the case of more than 2 components, provided that each component has a succesor (for the general case the reader is referred to [22, 10]). The representation theorem proves that each saturated BL-chain is an ordinal sum of MV, Gödel and Product chains, in an analogous way to the Mostert and Shields theorem for continuous t-norms. Along the paper, this kind of decomposition of a BL-chain will be referred as decomposition as ordinal sum of (basic) BL-chain components.

On the other hand Aglianó and Montagna in [1] give another decomposition of BL-chain as ordinal sums of Wajsberg hoops. Next we give the basic definitions about hoops and the decomposition theorem.

Definition 2.2. A structure $\mathbf{H} = (H, *, \to, 1)$ is a *hoop* if $*$ is a commutative binary operation on H with the unit 1 (i.e. $x * y = y * x$ and $1 * x = x$ for all x, y) and further $\to$ is a binary operation satisfying

$$x \to x = 1, \; x * (x \to y) = y * (y \to x), \; (x * y) \to z = x \to (y \to z)$$

for all $x, y, z \in H$.

In a hoop, the relation defined by $x \leq y$ iff $x \to y = 1$ is an ordering. Further basic properties of hoops are (see e.g. [2]):

(i) $z \leq x \to y$ iff $x * z \leq y$; (residuum)

(ii) $x * (y * z) = (x * y) * z$ (associativity)

(iii) $x \leq y$ implies $x * z \leq y * z$ (monotony)

(iv) $x * (x \to y) = x \wedge y$

(v) $1 \to x = x$

(vi) $x \leq 1$ (1 is maximal)

(vii) $x * y \leq x$.

From these properties we can see that hoops are also residuated structures but they have no 0 (lower bound) as constant. Interesting hoops which will be extensively used in this paper are the following:

- the hoop **2**, defined on the set of two idempotent elements $\{a, 1\}$, and coinciding with the 2-element Boolean algebra;
- *basic* hoops [2], which are hoops satisfying the equation $((x \to y) \to z) \to (((y \to x) \to z) \to z) = 1$;
- *Wajsberg* hoops[6] [18, 6], which are those hoops satisfying the equation $(x \to y) \to y = (y \to x) \to x$;
- and *cancellative* hoops [18], which are hoops in which $x * z \leq y * z$ implies $x \leq y$ for all x, y, z.

Actually, cancellative hoops also form a variety characterized by the equation $y \to (y * x) = x$, as it is shown in [7]. Wajsberg hoops can be bounded (having least element 0) or not. Bounded Wajsberg hoops coincide with MV-algebras. Non-bounded linearly-ordered Wajsberg hoops are cancellative and are product-chains without bottom element. In [1] it is proved a decomposition theorem of BL-chains as ordinal sums of Wajsberg hoops. The notion of ordinal sum for hoops is slightly different from the one for BL-chains.

Definition 2.3. [1] Let $\langle I, \leq \rangle$ be a totally ordered set, and for all $i \in I$ let $\mathbf{H_i} = (H_i, *_i, \to_i, 1)$ be a hoop such that for $i \neq j$, $H_i \cap H_j = \{1\}$. Then the ordinal sum of the family $\{\mathbf{H_i}\}_{\mathbf{i \in I}}$, written $\bigoplus_{i \in I} \mathbf{H_i}$, is the structure $(\bigcup_{i \in I} H_i, *, \to, 1)$, where the operations are defined as follows

$$x * y = \begin{cases} x *_i y & \text{if } x, y \in H_i \\ y & \text{if } x \in H_i \text{ and } y \in H_j \setminus \{1\}, \text{ with } i > j \\ x & \text{otherwise,} \end{cases}$$

$$x \to y = \begin{cases} x \to_i y & \text{if } x, y \in H_i \\ y & \text{if } x \in H_i \text{ and } y \in H_j, \text{ with } i > j \\ 1 & \text{otherwise} \end{cases}$$

One can notice that the main difference is that all the top elements of the components are identified with the top of the ordinal sum. Hence, for instance, when considering the decomposition of a BL-chain as an ordinal sum of Wajsberg hoops, the top of any component is the top of the BL-chain, and given two consecutive components, the bottom (if it exists) of the second

[6] They were originally called *Lukasiewicz* hoops in Ferreirim's thesis [18].

component is not in the first component (whose top is 1). In this way, a product BL-chain is the ordinal sum of **2** plus a cancellative component, and any Gödel BL-chain is the ordinal sum of as many components isomorphic to **2** as elements the chain has. Given two Wajsberg hoop components $\mathcal{A}$ and $\mathcal{B}$, we shall write $\mathcal{A} < \mathcal{B}$ to mean that every element of $\mathcal{A}$ (except from 1) precedes every element of $\mathcal{B}$.
The decompositions of BL-algebras of the form $[0,1]_*$ (t-norm algebras) as ordinal sums of Wajsberg hoops have the following peculiarities:

- There is always a first component, which is either the two-element BL-algebra **2** (this case occurs if $[0,1]_*$ is a SBL-algebra) or it is isomorphic to $\mathcal{L}$. (Note that decompositions of t-norm algebras as ordinal sums of Bl-chains need not have a first component).
- Any component which is a cancellative hoop is isomorphic to the cancellative hoop $\langle (0,1], \odot \rightarrow_{\odot} \rangle$ (such algebra will be denoted by $\mathcal{C}$).
- Any component which is not a cancellative hoop is isomorphic either to $\mathcal{L}$ or to **2**.
- Any component isomorphic to $\mathcal{C}$ (cannot be the first component and) is preceded by a component isomorphic to **2** (such a combination corresponds to a Product component in a ordinal sum decomposition with BL-chain components.)
- If $\mathcal{W}_i$ and $\mathcal{W}_j$ are two different components isomorphic to **2**, then between the bottoms of $\mathcal{W}_i$ and $\mathcal{W}_j$ there must be either a Łukasiewicz component, or a cancellative component, or continuum many components isomorphic to **2** (this corresponds to the case when the bottoms of $\mathcal{W}_i$ and $\mathcal{W}_j$ belong to the same Gödel component in a BL-chain decomposition).

As usual, throughout this paper, given two Wajsberg hoops, we shall use the notation $\mathcal{A} \oplus \mathcal{B}$ to denote the ordinal sum of $\mathcal{A}$ and $\mathcal{B}$ in this order, i.e. $\mathcal{A}$ is the first component of the sum. Moreover, for the sake of a easier notation, we introduce the following definition.

Definition 2.4. Let **M** and **N** be two classes of Wajsberg hoops. We define

$$\mathbf{M} \oplus \mathbf{N} = \{\mathcal{A} \oplus \mathcal{B} \mid \mathcal{A} \in \mathbf{M}, \mathcal{B} \in \mathbf{N}\}.$$

Definition 2.5. In the sequel, if **K** is any class of algebras of the same type, then $\mathbf{S}(\mathbf{K})$, $\mathbf{P}(\mathbf{K})$, $\mathbf{P}_u(\mathbf{K})$, $\mathbf{I}(\mathbf{K})$ and $\mathbf{H}(\mathbf{K})$ will denote the class of subalgebras, of direct products, of ultraproducts, of isomorphic images and of homomorphic images of algebras from $\mathbf{K}$ respectively, and $Var(\mathbf{K})$ will denote the variety generated by $\mathbf{K}$. Moreover, $Var(*)$ will denote the variety generated by $[0,1]_*$. Finally, if $\mathbf{O}$ is any combination of the operators $\mathbf{I}, \mathbf{S}, \mathbf{P}, \mathbf{H}, \mathbf{P}_u$, if $\mathbf{K_0}$ is a class of BL-algebras and $\mathbf{K_1}, \ldots, \mathbf{K_n}$ are classes of Wajsberg hoops, then, according to the above definition, we take

$$\mathbf{O}(\mathbf{K_0}) \oplus \ldots \oplus \mathbf{O}(\mathbf{K_n}) = \{\mathcal{W}_0 \oplus \ldots \oplus \mathcal{W}_n : \text{ for } i = 0, \ldots, n, \ \mathcal{W}_i \in \mathbf{O}(\mathbf{K_i})\}.$$

Definition 2.6. A BL-chain $\mathcal{A}$ is called *regular* if it is the ordinal sum of Wajsberg hoops of the form $\mathcal{L}$, $\mathcal{C}$ and $\mathbf{2}$, and $\mathcal{A}$ has a first component (which is either $\mathcal{L}$ or $\mathbf{2}$). The class of regular BL-chains will be called REG.
The class $\{[0,1]_* : *$ a continuous t-norm$\}$ is denoted by TN. Clearly $TN \subseteq REG$, but the opposite inclusion does not hold. Members of TN have been called *t-algebras* in [21] and *standard BL-algebras* in [24]. Here we will preferably use the term *t-norm algebra* to denote them (modulo isomorphism).
Furthermore, the subclass of REG consisting of those BL-chains which are *finite* ordinal sums of Wajsberg hoops will be denoted by Fin.

Definition 2.7. Let $\mathcal{A} \in REG$. Then $Fin(\mathcal{A})$ denotes the set of all finite ordinal sums of Wajsberg hoops $\mathcal{W}_0, \ldots, \mathcal{W}_n$ such that the following conditions hold:

- Each $\mathcal{W}_i$ is isomorphic either to $\mathbf{2}$, or to $\mathcal{C}$ or to $\mathcal{L}$.
- $\mathcal{W}_0$ is either $\mathbf{2}$ or $\mathcal{L}$.
- There are components $\mathcal{A}_0 < \ldots < \mathcal{A}_n$ of $\mathcal{A}$ such that $\mathcal{A}_0$ is the first component of $\mathcal{A}$, and for every i, if $\mathcal{W}_i$ is isomorphic to $\mathcal{L}$, then $\mathcal{A}_i$ is isomorphic to $\mathcal{L}$, if $\mathcal{W}_i$ is isomorphic to $\mathcal{C}$ then $\mathcal{A}_i$ is isomorphic either to $\mathcal{C}$ or to $\mathcal{L}$, and if $\mathcal{W}_i$ is isomorphic to $\mathbf{2}$ then $\mathcal{A}_i$ is isomorphic either to $\mathbf{2}$ or to $\mathcal{L}$.

In the sequel if $*$ is any continuous t-norm, we write $Fin(*)$ for $Fin([0,1]_*)$.

The next lemma is easy to demonstrate. To do so, we shall use the following fact, proved in [1]:

Theorem 2.8. *If $\mathcal{A}_0, \ldots, \mathcal{A}_n$ are linearly ordered hoops and $\mathcal{A}_0$ is bounded, then*

$$\mathbf{ISP}_u(\mathcal{A}_0 \oplus \ldots \oplus \mathcal{A}_n) = \mathbf{ISP}_u(\mathcal{A}_0) \oplus \ldots \oplus \mathbf{ISP}_u(\mathcal{A}_n),$$

where $\mathbf{ISP}_u(\mathcal{A}_0)$ is meant with respect to the operations of BL-algebras (hence 0 is a constant which is interpreted in $\mathcal{A}_0$) and for $i > 0$, $\mathbf{ISP}_u(\mathcal{A}_i)$ is meant with respect to the operations of hoops.

LEMMA 2.9. *Let* $\mathcal{A} \in REG$. *Then:*

(i) *Every finitely generated subalgebra of* $\mathcal{A}$ *is a subalgebra of at least one algebra in* $Fin(\mathcal{A})$.

(ii) *Every algebra in* $Fin(\mathcal{A})$ *is in* $\mathbf{ISP}_u(\mathcal{A})$.

(iii) $\mathbf{ISP}_u(\mathcal{A}) = \mathbf{ISP}_u(Fin(\mathcal{A}))$.

PROOF. (i). Let $a_0, \ldots, a_n$ be generators of a subalgebra $\mathcal{B}$ of $\mathcal{A}$ (without loss of generality, possibly adding 0 to the generators, we may assume that a_0 is in the first component), and let $\mathcal{W}_0, \ldots, \mathcal{W}_k$ be finitely many components of $\mathcal{A}$ such that $\{a_0, \ldots, a_n\} \subseteq \mathcal{W}_0 \cup \ldots \cup \mathcal{W}_k$. Without loss of generality we may assume that the components are enumerated in increasing order, i.e., if $i < j$ then $\mathcal{W}_i < \mathcal{W}_j$. Thus $\mathcal{W}_0$ is the first component of $\mathcal{A}$. Now by induction on the complexity we can see that for every term $t(x_1, \ldots, x_n)$ one has:

$$t(a_1, \ldots, a_n) \in \mathcal{W}_0 \oplus \ldots \oplus \mathcal{W}_k.$$

Since $\mathcal{B}$ is generated by $\{a_0, \ldots, a_n\}$, $\mathcal{B} \subseteq \mathcal{W}_0 \oplus \ldots \oplus \mathcal{W}_k$, and $\mathcal{W}_0 \oplus \ldots \oplus \mathcal{W}_k \in Fin(\mathcal{A})$. This proves (i).

(ii). It follows from [1] (see above Theorem 2.8) that if $\mathcal{U}_0$ is a Wajsberg algebra and $\mathcal{U}_1, \ldots, \mathcal{U}_k$ are Wajsberg hoops, then

$$\mathbf{ISP}_u(\mathcal{U}_0 \oplus \ldots \oplus \mathcal{U}_n) = \mathbf{ISP}_u(\mathcal{U}_0) \oplus \ldots \oplus \mathbf{ISP}_u(\mathcal{U}_n).$$

Now let $\mathcal{W}_0 \oplus \ldots \oplus \mathcal{W}_n \in Fin(\mathcal{A})$, and let $\mathcal{A}_0, \ldots, \mathcal{A}_n$ be components of $\mathcal{A}$ as in Definition 2.7. Recall that $\mathcal{C} \in \mathbf{ISP}_u(\mathcal{L})$, because any cancellative linearly ordered hoop is a subhoop of a linearly ordered Wajsberg algebra [7, Th. 1.16], and every linearly ordered Wajsberg algebra can be embedded in an ultrapower of $\mathcal{L}$ [12, 13]. Moreover, $\mathbf{2}$ is a subalgebra of $\mathcal{L}$, hence it is in $\mathbf{ISP}_u(\mathcal{L})$. It follows that for every i, $\mathcal{W}_i \in \mathbf{ISP}_u(\mathcal{A}_i)$, therefore

$$\mathcal{W}_0 \oplus \ldots \oplus \mathcal{W}_n \in \mathbf{ISP}_u(\mathcal{A}_0) \oplus \ldots \oplus \mathbf{ISP}_u(\mathcal{A}_n) = \mathbf{ISP}_u(\mathcal{A}_0 \oplus \ldots \oplus \mathcal{A}_n).$$

Since $\mathcal{A}_0 \oplus \ldots \oplus \mathcal{A}_n \in \mathbf{S}(\mathcal{A})$, the claim follows.

(iii). $Fin(\mathcal{A}) \subseteq \mathbf{ISP}_u(\mathcal{A})$ by (ii), therefore $\mathbf{ISP}_u(Fin(\mathcal{A})) \subseteq \mathbf{ISP}_u(\mathcal{A})$. On the other hand if $FG(\mathcal{A})$ denotes the class of finitely generated subalgebras of $\mathcal{A}$, then, by (i), $FG(\mathcal{A}) \subseteq \mathbf{S}(Fin(\mathcal{A}))$, and $\mathcal{A} \in \mathbf{ISP}_u(FG(\mathcal{A}))$. Thus $\mathcal{A} \in \mathbf{ISP}_u(Fin(\mathcal{A}))$, and $\mathbf{ISP}_u(\mathcal{A}) \subseteq \mathbf{ISP}_u(Fin(\mathcal{A}))$. ■

One may wonder whether $Fin(\cdot)$, as a function on algebras, commutes with the ordinal sum operation. And one can easily notice that this is not true, i.e. in general $Fin(\mathcal{A} \oplus \mathcal{B})$ is not equal to $Fin(\mathcal{A}) \oplus Fin(\mathcal{B})$, we only have $Fin(\mathcal{A}) \oplus Fin(\mathcal{B}) \subseteq Fin(\mathcal{A} \oplus \mathcal{B})$, because first components of algebras belonging to $Fin(\mathcal{B})$ can only be $\mathbf{2}$ or $\mathcal{L}$, but not $\mathcal{C}$. However, we can get the equality if we consider a weaker notion of $Fin(\cdot)$ by dropping this last condition.

Definition 2.10. In the sequel, we shall denote by REG' the class of all *linearly ordered hoops* which are ordinal sums (possibly without first component) of components isomorphic to $\mathcal{L}$, or to $\mathcal{C}$ or to $\mathbf{2}$.

Definition 2.11. Let $\mathcal{A} \in REG'$. Then $Fin'(\mathcal{A})$ denotes, besides the trivial hoop $\top = \{1\}$ consisting of only one element, the set of all finite ordinal sums of Wajsberg hoops $\mathcal{W}_0, \ldots, \mathcal{W}_n$ such that the following conditions hold:

- Each $\mathcal{W}_i$ is isomorphic either to $\mathbf{2}$, or to $\mathcal{C}$ or to $\mathcal{L}$.
- There are components $\mathcal{A}_0 < \ldots < \mathcal{A}_n$ of $\mathcal{A}$ such that for every i, if $\mathcal{W}_i$ is isomorphic to $\mathcal{L}$, then $\mathcal{A}_i$ is isomorphic to $\mathcal{L}$, if $\mathcal{W}_i$ is isomorphic to $\mathcal{C}$ then $\mathcal{A}_i$ is isomorphic either to $\mathcal{C}$ or to $\mathcal{L}$, and if $\mathcal{W}_i$ is isomorphic to $\mathbf{2}$ then $\mathcal{A}_i$ is isomorphic either to $\mathbf{2}$ or to $\mathcal{L}$.

The next decomposition properties are then easily proved from the definitions of Fin and Fin'.

Lemma 2.12. *Let $\mathcal{A}, \mathcal{B} \in REG'$. Then we have*

(i) $Fin'(\mathcal{A} \oplus \mathcal{B}) = Fin'(\mathcal{A}) \oplus Fin'(\mathcal{B})$*;*

(ii) *and moreover, if $\mathcal{A} \in REG$ then $Fin(\mathcal{A} \oplus \mathcal{B}) = Fin(\mathcal{A}) \oplus Fin'(\mathcal{B})$.*

3. A characterization of the set-theoretical inclusion relation between varieties generated by regular BL-chains

In this section we characterize the relation of inclusion between varieties of the form $Var(*)$, $Var(\circ)$, when $*$ and $\circ$ range over continuous t-norms, in terms of the inclusion of the sets $Fin(*)$ and $Fin(\circ)$. In fact we will prove a more general result: if $\mathcal{A}, \mathcal{B} \in REG$, then $Var(\mathcal{A}) \subseteq Var(\mathcal{B})$ iff $Fin(\mathcal{A}) \subseteq Fin(\mathcal{B})$. We start from the following:

Lemma 3.1. *Let $\mathcal{A} \in REG$. Then $Var(\mathcal{A}) = Var(Fin(\mathcal{A}))$.*

Proof. If $\mathcal{A} \in REG$ then the following equalities hold: $Var(\mathcal{A}) = Var(\mathbf{ISP}_u(\mathcal{A})) = Var(\mathbf{ISP}_u(Fin(\mathcal{A}))) = Var(Fin(\mathcal{A}))$, where the second equality holds by Lemma 2.9 (iii)). ■

COROLLARY 3.2. *Let* $\mathcal{A}, \mathcal{B} \in REG$. *Then* $Var(\mathcal{A}) \subseteq Var(\mathcal{B})$ *iff* $\mathcal{A} \in Var(\mathcal{B})$ *iff* $\mathcal{A} \in Var(Fin(\mathcal{B}))$ *iff* $Fin(\mathcal{A}) \subseteq Var(Fin(\mathcal{B}))$. *In particular, if* $*$ *and* $\circ$ *are continuous t-norms then* $Var(*) \subseteq Var(\circ)$ *iff* $[0,1]_* \in Var(\circ)$ *iff* $[0,1]_* \in Var(Fin(\circ))$ *iff* $Fin(*) \subseteq Var(Fin(\circ))$.

DEFINITION 3.3. We introduce the following terms:

$e_{\mathcal{L}}(x) : (x \to x^2) \vee ((x \to x^3) \to x^2)$

$e_{\mathcal{C}}(x) : x \to x^2$

$e_{\mathbf{2}}(x) : (x \to x^3) \to x^2$

LEMMA 3.4. *We have:*

1. *the equation* $e_{\mathcal{L}}(x) = 1$ *is valid in* **2** *and in cancellative hoops and it is not valid in any MV chain with more than two elements.*
2. *the equation* $e_{\mathcal{C}}(x) = 1$ *is valid in* **2** *and it is not valid neither in any MV chain with more than two elements nor in any non-trivial cancellative hoop.*
3. *the equation* $e_{\mathbf{2}}(x) = 1$ *is valid in any cancellative hoop and it is not valid neither in* **2** *nor in any MV chain.*

PROOF. The proof is easy, we shall only stress that in any MV-chain with more than two elements there always exists an element $0 < x < 1$ such that $(x \to x^3) \to x^2 < 1$. Namely, it is enough to take $x > 0$ such that $x^3 = x^2 = 0$, which can always be found. Then $(x \to x^3) \to x^2 = (x \to 0) \to 0 = x < 1$. ■

DEFINITION 3.5. Let $\mathcal{A} = \bigoplus_{i \in I} \mathcal{A}_i$ and $\mathcal{B} = \bigoplus_{i \in I} \mathcal{B}_i$ be two linearly ordered BL-algebras such that, for each $i \in I$, $\mathcal{A}_i$ and $\mathcal{B}_i$ are Wajsberg hoops, where I has a minimum i_0 and both $\mathcal{W}_{i_0}$ and $\mathcal{U}_{i_0}$ are bounded. We stress that we are assuming that the index set I is the same for $\mathcal{A}$ and $\mathcal{B}$. Then we say that $\mathcal{A}$ and $\mathcal{B}$ are of the same *type* if for each $i \in I$, both $\mathcal{A}_i$ and $\mathcal{B}_i$ are either MV-chains with more than two elements, or cancellative hoops, or isomorphic to **2**.

DEFINITION 3.6. Let $\mathcal{A} = \bigoplus_{i=0,n} \mathcal{A}_i$ be a BL-chain, and for each $i = 0 \ldots n$ let $e_i^{\mathcal{A}}$ be $e_{\mathcal{L}}$ if $\mathcal{A}_i$ is an MV algebra with more than two elements, be $e_{\mathcal{C}}$ if $\mathcal{A}_i$ is a non-trivial cancellative hoop and be $e_{\mathbf{2}}$ if $\mathcal{A}_i$ is **2**. Then we define the following equation

$$(e_{\mathcal{A}}):\quad [(\ \bigwedge_{i=0\ldots n-1}((x_{i+1} \to x_i) \to x_i)\ )\&(\neg\neg x_0 \to x_0)\ \to\ (\bigvee_{i=0\ldots n} x_i)] \vee [\bigvee_{i=0\ldots n} e_i^{\mathcal{A}}(x_i)] = 1\ .$$

The first disjunct of the left-hand side of this equation is an slight modification of equation λ_n of [1, Lemma 4.2], where it is shown that λ_n holds in an ordinal sum of linearly ordered Wjaberg hoops if and only if the number of componets is at most n. Notice that, by construction, the equation $(e_{\mathcal{A}})$ is not valid in $\mathcal{A}$. Namely, one can always find a sequence of values $0 \le x_0 < x_1 < \cdots < x_n < 1$ such that for each $i = 0, ..., n$, $x_i \in \mathcal{A}_i$, and thus $(\bigwedge_{i=0\ldots n-1}((x_{i+1} \to x_i) \to x_i)) \& (\neg\neg x_0 \to x_0) = 1$, but $\bigvee_{i=0\ldots n} x_i = x_n < 1$ and $e_i^{\mathcal{A}}(x_i)) < 1$ for all $i = 0, ..., n$.

Lemma 3.7.

(i) *Let $\mathcal{D} \in REG$, and let $\mathcal{A} \in Fin$. Then the equation $(e_{\mathcal{A}})$ is valid in all $\mathcal{B} \in Fin(\mathcal{D})$ iff $\mathcal{A} \notin Fin(\mathcal{D})$.*

(ii) *Let $\mathcal{B}$ and $\mathcal{D}$ be two BL-chains of the same type, and let $\mathcal{A} \in Fin$. Then the equation $(e_{\mathcal{A}})$ is valid in $\mathcal{B}$ iff the equation $(e_{\mathcal{A}})$ is valid in $\mathcal{D}$.*

Proof. (i) Let $\mathcal{A} = \bigoplus_{i=0,n} \mathcal{A}_i$. First of all observe that the claim trivially holds if the number of components of $\mathcal{D}$ is finite and smaller than $n+1$. Let us assume that $\mathcal{D}$ has at least $n+1$ components.

As for one direction, assume $\mathcal{A} \notin Fin(\mathcal{D})$ and take $\mathcal{B} = \bigoplus_{i=0,k} \mathcal{B}_i \in Fin(\mathcal{D})$. We want to show that the equation $(e_{\mathcal{A}})$ holds in $\mathcal{B}$. If $k < n$ then, again, $(e_{\mathcal{A}})$ holds in $\mathcal{B}$, so assume $n \le k$. Actually, we have only to care of validating the equation for all $(n+1)$-tuples $x_0 < x_1 < \ldots x_n < 1$ where every x_j belongs to a different component $\mathcal{B}_{i_j}$ and x_0 belongs to the first component, since for any other $(n+1)$-tuple the evaluation of the first disjunct of the equation $(e_{\mathcal{A}})$, i.e. $(\bigwedge_{i=0\ldots n-1}((x_{i+1} \to x_i) \to x_i))\&(\neg\neg x_0 \to x_0) \to (\bigvee_{i=0\ldots n} x_i)$, is already 1. For each $j = 0, \ldots, n$ let B_{i_j} be the component where x_j belongs to. Since $\mathcal{A} \notin Fin(\mathcal{D})$, there exists $0 \le j \le n$ such that one the following situations holds:

(a) $\mathcal{A}_j$ is $\mathcal{L}$ and $\mathcal{B}_{i_j}$ is either $\mathcal{C}$ or **2**, or

(b) $\mathcal{A}_j$ is $\mathcal{C}$ and $\mathcal{B}_{i_j}$ is **2**, or

(c) $\mathcal{A}_j$ is **2** and $\mathcal{B}_{i_j}$ is $\mathcal{C}$

If we are in case (a) then $e_j^{\mathcal{A}}(x_j) = e_{\mathcal{L}}(x_j) = 1$ for all $x_j \in \mathcal{B}_{i_j}$, if we are in case (b) then $e_j^{\mathcal{A}}(x_j) = e_{\mathcal{C}}(x_j) = 1$ for all $x_j \in B_{i_j}$, and if we are in case (c) $e_j^{\mathcal{A}}(x_j) = e_{\mathbf{2}}(x_j) = 1$ for all $x_j \in \mathcal{B}_{i_j}$. Hence, in any case, $e_j^{\mathcal{A}}(x_j) = 1$ and thus the equation $(e_{\mathcal{A}})$ is satisfied by the tuple $x_0, \ldots, x_n$. Therefore $(e_{\mathcal{A}})$ is valid in $\mathcal{B}$.

Reciprocally, it suffices to recall that the equation $(e_{\mathcal{A}})$ is not valid in $\mathcal{A}$ (see Lemma 3.4), therefore $\mathcal{A}$ cannot belong to $Fin(\mathcal{D})$.

(ii) If $\mathcal{B}$ and $\mathcal{D}$ are of the same type, then the cardinalities of the sets of Wajsberg components of both algebras must be the same. If $\mathcal{B}$ and $\mathcal{D}$ have less components than $\mathcal{A}$, then the equation $(e_{\mathcal{A}})$ trivially holds in both $\mathcal{B}$ and $\mathcal{D}$. Otherwise, the claim follows from Lemma 3.4, since (i) the fact that expression $e_{\mathcal{L}}(x)$ (resp. $e_{\mathcal{C}}(x)$ and $e_{\mathbf{2}}(x)$) gets value 1 for all x in a component $\mathcal{W}$ only depends on whether $\mathcal{W}$ is $\mathbf{2}$ or cancellative (resp. whether $\mathcal{W}$ is $\mathbf{2}$, or whether $\mathcal{W}$ is cancellative), and (ii) the relevant values of the variables involved $x_0, x_1, \ldots, x_n$ are only those taken in different components, with x_0 in the first component, otherwise the equation trivially holds. ■

Remark that, by Lemma 3.7, it follows that if $\circ$ is a continuous t-norm and $\mathcal{A} \in Fin$, then $(e_{\mathcal{A}})$ holds in all elements of $Fin(\circ)$ iff $\mathcal{A} \notin Fin(\circ)$.

COROLLARY 3.8. *Let $\mathcal{D} \in REG$, and let $\mathcal{A} \in Fin \cap \mathbf{ISP}_u(Fin(\mathcal{D}))$. Then $\mathcal{A} \in Fin(\mathcal{D})$. In other words, $Fin \cap \mathbf{ISP}_u(Fin(\mathcal{D})) = Fin(\mathcal{D})$. In particular, if $\circ$ is a continuous t-norm and $\mathcal{A} \in Fin \cap \mathbf{ISP}_u(Fin(\circ))$, then $\mathcal{A} \in Fin(\circ)$.*

PROOF. Suppose $\mathcal{A} \notin Fin(\mathcal{D})$. Then the equation $(e_{\mathcal{A}})$ is valid in any algebra of $Fin(\mathcal{D})$, hence it is also valid in any algebra of $\mathbf{ISP}_u(Fin(\mathcal{D}))$, hence it will also be valid in $\mathcal{A}$, which is a contradiction with the fact that $(e_{\mathcal{A}})$ is not valid in $\mathcal{A}$. ■

THEOREM 3.9. *Let $\mathcal{D}, \mathcal{E} \in REG$. Then $Var(\mathcal{D}) \subseteq Var(\mathcal{E})$ iff $Fin(\mathcal{D}) \subseteq Fin(\mathcal{E})$. Thus, in particular, if $*$ and $\circ$ are two continuous t-norms, then, $Var(*) \subseteq Var(\circ)$ iff $Fin(*) \subseteq Fin(\circ)$.*

PROOF. If $Fin(\mathcal{D}) \subseteq Fin(\mathcal{E})$ then we have $Var(\mathcal{D}) = Var(Fin(\mathcal{D})) \subseteq Var(Fin(\mathcal{E})) = Var(\mathcal{E})$. For the inverse implication, we recall that any variety $\mathbf{V}$ of BL-algebras is congruence-distributive, hence by Jónsson's lemma (see e.g. [9]), if $\mathbf{K}$ is any class such that $Var(\mathbf{K}) = \mathbf{V}$, then every subdirectly irreducible member of $\mathbf{V}$ is in $\mathbf{HSP}_u(\mathbf{K})$. It follows that every subdirectly irreducible member of $Var(\mathcal{E})$ is in $\mathbf{HSP}_u(Fin(\mathcal{E}))$. Now $\mathcal{L}$, $\mathcal{C}$ and $\mathbf{2}$ are simple (hence subdirectly irreducible) Wajsberg hoops. Moreover by [6], if $\mathcal{W}$ is a subdirectly irreducible Wajsberg hoop and $\mathcal{H}$ is any hoop, then the ordinal sum $\mathcal{H} \oplus \mathcal{W}$ is subdirectly irreducible. It follows that if $Var(\mathcal{D}) \subseteq Var(\mathcal{E})$, then $Fin(\mathcal{D}) \subseteq \mathbf{HSP}_u(Fin(\mathcal{E}))$.

Now let $\mathcal{A} = \bigoplus_{j=0}^{k} \mathcal{W}_j \in Fin(\mathcal{D})$ be arbitrary (as usual, $\mathcal{W}_j$ are the Wajsberg components of $\mathcal{A}$). Then there is $\mathcal{B} = \bigoplus_{i \in I} \mathcal{U}_i \in \mathbf{ISP}_u(Fin(\mathcal{E}))$ (where as usual $\mathcal{U}_i$ are the Wajsberg components of $\mathcal{B}$) and a filter F of $\mathcal{B}$ such that $\mathcal{A} \cong \mathcal{B}/F$. Without loss of generality we may suppose $F \neq \{1\}$ (otherwise $\mathcal{A} \in \mathbf{ISP}_u(Fin(\mathcal{E}))$, and $\mathcal{A} \in Fin(\mathcal{E})$ by Corollary 3.8), and $F \neq \mathcal{B}$ (otherwise $\mathcal{A}$ is a trivial algebra, which is impossible), so there are two possibilities:

(a) For every $i \in I$, either $\mathcal{U}_i \subseteq F$ or $\mathcal{U}_i \cap F = \{1\}$. In this case, it follows from the argument used in the proof of [1, Prop. 3.2] that $\mathcal{A} \cong \bigoplus_{i\in I:\mathcal{U}_i\cap F=\{1\}} \mathcal{U}_i \in \mathbf{IS}(\mathcal{B})$, $\mathcal{A} \in Fin \cap \mathbf{ISP}_u(Fin(\mathcal{E}))$, and $\mathcal{A} \in Fin(\mathcal{E})$ by Corollary 3.8.

(b) There is $i \in I$ such that $\{1\} \subset \mathcal{U}_i \cap F \subset \mathcal{U}_i$. In this case, letting $F_i = F \cap \mathcal{U}_i$, again by [1, Prop. 3.2], we have $\mathcal{A} \cong \bigoplus_{j<i} \mathcal{U}_j \oplus (\mathcal{U}_i/F_i)$. Clearly we may assume $i = k$, $\mathcal{U}_j = \mathcal{W}_j$ for $j < i$, and $\mathcal{W}_i = \mathcal{U}_i/F_i$. Since $\mathcal{W}_i$ is a quotient of $\mathcal{U}_i$, it follows that if $\mathcal{W}_i$ is $\mathcal{L}$ then $\mathcal{U}_i$ is an infinite linearly ordered Wajsberg chain, if $\mathcal{W}_i = \mathcal{C}$ then $\mathcal{U}_i$ is a linearly ordered cancellative hoop, and if $\mathcal{W}_i = \mathbf{2}$ then $\mathcal{U}_i$ is a bounded linearly ordered Wajsberg hoop.

If $\mathcal{W}_i = \mathbf{2}$, then $\mathcal{W}_i$ is a subalgebra $\mathcal{U}_i$ and thus $\mathcal{A} \in \mathbf{ISP}_u(Fin(\mathcal{E}))$ and, by Corollary 3.8, $\mathcal{A} \in Fin(\mathcal{E})$. Otherwise, either $\mathcal{W}_i$ is $\mathcal{L}$ and $\mathcal{U}_i$ is an infinite MV chain, or $\mathcal{W}_i$ is $\mathcal{C}$ and $\mathcal{U}_i$ is a cancellative hoop, so in any case $\mathcal{W}_i$ and $\mathcal{U}_i$ are of the same type and so $\mathcal{A}$ and $\mathcal{B}$ are. Now suppose $\mathcal{A} \notin Fin(\mathcal{E})$, then by (i) of Lemma 3.7, $(e_{\mathcal{A}})$ is valid in $Fin(\mathcal{E})$, hence valid in $\mathbf{ISP}_u(Fin(\mathcal{E}))$, hence valid in $\mathcal{B}$. But by (ii) of Lemma 3.7, $(e_{\mathcal{A}})$ must be also valid in $\mathcal{A}$, which is a contradiction. ■

COROLLARY 3.10. *Let $\mathcal{A}, \mathcal{B} \in REG$. Then $\mathcal{A} \in Var(\mathcal{B})$ iff $\mathcal{A} \in \mathbf{ISP}_u(\mathcal{B})$. Thus if $\mathcal{A}$ satisfies all equations valid in $\mathcal{B}$, it also satisfies all universal formulas valid in $\mathcal{B}$.*

PROOF. By Theorem 3.9, if $\mathcal{A} \in Var(\mathcal{B})$ then $Fin(\mathcal{A}) \subseteq Fin(\mathcal{B})$, and $\mathbf{ISP}_u(Fin(\mathcal{A})) \subseteq \mathbf{ISP}_u(Fin(\mathcal{B}))$. By Lemma 2.9, it follows that $\mathbf{ISP}_u(\mathcal{A}) \subseteq \mathbf{ISP}_u(\mathcal{B})$. The other implication is trivial. ■

4. Canonical regular BL-chains and embedding algorithm

Thanks to Theorem 3.9, in the first part of this section we prove that regular BL-chains admit a kind of canonical form, in terms of their generated varieties, as a *finite* ordinal sum whose components may be either basic components (i.e. $\mathbf{2}, \mathcal{C}, \mathcal{L}$ components) or complex components consisting in turn of ordinal sums of infinite copies of $\mathbf{2}$, $\mathcal{C}$, Π and $\mathcal{L}$ components. This canonical representation is used in the second part of the section to develop an algorithm to check when the variety generated by a regular BL-chain $\mathcal{A}$ is included in the variety generated by another regular BL-chain $\mathcal{B}$.

4.1. Canonical representation of regular BL-chains

DEFINITION 4.1. In the sequel, $\mathcal{C}^\infty$ denotes the ordinal sum of ω copies of $\mathcal{C}$, Π^∞ denotes the ordinal sum of ω copies of Π, and $\mathcal{L}^\infty$ denotes the ordinal sum of ω copies of $\mathcal{L}$. (Recall that $\mathcal{G}$ denotes in fact the ordinal sum of continuum many copies of $\mathbf{2}$.)

DEFINITION 4.2. Let $\mathcal{A} = \bigoplus_{i\in I} \mathcal{A}_i \in REG'$, where for all $i \in I$, $\mathcal{A}_i \in \{\mathcal{C}, \mathbf{2}\}$. We say that $\mathcal{A}$ has *infinitely many alternations of* $\mathcal{C}$ and $\mathbf{2}$ if for every $n \in \mathbb{N}$ there are $i_0 < i_1 < \ldots < i_n \in I$ such that for $j = 0, \ldots, n-1$, if $\mathcal{A}_{i_j} = \mathcal{C}$ then $\mathcal{A}_{i_{j+1}} = \mathbf{2}$, and if $\mathcal{A}_{i_j} = \mathbf{2}$, then $\mathcal{A}_{i_{j+1}} = \mathcal{C}$.

LEMMA 4.3. *Let $\mathcal{A} \in REG'$ with no $\mathcal{L}$ component. Then:*

(i) *If $\mathcal{A}$ has infinitely many $\mathbf{2}$ components and no $\mathcal{C}$ component, then $Fin'(\mathcal{A}) = Fin'(\mathcal{G}) = Fin(\mathcal{G})$*

(ii) *If $\mathcal{A}$ has infinitely many $\mathcal{C}$ components and no $\mathbf{2}$ component, then $Fin'(\mathcal{A}) = Fin'(\mathcal{C}^\infty)$.*

(iii) *If $\mathcal{A}$ has infinitely many alternations of $\mathcal{C}$ and $\mathbf{2}$, then $Fin'(\mathcal{A}) = Fin'(\Pi^\infty)$.*

Furthermore, if $\mathcal{A} \in REG$, then in cases (i) and (iii), $Fin'(\mathcal{A}) = Fin(\mathcal{A})$.

PROOF. It reduces to check that $Fin'(\mathcal{A})$ consists respectively of all finite ordinal sums of $\mathbf{2}$ components in case (i), of all finite ordinal sums of $\mathcal{C}$ components in case (ii), and of all finite ordinal sums of $\mathbf{2}$ and $\mathcal{C}$ components in case (iii). ■

LEMMA 4.4. *Let $\mathcal{A} \in REG'$ with no $\mathcal{L}$ component, and let $\mathcal{B} \in REG$ and $\mathcal{D} \in REG'$ be arbitrary. Then:*

(i) *If $\mathcal{A}$ has infinitely many $\mathbf{2}$ components and no $\mathcal{C}$ component, then $Var(\mathcal{B} \oplus \mathcal{A} \oplus \mathcal{D}) = Var(\mathcal{B} \oplus \mathcal{G} \oplus \mathcal{D})$.*

(ii) *If $\mathcal{A}$ has infinitely many $\mathcal{C}$ components and no $\mathbf{2}$ component, then $Var(\mathcal{B} \oplus \mathcal{A} \oplus \mathcal{D}) = Var(\mathcal{B} \oplus \mathcal{C}^\infty \oplus \mathcal{D})$.*

(iii) *If $\mathcal{A}$ has infinitely many alternations of $\mathcal{C}$ and $\mathbf{2}$, then $Var(\mathcal{B}\oplus\mathcal{A}\oplus\mathcal{D}) = Var(\mathcal{B} \oplus \Pi^\infty \oplus \mathcal{D})$.*

Moreover, if $\mathcal{A} \in REG$*, then in case (i)* $Var(\mathcal{A} \oplus \mathcal{D}) = Var(\mathcal{G} \oplus \mathcal{D})$*, and in case (iii)* $Var(\mathcal{A} \oplus \mathcal{D}) = Var(\Pi^\infty \oplus \mathcal{D})$*.*

PROOF. The equalities easily follow from Lemma 4.3, because by Theorem 3.9 it suffices to show, for instance in case (i), that $Fin(\mathcal{B} \oplus \mathcal{A} \oplus \mathcal{D}) = Fin(\mathcal{B} \oplus \mathcal{G} \oplus \mathcal{D})$, and by Lemma 2.12 this holds since $Fin'(\mathcal{A}) = Fin'(\mathcal{G})$. Analogously for all the remaining cases. ∎

DEFINITION 4.5. A regular BL-algebra $\mathcal{H}$ is said to be *canonical* iff either $\mathcal{H} = \mathcal{L}^\infty$, or $\mathcal{H} = \mathbf{2} \oplus \mathcal{L}^\infty$, or $\mathcal{H}$ is a *finite* ordinal sum of components of the form $\mathcal{L}$, $\mathbf{2}$, $\mathcal{G}$, $\mathcal{C}$, $\mathcal{C}^\infty$ and Π^∞, where

(i) each component $\mathcal{G}$ is *not* preceded and *not* followed by $\mathbf{2}$ or by another $\mathcal{G}$;

(ii) each component $\mathcal{C}^\infty$ is *not* preceded and *not* followed by $\mathcal{C}$ or by another $\mathcal{C}^\infty$.

(iii) each component Π^∞ is *not* preceded and *not* followed by $\mathbf{2}$, $\mathcal{G}$, $\mathcal{C}$, $\mathcal{C}^\infty$ or by another Π^∞.

Due to Lemma 4.4, if we restrict ourselves to t-norm algebras, it turns out that their corresponding canonical regular algebras are indeed t-norm algebras (i.e ordinal sums of $\mathcal{G}$, Π and $\mathcal{L}$ components) with the only exception of those t-norm algebras that generate the whole subvariety **SBL** (those with infinitely-many $\mathcal{L}$ components but not starting with $\mathcal{L}$): $\mathbf{2} \oplus \mathcal{L}^\infty$ is not a t-norm algebra, so it is replaced by $\Pi \oplus \mathcal{L}^\infty$, that also generates the variety of SBL-algebras.

DEFINITION 4.6. A BL-chain $\star$ is said to be a *canonical* t-norm algebra iff either $\star = \mathcal{L}^\infty$, or $\star = \Pi \oplus \mathcal{L}^\infty$, or $\star$ is a *finite* ordinal sum of components of the form $\mathcal{L}$, Π, $\mathcal{G}$ and Π^∞, where each component $\mathcal{G}$ is *not* preceded and *not* followed by another $\mathcal{G}$, and each component Π^∞ is *not* preceded and *not* followed by $\mathcal{G}$, or by Π or by another Π^∞.

As a matter of example, $\mathbf{2} \oplus \Pi^\infty \oplus \mathcal{G} \oplus \mathcal{C}^\infty$ is not a canonical regular algebra, $\mathcal{L} \oplus \mathcal{C} \oplus \mathcal{L} \oplus \Pi^\infty$ is a canonical regular algebra but not a canonical t-norm algebra, and finally, $\mathcal{G} \oplus \mathcal{L} \oplus \Pi^\infty$ is a canonical t-norm algebra.

THEOREM 4.7. (i) *For every regular BL-algebra* $\mathcal{H}$ *there is a canonical regular BL-algebra* $\mathcal{K}$ *such that* $Var(\mathcal{H}) = Var(\mathcal{K})$*.*

(ii) *For every continuous t-norm* $\star$ *there is a canonical continuous t-norm* $\circ$ *such that* $Var(\star) = Var(\circ)$*.*

PROOF. We consider the case of regular algebras and we distinguish several cases:

1. If $\mathcal{H}$ has infinitely many $\mathcal{L}$ components and the first component is $\mathcal{L}$, then $Var(\mathcal{H}) = Var(\mathcal{L}^{\infty})$ is the variety of BL-algebras (cf. [1]).
2. If $\mathcal{H}$ has infinitely many $\mathcal{L}$ components and $\mathcal{L}$ is not the first component, then $Var(\mathcal{H}) = Var(\mathbf{2} \oplus \mathcal{L}^{\infty})$ is the variety of SBL-algebras.
3. If $\mathcal{H}$ has finitely many $\mathcal{L}$ components, say $\mathcal{L}_1, \ldots, \mathcal{L}_n$, then $\mathcal{H}$ is of the form
$$\mathcal{H} = \mathcal{A}_1 \oplus \mathcal{L}_1 \oplus \mathcal{A}_2 \ldots \mathcal{A}_n \oplus \mathcal{L}_n \oplus \mathcal{A}_{n+1}.$$
If $\mathcal{A}_i$ is *not* a finite (or empty) ordinal sum of components isomorphic to $\mathbf{2}$ or to $\mathcal{C}$, then it must contain infinitely many $\mathbf{2}$ or infinitely many $\mathcal{C}$ components (or both) and no $\mathcal{L}$ component. We distinguish the following cases:

    (a) $\mathcal{A}_i$ has finitely many occurrences of $\mathcal{C}$ and infinitely many occurrences of $\mathbf{2}$. Then we replace any maximal ordinal sum of infinitely many consecutive $\mathbf{2}$ components in $\mathcal{A}_i$ by $\mathcal{G}$.

    (b) $\mathcal{A}_i$ has finitely many occurrences of $\mathbf{2}$ and infinitely many occurrences of $\mathcal{C}$. Then we replace any maximal ordinal sum of infinitely many consecutive $\mathcal{C}$ components in $\mathcal{A}_i$ by $\mathcal{C}^{\infty}$.

    (c) $\mathcal{A}_i$ has infinitely many occurrences of $\mathcal{C}$ and infinitely many occurrences of $\mathbf{2}$, but only finitely many alternations of $\mathcal{C}$ and $\mathbf{2}$. Then we replace any maximal ordinal sum of infinitely many consecutive $\mathbf{2}$ components in $\mathcal{A}_i$ by $\mathcal{G}$, and any maximal ordinal sum of consecutive $\mathcal{C}$ components in $\mathcal{A}_i$ by $\mathcal{C}^{\infty}$.

    (d) $\mathcal{A}_i$ has infinitely many alternations of $\mathcal{C}$ and $\mathbf{2}$. Then we replace $\mathcal{A}_i$ by Π^{∞}.

    After performing operations (a), (b), (c) and (d) for all $\mathcal{A}_i$, we obtain a canonical regular BL-algebra $\mathcal{H}$ which, by Lemma 4.4, generates the same variety as $\mathcal{H}$.

As for the case of $\star$ being a continuous t-norm, the above first case is the same, the second one reads

2'. If $\star$ has infinitely many $\mathcal{L}$ components and $\mathcal{L}$ is not the first component, then $Var(\star) = Var(\Pi \oplus \mathcal{L}^{\infty})$ is the variety of SBL-algebras.

and the third one can be simplified as follows:

3'. If $\star$ has finitely many $\mathcal{L}$ components, let $[a_1, b_1]$, ..., $[a_n, b_n]$ be such $\mathcal{L}$ components enumerated in increasing order. Let $\star_i$ be any of the components $[0, a_1]$, $[b_1, a_2]$, ..., $[b_{n-1}, a_n]$ and $[b_n, 1]$. If $\star_i$ is *not* a finite (or empty) ordinal sum of components isomorphic to Π or to $\mathcal{G}$ (after having glued together all contiguous $\mathcal{G}$ components), then it must contain infinitely many Π components and no $\mathcal{L}$ component. In this case we replace $\star_i$ by (an isomorphic copy of) Π^∞.

After we have performed this operation for all $\star_i$, we obtain a canonical t-norm $\circ$. By an iterated application of Lemma 4.4, we see that $Var(\star) = Var(\circ)$. ■

As an example, by applying the procedure described in the last theorem to the BL-algebra $\mathcal{H} = \mathbf{2} \oplus \Pi^\infty \oplus \mathcal{G} \oplus \mathcal{C}^\infty$, one obtains $\mathcal{K} = \Pi^\infty$ as canonical regular algebra such that $Var(\mathcal{H}) = Var(\mathcal{K})$. Notice that in this case, $\mathcal{K}$ is indeed a t-norm BL-algebra, while $\mathcal{H}$ is not.

THEOREM 4.8. *Let* $\mathcal{H} = \bigoplus_{i=0,n} \mathcal{H}_i$ *and* $\mathcal{K} = \bigoplus_{i=0,m} \mathcal{K}_j$ *be two canonical regular BL-chains. Then* $Var(\mathcal{H}) = Var(\mathcal{K})$ *if and only if* $n = m$ *and* $\mathcal{H}_i = \mathcal{K}_i$ *for each* $i = 1, \ldots, n$.

PROOF. Assume $Var(\mathcal{H}) = Var(\mathcal{K})$. We can restrict ourselves to the case of both $\mathcal{H}$ and $\mathcal{K}$ having a finite number of $\mathcal{L}$ components, otherwise either $\mathcal{H} = \mathcal{K} = \mathcal{L}^\infty$ or $\mathcal{H} = \mathcal{K} = \mathbf{2} \oplus \mathcal{L}^\infty$. We shall prove several claims.

CLAIM 1: $\mathcal{H}$ and $\mathcal{K}$ have the same number of $\mathcal{L}$ components and the same first Wajsberg component.

PROOF: We prove that otherwise $Fin(\mathcal{H}) \neq Fin(\mathcal{K})$. Let l_H and l_K the number of $\mathcal{L}$ components in $\mathcal{H}$ and $k_\mathcal{K}$ respectively. If $l_H > l_K$, then any ordinal sum of $Fin(\mathcal{H})$ with l_H components $\mathcal{L}$ will not belong to $Fin(\mathcal{K})$. The case $l_H < l_K$ is analogous. Finally, consider the case $l_H = l_K$ but with $\mathcal{H}$ and $\mathcal{K}$ starting with different Wajsberg components. Without loss of generality, assume $\mathcal{H}$ starts with $\mathbf{2}$ and $\mathcal{K}$ not. Then $\mathbf{2} \oplus \mathcal{L} \oplus \overset{(l_H)}{\ldots\ldots} \oplus \mathcal{L}$ will belong to $Fin(\mathcal{H})$ and not to $Fin(\mathcal{K})$.

From now on, let us denote by $\mathcal{L}_1, \ldots, \mathcal{L}_k$ the k components $\mathcal{L}$ appearing in both the canonical decompositions of $\mathcal{H}$ and $\mathcal{K}$. Then, let $\mathcal{U}$ and $\mathcal{V}$ non-empty ordinal sums with no $\mathcal{L}$ component such that $\mathcal{H} = \ldots \oplus \mathcal{L}_i \oplus \mathcal{U} \oplus \mathcal{L}_{i+1} \oplus \ldots$ and $\mathcal{K} = \ldots \oplus \mathcal{L}_i \oplus \mathcal{V} \oplus \mathcal{L}_{i+1} \oplus \ldots$.

CLAIM 2: $Fin'(\mathcal{U}) = Fin'(\mathcal{V})$.

PROOF: If $Fin'(\mathcal{U}) \neq Fin'(\mathcal{V})$, let e.g. $\mathcal{A}$ be such $\mathcal{A} \in Fin'(\mathcal{V})$ and $\mathcal{A} \notin Fin'(\mathcal{U})$. Then $\mathcal{B} = \mathcal{L}_1 \oplus \ldots \oplus \mathcal{L}_i \oplus \mathcal{A} \oplus \mathcal{L}_{i+1} \oplus \ldots \oplus \mathcal{L}_k$ will belong to $Fin(\mathcal{K})$ but not to $Fin(\mathcal{H})$, hence it would be $Fin(\mathcal{K}) \neq Fin(\mathcal{H})$.

Hence we can assume $Fin'(\mathcal{U}) = Fin'(\mathcal{V})$. Now if both $\mathcal{U}$ and $\mathcal{V}$ have infinitely-many alternations, then $\mathcal{U} = \mathcal{V} = \Pi^{\infty}$.

CLAIM 3: If $\mathcal{U}$ has finitely-many alternations $\mathbf{2}/\mathcal{C}$ (resp. $\mathcal{C}/\mathbf{2}$) then so $\mathcal{V}$ has, and viceversa. Moreover, if the first alternation in $\mathcal{U}$ is of type $\mathbf{2}/\mathcal{C}$ (resp. $\mathcal{C}/\mathbf{2}$), so it is in $\mathcal{V}$.

PROOF: If $\mathcal{V}$ has more alternations $\mathbf{2}/\mathcal{C}$ than $\mathcal{U}$, we can always consider a sufficiently large ordinal sum $\mathcal{A} = \mathbf{2} \oplus \mathcal{C} \oplus \ldots \mathbf{2} \oplus \mathcal{C}$ such that $\mathcal{B} = \mathcal{L}_1 \oplus \ldots \oplus \mathcal{L}_i \oplus \mathcal{A} \oplus \mathcal{L}_{i+1} \oplus \ldots \oplus \mathcal{L}_k$ will belong to $Fin(\mathcal{K})$ but not to $Fin(\mathcal{H})$. Here we have assumed that $\mathcal{H}$ and $\mathcal{K}$ start with a $\mathcal{L}$ component, otherwise, if they start with a $\mathbf{2}$ component, one has to add such component at the beginning of $\mathcal{B}$. The case of alternations $\mathcal{C}/\mathbf{2}$ is dealt with in the same way. Finally, if $\mathcal{U}$ and $\mathcal{V}$ have the same number of alternations (both $\mathbf{2}/\mathcal{C}$ and $\mathcal{C}/\mathbf{2}$), say s, but, e.g., $\mathcal{U}$ starts with $\mathbf{2}$ and $\mathcal{V}$ with $\mathcal{C}$, then $(\bigoplus_{i=1,s}(\mathbf{2} \oplus \mathcal{C})) \oplus \mathbf{2}$ would belong to $Fin'(\mathcal{U})$ but not to $Fin'(\mathcal{V})$, contradiction with Claim 2.

Hence we can restrict ourselves to the case where both $\mathcal{U}$ and $\mathcal{V}$ have the same number (and type) of alternations. Without loss of generality, assume the first alternation in both is of the type $\mathbf{2}/\mathcal{C}$. If t is the number of such alternations, then we can decompose $\mathcal{U}$ and $\mathcal{V}$ in terms of Wajsberg components in the following form:

$$\mathcal{U} = \bigoplus_{i=1,t}(\mathcal{D}_i^1 \oplus \mathbf{2} \oplus \mathcal{C} \oplus \mathcal{E}_i^1),$$

$$\mathcal{V} = \bigoplus_{i=1,t}(\mathcal{D}_i^2 \oplus \mathbf{2} \oplus \mathcal{C} \oplus \mathcal{E}_i^2)$$

where the $\mathcal{D}_i^j$'s are ordinal sums of copies of $\mathbf{2}$'s and the $\mathcal{E}_i^j$'s are ordinal sums of copies of $\mathcal{C}$'s, possibly empty.

CLAIM 4: for each i, $\mathcal{D}_i^1 = \mathcal{D}_i^2$ and $\mathcal{E}_i^1 = \mathcal{E}_i^2$.

PROOF: it is easy to see that if for some i either $\mathcal{D}_i^1 \neq \mathcal{D}_i^2$ or $\mathcal{E}_i^1 \neq \mathcal{E}_i^2$, then $Fin'(\mathcal{D}_i^1) \neq Fin'(\mathcal{D}_i^2)$ or $Fin'(\mathcal{E}_i^1) \neq Fin'(\mathcal{E}_i^2)$ respectively. Assume w.l.g $i = 1$, $\mathcal{D}_i^1 \neq \mathcal{D}_i^2$ and let $\mathcal{A} \in Fin'(\mathcal{D}_i^1)$ and $\mathcal{A} \notin Fin'(\mathcal{D}_i^2)$. Then we easily have that

$$\mathcal{A} \oplus (\bigoplus_{i=1,t}(\mathbf{2} \oplus \mathcal{C}))$$

belongs to $Fin(\mathcal{U})$ and not to $Fin(\mathcal{V})$, contradiction.

Finally, from Claims 1, 2, 3 and 4, it follows that $Var(\mathcal{H}) = Var(\mathcal{K})$ only if they have exactly the same components in their canonical decomposition, i.e. only if $\mathcal{H} = \mathcal{V}$. ∎

Another interesting corollary of the above theorem is the following.

COROLLARY 4.9. *The set of varieties generated by regular BL-algebras is countable. In particular, the set of the ones generated by t-norm BL-algebras is so.*

REMARK. Corollary 4.9 referring to varieties generated by t-norm BL-algebras can also be derived from the paper [25] where it is shown that every variety generated by a t-norm BL-algebra is Co-NP complete, using the fact that there are only countably many Co-NP complete sets.

4.2. Embedding algorithm

There is an easy algorithm to decide whether given two canonical BL-algebras $\mathcal{A}$ and $\mathcal{B}$, one has: $\mathcal{A} \in Var(\mathcal{B})$. Note that this occurs just when $\mathcal{A}$ can be embedded into an ultrapower of $\mathcal{B}$, hence in this case $\mathcal{A}$ satisfies not only all identities valid in $\mathcal{B}$, but also all universal formulas which are valid there. So the algorithm really checks if $\mathcal{A} \in \mathbf{ISP}_u(\mathcal{B})$. Thus our algorithm will be called *embedding algorithm*.
In the following we write $\mathcal{A} \preceq \mathcal{B}$ for $\mathcal{A} \in \mathbf{ISP}_u(\mathcal{B})$.

LEMMA 4.10.

- $\mathcal{C} \preceq \mathcal{L}$ *(here, $\mathcal{C}$ and $\mathcal{L}$ are considered as Wajsberg hoops), and* $\mathbf{2} \preceq \mathcal{L}$.
- *If $\mathcal{E}$ is any finite ordinal sum of linearly ordered hoops from the set* $\{\Pi^\infty, \mathcal{G}, \mathcal{C}, \mathcal{C}^\infty, \mathbf{2}\}$*, the first one bounded, then* $\mathcal{L} \not\preceq \mathcal{E}$.
- *If $\mathcal{E}$ is any finite ordinal sum of $\mathcal{G}$, $\mathcal{C}$, $\mathcal{C}^\infty$ and $\mathbf{2}$, then* $\mathcal{E} \preceq \Pi^\infty$.
- *If $\mathcal{E}$ is any finite ordinal sum of components isomorphic to $\mathbf{2}$, then* $\mathcal{E} \preceq \mathcal{G}$.
- *If $\mathcal{E}$ is any finite ordinal sum of components isomorphic to $\mathcal{C}$, then* $\mathcal{E} \preceq \mathcal{C}^\infty$.
- *If $\mathcal{E}$ is any finite ordinal sum of $\mathcal{C}$, $\mathcal{C}^\infty$, $\mathbf{2}$, $\mathcal{L}$ and $\mathcal{G}$, then* $\Pi^\infty \not\preceq \mathcal{E}$.
- *If $\mathcal{E}$ is any finite ordinal sum of $\mathbf{2}$, $\mathcal{L}$, $\mathcal{C}$ and $\mathcal{C}^\infty$, then* $\mathcal{G} \not\preceq \mathcal{E}$.
- *If $\mathcal{E}$ is any finite ordinal sum of $\mathbf{2}$, $\mathcal{L}$, $\mathcal{C}$ and $\mathcal{G}$, then* $\mathcal{C}^\infty \not\preceq \mathcal{E}$.
- *If $\mathcal{E}$ is any ordinal sum of two or more linearly ordered hoops, then* $\mathcal{E} \not\preceq \mathcal{L}$, $\mathcal{E} \not\preceq \mathcal{C}$ *and* $\mathcal{E} \not\preceq \mathbf{2}$.

Using Lemma 4.10 and Theorem2.8, we develop our algorithm. But first we introduce the following definition.

DEFINITION 4.11. Let $\mathcal{A}$ and $\mathcal{B}$ be two canonical regular BL-chains, $\mathcal{B}$ with at least two components. We say that $\mathcal{B}$ *maximally embeds* in $\mathcal{A}$, written $\mathcal{A} \looparrowright \mathcal{B}$, if $\mathcal{A} \preceq \mathcal{B}$ but $\mathcal{A} \not\preceq \mathcal{B}^-$, where $\mathcal{B}^-$ is the canonical BL-chain resulting from $\mathcal{B}$ by deleting its last (canonical) component. If $\mathcal{B}$ has a single component, then we simply define $\mathcal{A} \looparrowright \mathcal{B}$ if $\mathcal{A} \preceq \mathcal{B}$.

Embedding algorithm. Let $\mathcal{A}, \mathcal{B}$ be canonical regular BL-chains.

If $\mathcal{B} = \mathcal{L}^\infty$, then $\mathcal{A} \in \mathbf{ISP}_u(\mathcal{B})$. If $\mathcal{B} = \mathbf{2} \oplus \mathcal{L}^\infty$, then $\mathcal{A} \in \mathbf{ISP}_u(\mathcal{B})$ iff $\mathcal{A}$ is of the form $\mathbf{2} \oplus \mathcal{H}$ (i.e., if the first component of $\mathcal{A}$ is not $\mathcal{L}$).

If $\mathcal{A} \in \{\mathcal{L}^\infty, \mathcal{G} \oplus \mathcal{L}^\infty\}$ and $\mathcal{B} \notin \{\mathcal{L}^\infty, \mathcal{G} \oplus \mathcal{L}^\infty\}$, then $\mathcal{A} \notin \mathbf{ISP}_u(\mathcal{B})$.

It remains to consider the case where $\mathcal{A} = \bigoplus_{i=0}^n \mathcal{A}_i$ and $\mathcal{B} = \bigoplus_{j=0}^m \mathcal{B}_j$, where for all $i \leq n$ and for all $j \leq m$, $\mathcal{A}_i, \mathcal{B}_j \in \{\mathbf{2}, \mathcal{L}, \mathcal{G}, \mathcal{C}, \mathcal{C}^\infty, \Pi^\infty\}$.

Say that $\bigoplus_{i=0}^k \mathcal{A}_i$ *twice-maximally embeds in* $\bigoplus_{j=0}^r \mathcal{B}_j$ if $\bigoplus_{i=0}^k \mathcal{A}_i \looparrowright \bigoplus_{j=0}^r \mathcal{B}_j$ and if $k < r$ then $\bigoplus_{i=0}^{k+1} \mathcal{A}_i \not\preceq \bigoplus_{j=0}^r \mathcal{B}_j$. At each step s of our construction, we either decide whether $\mathcal{A} \in \mathbf{ISP}_u(\mathcal{B})$ or not, or else we produce indexes $i_s \leq k$, $j_s \leq r$ such that $\bigoplus_{i=0}^{i_s} \mathcal{A}_i$ twice-maximally embeds into $\bigoplus_{j=0}^{j_s} \mathcal{B}_j$, and if $s > 0$, then $i_s > i_{s-1}$ and $j_s > j_{s-1}$.

STEP 0. If $\mathcal{A}_0 = \mathcal{L}$ and $\mathcal{B}_0 \neq \mathcal{L}$, then $\mathcal{A} \not\preceq \mathcal{B}$. Otherwise, let j_0 be minimal such that $\mathcal{A}_0 \preceq \bigoplus_{j=0}^{j_0} \mathcal{B}_j$ (if such a j does not exist, then $\mathcal{A} \not\preceq \mathcal{B}$). Such j_0 is determined as follows:

- If $\mathcal{A}_0 = \mathcal{L}$ and $\mathcal{B}_0 = \mathcal{L}$, then $j_0 = 0$.
- If $\mathcal{A}_0 = \mathbf{2}$, then $j_0 = 0$.
- If $\mathcal{A}_0 = \mathcal{G}$, then j_0 is the minimum j such that either $\mathcal{B}_j = \mathcal{G}$ or $\mathcal{B}_j = \Pi^\infty$ (if such a j does not exists, then $\mathcal{A} \not\preceq \mathcal{B}$).
- If $\mathcal{A}_0 = \Pi^\infty$, then j_0 is the minimum j such that $\mathcal{B}_j = \Pi^\infty$, (if such a j does not exist, then $\mathcal{A} \not\preceq \mathcal{B}$).

Now we define $i_0 = 0$ unless (j_0 is defined and) either: (i) $\mathcal{B}_{j_0} = \Pi^\infty$ and $\mathcal{A}_0 \neq \Pi^\infty$, in this case, i_0 is the maximum i such that for all $h \leq i$, $\mathcal{A}_h \neq \mathcal{L}$; or (ii), $\mathcal{B}_{j_0} = \mathcal{G}$ and $\mathcal{A}_0 = \mathbf{2}$, and in this case, i_0 is the maximum i such that for all $h \leq i$, $\mathcal{A}_h = \mathbf{2}$.

STEP $s+1$. Suppose that the algorithm did not stop within stage s, and that $i_s \leq n$ and $j_s \leq m$ have been produced such that $\bigoplus_{i=0}^{i_s} \mathcal{A}_i$ twice-maximally embeds into $\bigoplus_{i=0}^{j_s} \mathcal{B}_j$.

- If $i_s = n$, then $\mathcal{A} \preceq \mathcal{B}$.
- If $i_s < n$ and $j_s = m$, then $\mathcal{A} \not\preceq \mathcal{B}$.

In the next cases we assume $i_s < n$ and $j_s < m$. Let $k = i_s + 1$. Then:

- If $\mathcal{A}_k = \mathcal{L}$ then j_{s+1} is the minimum $h > j_s$ such that $\mathcal{B}_h = \mathcal{L}$ (if such a j does not exists, then $\mathcal{A} \not\preceq \mathcal{B}$).
- If $\mathcal{A}_k = \mathbf{2}$, then j_{s+1} is the minimum $h > j_s$ such that $\mathcal{B}_h \neq \mathcal{C}$ and $\mathcal{B}_h \neq \mathcal{C}^\infty$ (if such a j does not exists, then $\mathcal{A} \not\preceq \mathcal{B}$).
- If $\mathcal{A}_k = \mathcal{C}$, then j_{s+1} is the minimum $h > j_s$ such that $\mathcal{B}_h \neq \mathbf{2}$ (if such a j does not exists, then $\mathcal{A} \not\preceq \mathcal{B}$).
- If $\mathcal{A}_k = \mathcal{G}$, then j_{s+1} is the minimum $h > j_s$ such that either $\mathcal{B}_h = \mathcal{G}$ or $\mathcal{B}_j = \Pi^\infty$ (if such a h does not exists, then $\mathcal{A} \not\preceq \mathcal{B}$).
- If $\mathcal{A}_k = \mathcal{C}^\infty$, then j_{s+1} is the minimum $h > j_s$ such that $\mathcal{B}_h = \mathcal{C}^\infty$ or $\mathcal{B}_h = \Pi^\infty$, (if such a h does not exist, then $\mathcal{A} \not\preceq \mathcal{B}$).
- If $\mathcal{A}_k = \Pi^\infty$, then j_{s+1} is the minimum $h > j_s$ such that $\mathcal{B}_h = \Pi^\infty$, (if such a h does not exist, then $\mathcal{A} \not\preceq \mathcal{B}$).

Now we define $i_{s+1} = i_s + 1$ unless (j_{s+1} is defined and)

- $\mathcal{B}_{j_{s+1}} = \Pi^\infty$ and $\mathcal{A}_{i_s+1} \neq \Pi^\infty$: in this case, i_{s+1} is the maximum $i > i_s$ such that for all $i_s < h \leq i$, $\mathcal{A}_h \neq \mathcal{L}$.
- $\mathcal{B}_{j_{s+1}} = \mathcal{C}^\infty$ and $\mathcal{A}_{i_s+1} \neq \mathcal{C}^\infty$: in this case, i_{s+1} is the maximum $i > i_s$ such that for all $i_s < h \leq i$, $\mathcal{A}_h = \mathcal{C}$.
- $\mathcal{B}_{j_{s+1}} = \mathcal{G}$ and $\mathcal{A}_{i_s+1} \neq \mathcal{G}$: in this case, i_{s+1} is the maximum $i > i_s$ such that for all $i_s < h \leq i$, $\mathcal{A}_h = \mathbf{2}$.

Clearly the algorithm terminates, and by Lemma 4.10 and by Theorem 2.8 it always gives the correct answer. Moreover, (if we consider $\mathcal{G} \oplus \mathcal{L}$ and if we identify $\mathbf{2} \oplus \mathcal{C}$ and Π), the algorithm for canonical t-norm BL-algebras becomes a special case of the algorithm shown above. Finally it is worth to remark that the algorithm works in polynomial (in fact, linear) time.

EXAMPLE 1. Consider the canonical regular algebras $\mathcal{A} = \mathbf{2} \oplus \mathbf{2} \oplus \mathcal{L} \oplus \mathbf{2} \oplus \mathcal{C}^\infty \oplus \mathcal{G}$ $(n = 5)$ and $\mathcal{B} = \Pi^\infty \oplus \mathcal{L} \oplus \Pi^\infty$ $(m = 2)$, and let us check whether $\mathcal{A} \preceq \mathcal{B}$ using the above algorithm.

STEP 0: $j_0 = 0, i_0 = 1$; $\mathbf{2} \oplus \mathbf{2}$ twice-maximally embeds in Π^∞

STEP 1: $j_1 = 1, i_1 = 2$; $\mathbf{2} \oplus \mathbf{2} \oplus \mathcal{L}$ twice-maximally embeds in $\Pi^\infty \oplus \mathcal{L}$

STEP 2: $j_2 = 2, i_2 = 5$; $\mathbf{2} \oplus \mathbf{2} \oplus \mathcal{L} \oplus \mathbf{2} \oplus \mathcal{C}^\infty \oplus \mathcal{G}$ twice-maximally embeds in $\Pi^\infty \oplus \mathcal{L} \oplus \Pi^\infty$

STEP 3: since $5 = i_2 = n$, then $\mathcal{A} \preceq \mathcal{B}$. STOP.

Now consider the algebras $\mathcal{A} = \mathbf{2} \oplus \mathcal{L} \oplus \mathcal{C} \oplus \mathbf{2}$ $(n = 3)$ and $\mathcal{B} = \Pi^\infty \oplus \mathcal{L} \oplus \mathcal{G} \oplus \mathcal{C}^\infty$ $(m = 3)$ and let us run the algorithm to check whether $\mathcal{A} \preceq \mathcal{B}$.

STEP 0: $j_0 = 0, i_0 = 0$; $\mathbf{2}$ twice-maximally embeds in Π^∞

STEP 1: $j_1 = 1, i_1 = 1$; $\mathbf{2} \oplus \mathcal{L}$ twice-maximally embeds in $\Pi^\infty \oplus \mathcal{L}$

STEP 2: $j_2 = 3, i_2 = 2$; $\mathbf{2} \oplus \mathcal{L} \oplus \mathcal{C}$ twice-maximally embeds in $\Pi^\infty \oplus \mathcal{L} \oplus \mathcal{G} \oplus \mathcal{C}^\infty$

STEP 3: since $2 = i_2 < n = 3$ and $3 = j_2 = m$, then $\mathcal{A} \not\preceq \mathcal{B}$. STOP.

Hence, in this example we have checked that $\mathbf{2} \oplus \mathbf{2} \oplus \mathcal{L} \oplus \mathbf{2} \oplus \mathcal{C}^\infty \oplus \mathcal{G} \preceq \Pi^\infty \oplus \mathcal{L} \oplus \Pi^\infty$ while $\mathbf{2} \oplus \mathcal{L} \oplus \mathcal{C} \oplus \mathbf{2} \not\preceq \Pi^\infty \oplus \mathcal{L} \oplus \mathcal{G} \oplus \mathcal{C}^\infty$. □

5. Axiomatizations of varieties generated by a standard BL-algebra

5.1. General properties

We start with some notation.

- Given $\mathcal{A} \in REG$ we denote by $\mathcal{A}^\perp$ the set $Fin \setminus Fin(\mathcal{A})$.
- Given $\mathbf{M} \subseteq REG$ we denote by $Min(\mathbf{M})$ the set of minimal elements of $\mathbf{M}$ with respect to the relation $\mathcal{B} \preceq \mathcal{D}$ iff $\mathcal{B} \in \mathbf{ISP}_u(\mathcal{D})$. In particular, the sets $Min(\mathcal{A}^\perp)$, for $\mathcal{A} \in REG$, will play a major role in the rest of this section.

Note that if $\mathcal{A} \in Fin$ then $Fin(\mathcal{A}) = Fin \cap \mathbf{ISP}_u(\mathcal{A})$ is finite. Namely, each component of an element of $Fin(\mathcal{A})$ can be either $\mathcal{C}$ or $\mathcal{L}$ or $\mathbf{2}$, and moreover elements of $Fin(\mathcal{A})$ have at most the same number of components as $\mathcal{A}$. Hence, if $\mathcal{D} \in \mathcal{A}^\perp$ there is $\mathcal{H} \in Min(\mathcal{A}^\perp)$ such that $\mathcal{H} \preceq \mathcal{D}$.

The relation $\preceq$ can be extended to classes of algebras: if $\mathbf{M}$ and $\mathbf{M}'$ are two classes of algebras, we shall write $\mathbf{M} \preceq \mathbf{M}'$ iff for all $\mathcal{D} \in \mathbf{M}'$ there exists $\mathcal{B} \in \mathbf{M}$ such that $\mathcal{B} \preceq \mathcal{D}$. For instance, for any $\mathcal{A} \in REG$, we have $Min(\mathcal{A}^\perp) \preceq \mathcal{A}^\perp$.

The following lemma will be useful later.

LEMMA 5.1. (i) *Let $\mathcal{D}$, $\mathcal{E} \in Fin$ such that $\mathcal{D} \preceq \mathcal{E}$. Then every algebra which satisfies the equation $(e_\mathcal{D})$ satisfies the equation $(e_\mathcal{E})$ as well.*

(ii) *Let* $\mathbf{M}, \mathbf{M}' \subseteq Fin$ *such that* $\mathbf{M} \preceq \mathbf{M}'$. *Then, every algebra which satisfies the set of equations* $\{(e_{\mathcal{D}}) \mid \mathcal{D} \in \mathbf{M}\}$ *will also satisfy the equations* $\{(e_{\mathcal{E}}) \mid \mathcal{E} \in \mathbf{M}'\}$.

PROOF. i) First of all, note that from Lemma 3.4, for any $\mathcal{T}_1, \mathcal{T}_2 \in \{\mathbf{2}, \mathcal{C}, \mathcal{L}\}$, if $\mathcal{T}_1 \preceq \mathcal{T}_2$ and $e_{\mathcal{T}_1}(x) = 1$ then $e_{\mathcal{T}_2}(x) = 1$ as well[7]. Assume $\mathcal{E} = \mathcal{E}_0 \oplus ... \oplus \mathcal{E}_n$ and $\mathcal{D} = \mathcal{D}_0 \oplus ... \oplus \mathcal{D}_m$, with $m \leq n$, and let $\mathcal{A} = \mathcal{A}_0 \oplus ... \oplus \mathcal{A}_k$, with $k \geq n$, such that $\mathcal{A}$ does not satisfy the equation $(e_{\mathcal{E}})$. Then there exist a $(n+1)$-tuple in $\mathcal{A}$, $x_0 < x_1 < ... < x_n$, with $x_n < 1$, where $x_0 \in \mathcal{A}_0$ and the rest of x_i's belonging to different components of $\mathcal{A}$, such that $e_i^{\mathcal{E}}(x_i) < 1$ for all $i = 1, ..., n$. Now, since $\mathcal{D} \preceq \mathcal{E}$, then $\mathcal{D}_0 \preceq \mathcal{E}_0$, and there exists a subset of indices $\{j_1, ..., j_m\} \subseteq \{1, ..., n\}$ such that $\mathcal{D}_i \preceq \mathcal{E}_{j_i}$ for all $i = 1, ..., m$. Then, by the above remark, it is clear that, since $e_{j_i}^{\mathcal{E}}(x) < 1$ implies $e_i^{\mathcal{D}}(x) < 1$, the equation $(e_{\mathcal{D}})$ will not be satisfied in $\mathcal{A}$ as well.

ii) Assume an algebra satisfies the equations $\{(e_{\mathcal{D}}) \mid \mathcal{D} \in \mathbf{M}\}$. For all $\mathcal{E} \in \mathbf{M}'$, there exists $\mathcal{D} \in \mathbf{M}$ with $\mathcal{D} \preceq \mathcal{E}$. Then the algebra satisfies the equation $(e_{\mathcal{D}})$ and hence the equation $(e_{\mathcal{E}})$ as well. ∎

Some interesting examples of $Min(\mathcal{A}^{\perp})$ are the following ones:

1. Let $\mathcal{A} \in REG$ be any ordinal sum of infinitely many components isomorphic to $\mathcal{L}$ whose first component is $\mathcal{L}$. Then, $\mathcal{A}^{\perp} = Min(\mathcal{A}^{\perp}) = \emptyset$ and $Min((\mathbf{2} \oplus \mathcal{A})^{\perp}) = \{\mathcal{L}\}$. (Remember that for any chain $\mathcal{A}$ of this kind, $\mathcal{A}$ itself generates the whole variety **BL** and $\mathbf{2} \oplus \mathcal{A}$ generates the whole variety **SBL**, see cf. [1].)
2. $Min(\mathcal{L}^{\perp}) = \{\mathbf{2} \oplus \mathbf{2}, \mathbf{2} \oplus \mathcal{C}\}$.
3. Let $\mathcal{A} \in REG$ be any ordinal sum with infinitely many product components and with no Lukasiewicz component. Then $Min(\mathcal{A}^{\perp}) = \{\mathcal{L}, \mathbf{2} \oplus \mathcal{L}\}$. (Remember that all theses chains generate the same variety.)
4. Let $\mathcal{G}$ be an infinite Gödel chain. Then $Min(\mathcal{G}^{\perp}) = \{\mathcal{L}, \mathbf{2} \oplus \mathcal{C}\}$.
5. Let $\mathcal{P}$ be a product chain. The $Min(\mathcal{P}^{\perp}) = \{\mathcal{L}, \mathbf{2} \oplus \mathbf{2}, \mathbf{2} \oplus \mathcal{C} \oplus \mathcal{C}\}$.

The following theorem shows why, for a given regular BL-chain $\mathcal{A}$, the sets $A^{\perp}$ and, mainly, $Min(\mathcal{A}^{\perp})$ play a fundamental role.

THEOREM 5.2. *Let* $\mathcal{A}$ *be a regular BL-chain. Then:*

[7] Caution: here $e_{\mathcal{T}_1}(x)$ and $e_{\mathcal{T}_2}(x)$ refer to the three expressions defined in Definition 3.3 and not to two instances of the equation in Definition 3.6.

(i) $Var(\mathcal{A})$ *is axiomatized by* $AX(\mathcal{A}) = \{e_{\mathcal{B}} : \mathcal{B} \in \mathcal{A}^{\perp}\}$.

(ii) $Var(\mathcal{A})$ *is axiomatized by* $AX_0(\mathcal{A}) = \{e_{\mathcal{B}} : \mathcal{B} \in Min(\mathcal{A}^{\perp})\}$.

Proof. (i). If $\mathcal{B} \in \mathcal{A}^{\perp}$, then by Lemma 3.7, $e_{\mathcal{B}}$ is valid in (every element of) $Fin(\mathcal{A})$, hence it is valid in every element of $Var(Fin(\mathcal{A})) = Var(\mathcal{A})$. It follows that every member of $Var(\mathcal{A})$ satisfies $AX(\mathcal{A})$. For the other direction, first note that it is sufficient to prove:

Claim A. Any subdirectly irreducible BL-algebra $\mathcal{H} \notin Var(\mathcal{A})$ does not satisfy $AX(\mathcal{A})$.

Indeed, any BL-algebra $\mathcal{H} \notin Var(\mathcal{A})$ is isomorphic to a subdirect product of subdirectly irreducible BL-algebras, and at least one of them is not in $Var(\mathcal{A})$. Now if Claim A holds then at least one subdirectly irreducible factor $\mathcal{H}_i$ of $\mathcal{H}$ does not satisfy $AX(\mathcal{A})$, and since $\mathcal{H}_i$ is a quotient of $\mathcal{H}$, $\mathcal{H}$ does not satisfy $AX(\mathcal{A})$ either.

We proceed with the proof of Claim A. If $\mathcal{H}$ is a subdirectly irreducible BL-chain, then it is the ordinal sum of an indexed family of Wajsberg hoops, the first one bounded. Now let us replace every Wajsberg algebra with more than two elements occurring as a summand in this ordinal sum by $\mathcal{L}$, and every cancellative hoop by $\mathcal{C}$. Then we obtain a regular BL-chain $\mathcal{H}'$. Note that by Lemma 3.7, for $\mathcal{D} \in Fin$, $e_{\mathcal{D}}$ holds in $\mathcal{H}$ iff $e_{\mathcal{D}}$ holds in $\mathcal{H}'$, as $\mathcal{H}$ and $\mathcal{H}'$ are of the same type. To conclude the proof it is sufficient to prove:

Claim B. $\mathcal{H}' \notin Var(\mathcal{A})$.

Indeed, since $\mathcal{H}' \in REG$, by Theorem 3.9, from Claim B we obtain that $Fin(\mathcal{H}') \nsubseteq Fin(\mathcal{A})$, therefore there is $\mathcal{D} \in Fin(\mathcal{H}') \setminus Fin(\mathcal{A})$. But then $e_{\mathcal{D}} \in AX(\mathcal{A})$, $\mathcal{D} \not\models e_{\mathcal{D}}$, $\mathcal{H}' \not\models e_{\mathcal{D}}$ (as $\mathcal{D} \in \mathbf{ISP}_u(\mathcal{H}')$), and finally $\mathcal{H} \not\models e_{\mathcal{D}}$, by Lemma 3.7, as $\mathcal{H}$ and $\mathcal{H}'$ are of the same type.

We thus proceed with the proof of Claim B. Since $\mathcal{H} \notin Var(\mathcal{A})$, there is an equation, $e(x_0, \dots, x_n)$ say, which is true in $\mathcal{A}$ but not in $\mathcal{H}$. Now if $\mathcal{W}_{i_0}, \dots, \mathcal{W}_{i_k}$ are the components which $0, x_0, \dots, x_n$ belong to, enumerated in increasing order, (thus $0 \in \mathcal{W}_{i_0}$, and $\mathcal{W}_{i_0}$ is the first component) then $e(x_0, \dots, x_n)$ is invalidated in $\mathcal{W}_{i_0} \oplus \dots \oplus \mathcal{W}_{i_n}$. Let $\mathcal{U}_0, \dots, \mathcal{U}_n$ be the corresponding Wajsberg components of $\mathcal{H}'$. Then $\mathcal{W}_{i_j} \in \mathbf{ISP}_u(\mathcal{U}_j)$ for $j = 0, \dots, n$, and

$$\mathcal{W}_{i_0} \oplus \dots \oplus \mathcal{W}_{i_n} \in \mathbf{ISP}_u(\mathcal{U}_0 \oplus \dots \oplus \mathcal{U}_n) \subseteq \mathbf{ISP}_u(\mathcal{H}').$$

It follows that $e(x_0, \dots, x_n)$ can be invalidated in some element of $\mathbf{ISP}_u(\mathcal{H}')$, therefore it can be invalidated in $\mathcal{H}'$. Since $e(x_0, \dots, x_n)$ is valid in $\mathcal{A}$, $\mathcal{H}' \notin Var(\mathcal{A})$.

(ii). Clearly $AX(\mathcal{A})$ contains $AX_0(\mathcal{A})$, so we only have to prove that every BL-algebra satisfying $AX_0(\mathcal{A})$ satisfies $AX(\mathcal{A})$ as well. But, since $Min(\mathcal{A}^\perp) \preceq \mathcal{A}^\perp$, the claim easily follows by (ii) of Lemma 5.1. ∎

Finally, as a consequence of (ii) of Theorem 5.2, we end up with the following characterization result of the inclusion between varieties in terms of the $\preceq$ relation between corresponding sets of minimal algebras.

COROLLARY 5.3. *Let* $\mathcal{A}, \mathcal{B} \in REG$. *Then* $Var(\mathcal{A}) \subseteq Var(\mathcal{B})$ *iff* $Min^\perp(\mathcal{A}) \preceq Min^\perp(\mathcal{B})$.

PROOF. As for one direction, assume $Var(\mathcal{A}) \subseteq Var(\mathcal{B})$, hence $Fin(\mathcal{A}) \subseteq Fin(\mathcal{B})$ and $\mathcal{A}^\perp \supseteq \mathcal{B}^\perp$. Assume $\mathcal{D} \in Min^\perp(\mathcal{B})$. Since $Min^\perp(\mathcal{B}) \subseteq B^\perp \subseteq A^\perp$, $\mathcal{D} \in \mathcal{A}^\perp$ as well, hence there is $\mathcal{E} \in Min^\perp(\mathcal{A})$ with $\mathcal{E} \preceq \mathcal{D}$, i.e. we have that $Min^\perp(\mathcal{A}) \preceq Min^\perp(\mathcal{B})$.

As for the other direction, if $Min^\perp(\mathcal{A}) \preceq Min^\perp(\mathcal{B})$, by of ii) of Lemma 5.1 any algebra satisfying the equations $AX_0(\mathcal{A}) = \{e_\mathcal{D} \mid \mathcal{D} \in Min^\perp(\mathcal{A})\}$ will also satisfy the equations $AX_0(\mathcal{B}) = \{e_\mathcal{D} \mid \mathcal{D} \in Min^\perp(\mathcal{B})\}$. But, by Theorem 5.2, these sets of equations axiomatize the varieties $Var(\mathcal{A})$ and $Var(\mathcal{B})$ respectively, thus any algebra of $Var(\mathcal{A})$ must also be in $Var(\mathcal{B})$. ∎

Theorem 5.2 is very important for our purposes since, as we shall see in the next subsections, if $\mathcal{H}$ is a BL-algebra from REG (in particular a t-norm BL-algebra) then $Min(\mathcal{H}^\perp)$ is always finite, therefore, by Theorem 5.2, $Var(\mathcal{H})$ is finitely axiomatizable. And moreover, we shall also show an algorithm which finds all elements of $Min(*^\perp)$, and hence, which leads to a finite set of equations axiomatizing $Var(*)$.

In the remaining part of the section, we show an effective way of computing $Min(\mathcal{A}^\perp)$ for any *canonical* $\mathcal{A} \in REG$. We start by the simpler case of algebras from Fin, i.e. those regular BL-algebras which are isomorphic to finite ordinal sums of $\mathbf{2}, \mathcal{C}, \mathcal{L}$ components. Then we extend the algorithm to the case of arbitrary canonical regular algebras BL-algebras and finally we consider the particular case of canonical t-norm BL-chains.

5.2. The case of regular BL-chains with finitely many components

Actually, the problem of axiomatizing the varieties generated by continuous t-norms with finitely many Wajsberg hoop components was already solved in [1]. Indeed, as we shall see, it is not difficult to show that $Min(\mathcal{A}^\perp)$ is finite for every $\mathcal{A} \in Fin$, hence $Var(\mathcal{A})$ is finitely axiomatizable. Here, besides, we shall describe an effective method for finding the elements of $Min(\mathcal{A}^\perp)$, and thus for finding a set of equations axiomatizing $Var(\mathcal{A})$.

We start by introducing the following definition related to the notion of maximal embeddability introduced in Definition 4.11.

Definition 5.4. Let $\mathcal{A}, \mathcal{B} \in Fin$, where $\mathcal{A} = \mathcal{A}_0 \oplus ... \oplus \mathcal{A}_n$, with $\mathcal{A}_i \in \{\mathbf{2}, \mathcal{C}, \mathcal{L}\}$, for $i = 0, \ldots, n$. Then we define the degree of "maximal embeddability" of $\mathcal{B}$ in $\mathcal{A}$ as follows:

$$g(\mathcal{B} \looparrowright \mathcal{A}) = \begin{cases} k, & \text{if } \mathcal{B} \in Fin(\mathcal{A}) \text{ and } \mathcal{B} \looparrowright \mathcal{A}_0 \oplus ... \oplus \mathcal{A}_k, \text{ with } k \leq n \\ n+1, & \text{if } \mathcal{B} \notin Fin(\mathcal{A}) \end{cases}$$

Of course, $g(\mathcal{B} \looparrowright \mathcal{A}) = n$ iff $\mathcal{B} \looparrowright \mathcal{A}$.

Let $\mathcal{A}$ be as in the above definition and let $\mathcal{D} \in Min(\mathcal{A}^{\perp})$ be of the form $\mathcal{D} = \mathcal{D}' \oplus \mathcal{W}$, where $\mathcal{W}$ is either $\mathbf{2}$, $\mathcal{C}$ or $\mathcal{L}$. If $\mathcal{D}'$ is empty then necessarily $\mathcal{W}$ is isomorphic to $\mathcal{L}$ and $\mathcal{L} \notin Fin(\mathcal{A}_0)$, hence $\mathcal{A}_0$ is isomorphic to a $\mathbf{2}$ component. Assume $\mathcal{D}'$ is non-empty, then it is interesting to remark the following facts concerning $\mathcal{D}'$ and $\mathcal{W}$:

- $\mathcal{D}' \in Fin(\mathcal{A})$.

  Otherwise, if $\mathcal{D}' \notin Fin(\mathcal{A})$ then, since $\mathcal{D}' \prec \mathcal{D}$, $\mathcal{D}$ would not belong to $Min(\mathcal{A}^{\perp})$, contradiction.
- Let $g(\mathcal{D}' \looparrowright \mathcal{A}) = k$, i.e. $\mathcal{D}' \looparrowright \mathcal{A}_1 \oplus ... \oplus \mathcal{A}_k$. Let $\mathcal{A}' = \mathcal{A}_1 \oplus ... \oplus \mathcal{A}_k$ and $\mathcal{A}'' = \mathcal{A}_{k+1} \oplus ... \oplus \mathcal{A}_n$, so $\mathcal{A} = \mathcal{A}' \oplus \mathcal{A}''$. Then, obviously, $\mathcal{W} \notin Fin'(\mathcal{A}'')$.
- Moreover, $\mathcal{D}'$ is minimal in $Fin(\mathcal{A}')$, w.r.t. $\preceq$, among those that maximally embed in $\mathcal{A}'$.

  Namely, if not, let $\mathcal{E} \in Fin(\mathcal{A}')$ such that $\mathcal{E} \prec \mathcal{D}'$ and $\mathcal{E} \looparrowright \mathcal{A}'$. Then we would have that $\mathcal{E} \oplus \mathcal{W} \notin Fin(\mathcal{A})$, $\mathcal{E} \oplus \mathcal{W} \prec \mathcal{D}' \oplus \mathcal{W} = \mathcal{D}$, hence it would imply that $\mathcal{D} \notin Min(\mathcal{A}^{\perp})$, contradiction.

As a direct consequence of above first property we have that any element of $Min(\mathcal{A}^{\perp})$ has at most the same number of components as $\mathcal{A}$ plus one, hence $Min(\mathcal{A}^{\perp})$ is finite.

Proposition 5.5. *If $\mathcal{A} \in Fin$, then $Min(\mathcal{A}^{\perp})$ is finite.*

This basic fact, together with the rest of the above properties, allows us to devise a systematic method to find all elements of $Min(\mathcal{A}^{\perp})$. The idea is, starting from the empty ordinal sum, to successively consider ordinal sums $\mathcal{D}$ (starting with $\mathbf{2}$ or $\mathcal{L}$) of growing length by, step-by-step, expanding them with one Wajsberg component ($\mathbf{2}$, $\mathcal{C}$ or $\mathcal{L}$), until $\mathcal{D} \notin Fin(\mathcal{A})$. Remember that if $\mathcal{D}$ has more components than $\mathcal{A}$ then necessarily $\mathcal{D} \in \mathcal{A}^{\perp}$. However not all possible expansions are worth to be considered, indeed

i) if $\mathcal{D}$ maximally embeds into some $\mathcal{A}'$, where $\mathcal{A} = \mathcal{A}' \oplus \mathcal{A}''$, check if $\mathcal{D}$ is minimal among those previously considered and maximally embedding in $\mathcal{A}'$, if it is so, then $\mathcal{D}$ can be further expanded, otherwise not;

ii) if $\mathcal{D}$ does not embed in $\mathcal{A}$ (i.e. if $\mathcal{D} \notin Fin(\mathcal{A})$), then it is a candidate to belong to $Min(\mathcal{A}^{\perp})$, unless there is another previously found candidate $\mathcal{D}'$ such that $\mathcal{D}' \prec \mathcal{D}$.

This process naturally ends when there is left no ordinal sum to be further expanded.

This method of finding $Min(\mathcal{A}^{\perp})$ can be thought as expanding a tree, where the root node is the empty ordinal sum, and each node is an ordinal sum that corresponds to a possible expansion of its parent. Moreover, during the building procedure, and following the above criteria, nodes are successively either closed or expanded until they are checked not to belong to $Fin(\mathcal{A})$. Let us show a short example of how this method works.

Example 2. Let $\mathcal{A} = \Pi \oplus \mathcal{L} = \mathbf{2} \oplus \mathcal{C} \oplus \mathcal{L}$. The process of finding $Min(\mathcal{A}^{\perp})$ follows the next steps, and the expanded tree is depicted in Figure 1.

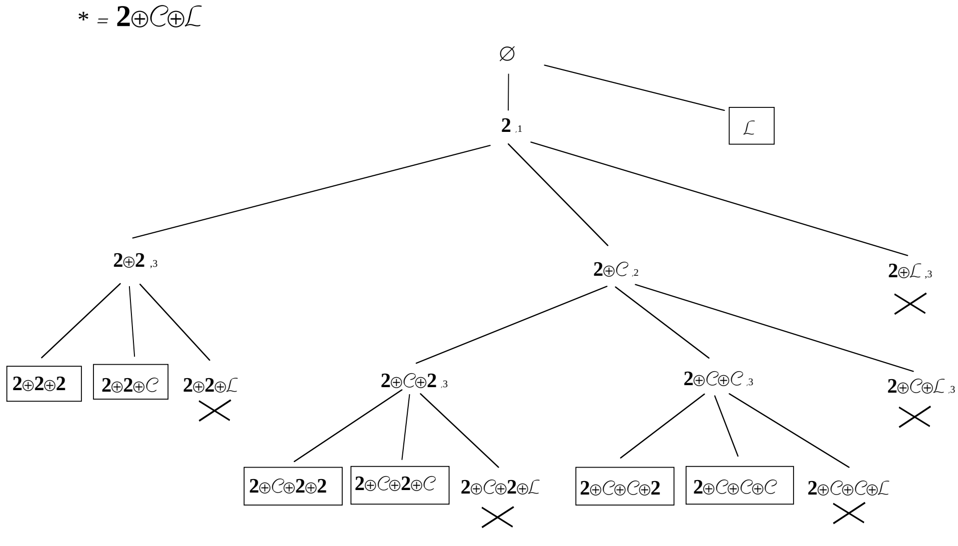

Figure 1. Expanded tree for $* = \mathbf{2} \oplus \mathcal{C} \oplus \mathcal{L}$.

1. We start by considering $\mathcal{U}_1 = \mathbf{2}$ and $\mathcal{U}_2 = \mathcal{L}$. Since $\mathbf{2} \in Fin(\mathcal{A})$ and $\mathcal{L} \notin Fin(\mathcal{A})$ then we already know that $\mathcal{L} \in Min(\mathcal{A}^{\perp})$ and we go on by expanding $\mathcal{U}_1$.

2. Let $\mathcal{U}_{11} = \mathbf{2} \oplus \mathbf{2}$, $\mathcal{U}_{12} = \mathbf{2} \oplus \mathcal{C}$ and $\mathcal{U}_{13} = \mathbf{2} \oplus \mathcal{L}$. Now, it turns out that both $\mathcal{U}_{11}$ and $\mathcal{U}_{13}$ maximally embed in $\mathcal{A}$, but $\mathcal{U}_{12}$ does not, namely $g(\mathcal{U}_{12} \looparrowright \mathcal{A}) = 2$. At this point we can discard $\mathcal{U}_{13}$, since $\mathcal{U}_{11} \prec U_{13}$, and continue expanding $\mathcal{U}_{11}$ and $\mathcal{U}_{12}$:
   (a) Let $\mathcal{U}_{111} = \mathbf{2} \oplus \mathbf{2} \oplus \mathbf{2}, \mathcal{U}_{112} = \mathbf{2} \oplus \mathbf{2} \oplus \mathcal{C}, \mathcal{U}_{113} = \mathbf{2} \oplus \mathbf{2} \oplus \mathcal{L}$. None of them belongs to $Fin(A)$ since $\mathcal{U}_{11} \looparrowright \mathcal{A}$, but only $\mathcal{U}_{111}$ and $\mathcal{U}_{112}$ will be in $Min(\mathcal{A}^{\perp})$ since $\mathcal{U}_{111}, \mathcal{U}_{112} \prec \mathcal{U}_{113}$.
   (b) Let $\mathcal{U}_{121} = \mathbf{2} \oplus \mathcal{C} \oplus \mathbf{2}$, $\mathcal{U}_{122} = \mathbf{2} \oplus \mathcal{C} \oplus \mathcal{C}$, $\mathcal{U}_{123} = \mathbf{2} \oplus \mathcal{C} \oplus \mathcal{L}$. All of them maximally embed in $\mathcal{A}$, but we discard $\mathcal{U}_{123}$, since $\mathcal{U}_{121}, \mathcal{U}_{122} \prec \mathcal{U}_{123}$, and continue expanding $\mathcal{U}_{121}$ and $\mathcal{U}_{122}$:
      i. Let $\mathcal{U}_{1211} = \mathbf{2} \oplus \mathcal{C} \oplus \mathbf{2} \oplus \mathbf{2}$, $\mathcal{U}_{1212} = \mathbf{2} \oplus \mathcal{C} \oplus \mathbf{2} \oplus \mathcal{C}$, $\mathcal{U}_{1213} = \mathbf{2} \oplus \mathcal{C} \oplus \mathbf{2} \oplus \mathcal{L}$. None of them belongs to $Fin(A)$ but only $\mathcal{U}_{1211}$ and $\mathcal{U}_{1211}$ are minimal.
      ii. Analogously, let $\mathcal{U}_{1221} = \mathbf{2} \oplus \mathcal{C} \oplus \mathcal{C} \oplus \mathbf{2}$, $\mathcal{U}_{1222} = \mathbf{2} \oplus \mathcal{C} \oplus \mathcal{C} \oplus \mathcal{C}$, $\mathcal{U}_{1223} = \mathbf{2} \oplus \mathcal{C} \oplus \mathcal{C} \oplus \mathcal{L}$. None of them belongs to $Fin(A)$ but only $\mathcal{U}_{1221}$ and $\mathcal{U}_{1222}$ are minimal.

3. The process ends since there is no further ordinal sum to be expanded and we get $Min(\mathcal{A}^{\perp}) = \{\mathcal{U}_2, \mathcal{U}_{111}, , \mathcal{U}_{112}, \mathcal{U}_{1211}, \mathcal{U}_{1212}, \mathcal{U}_{1221}, \mathcal{U}_{1222}\}$, i.e. $Min(\mathcal{A}^{\perp}) = \{\mathcal{L}, \mathbf{2} \oplus \mathbf{2} \oplus \mathbf{2}, \mathbf{2} \oplus \mathbf{2} \oplus \mathcal{C}, \mathbf{2} \oplus \mathcal{C} \oplus \mathbf{2} \oplus \mathbf{2}, \mathbf{2} \oplus \mathcal{C} \oplus \mathbf{2} \oplus \mathcal{C}, \mathbf{2} \oplus \mathcal{C} \oplus \mathcal{C} \oplus \mathbf{2}, \mathbf{2} \oplus \mathcal{C} \oplus \mathcal{C} \oplus \mathcal{C}\}$.

□

The general procedure of finding $Min(\mathcal{A}^{\perp})$ can be described by the following algorithm, where $\mathcal{U}_{\emptyset}$ will denote the empty ordinal sum, and by convention we take $g(\mathcal{U}_{\emptyset} \looparrowright \mathcal{A}) = 0$.

procedure $find\_Min^{\perp}(\mathcal{A})$
% input: $\mathcal{A} = \mathcal{A}_1 \oplus ... \oplus A_n$, where $\mathcal{A}_1 \in \{\mathbf{2}, \mathcal{L}\}$ and $\mathcal{A}_i \in \{\mathbf{2}, \mathcal{C}, \mathcal{L}\}$ for $i > 1$
% output: *minimal_list* – list in which minimal elements of $\mathcal{A}^{\perp}$ are stored
% auxiliary list: *open_list* – list containing nodes to be expanded
    $n = length(\mathcal{A})$;
    $open\_list = [\mathcal{U}_{\emptyset}]$;
    $minimal\_list = [\ ]$;
    do while $open\_list \neq [\ ]$
        $\mathcal{U} = first(open\_list)$;
        $k = g(\mathcal{U} \looparrowright \mathcal{A})$;
        if $k = 0$ then $expanded\_nodes = \{\mathcal{U} \oplus \mathbf{2}, \mathcal{U} \oplus \mathcal{L}\}$;
        if $0 < k < n$ then $expanded\_nodes = \{\mathcal{U} \oplus \mathbf{2}, \mathcal{U} \oplus \mathcal{C}, \mathcal{U} \oplus \mathcal{L}\}$;
        if $k = n$ then $expanded\_nodes = \{\mathcal{U} \oplus \mathbf{2}, \mathcal{U} \oplus \mathcal{C}\}$;

```
        for all U' ∈ expanded_nodes do
                l = g(U' ↬ A);
                if l ≤ n then open_list = update(open_list,U');
                if l = n + 1 then minimal_list = update(minimal_list,U');
        end for
        open_list = remove(open_list,U);
    end do
end procedure

procedure update(list,U)
% inputs: list to be possibly updated with node U
% output: list after being updated
    for all W ∈ list do
        if g(W ↬ A) = g(U ↬ A) then do
                if W ⪯ U then return list;
                if U ≺ W then list = remove(list, W);
        end do
    end for
    list = append(list,U);
    return list;
end procedure
```

REMARK: Already in this simple case the algorithm is exponential in the worst case, because if $\mathcal{A}$ is the sum of n $\mathcal{L}$-components, then $Min(\mathcal{A}^{\perp})$ consists of all ordinal sums of $n+1$ components in $\{\mathcal{C}, \mathbf{2}\}$ such that the first summand is $\mathbf{2}$, and there are 2^n such ordinal sums.

5.3. The case of arbitrary canonical ordinal sums

Finally, let us consider the problem of finding $Min(\mathcal{A}^{\perp})$ for an arbitrary *canonical* regular BL-chain $\mathcal{A}$. We can restrict ourselves to the case of $\mathcal{A}$ having finitely-many Lukasiewicz components, since if $\mathcal{A}$ has infinitely many components then either $Var(\mathcal{A}) = \mathbf{BL}$ or $Var(\mathcal{A}) = \mathbf{SBL}$, as already mentioned in Subsection 4.2.

In contrast to the previous case, here $Fin(\mathcal{A})$ may be infinite, since components $\mathcal{G}$, $\mathcal{C}^{\infty}$ and Π^{∞} actually are infinite ordinal sums of Wajsberg hoops $\mathbf{2}$ or $\mathcal{C}$. However we shall see that $Min(\mathcal{A}^{\perp})$ remains finite.

The notions related to *maximal embeddability* $\looparrowright$ given in Definition 5.4 for algebras of Fin, naturally extend to this case by considering $\mathcal{G}$, $\mathcal{C}^{\infty}$ and Π^{∞} components as proper components (and not as sums of infinitely-many elementary components), for which it obviously holds that

$\mathbf{2} \in Fin(\mathcal{G})$ and $\mathcal{C}, \mathcal{L} \notin Fin(\mathcal{G})$;

$\mathcal{C} \in Fin(\mathcal{C}^\infty)$ and $\mathbf{2}, \mathcal{L} \notin Fin(\mathcal{C}^\infty)$; and

$\mathbf{2}, \mathcal{C} \in Fin(\Pi^\infty)$ and $\mathcal{L} \notin Fin(\Pi^\infty)$.

However, there is a relevant difference with respect to the previous case. Here, due to the possible presence of $\mathcal{G}$, $\mathcal{C}^\infty$ or Π^∞ components, we can have infinitely-many chains from Fin maximally embedding in a same canonical chain. For instance, for arbitrary $n \geq 1$, all the regular BL-chains $\mathcal{L} \oplus \mathbf{2} \oplus \overset{n}{\ldots} \oplus \mathbf{2}$, maximally embed in $\mathcal{L} \oplus \mathcal{G}$. Analogously, we have that $g(\mathcal{L} \oplus \mathbf{2} \oplus \overset{n}{\ldots} \oplus \mathbf{2} \hookrightarrow \mathcal{L} \oplus \mathcal{G} \oplus \mathcal{L}) = 2$ for any $n \geq 1$. To distinguish these situations, and to identify the cases where the embedded chain is minimal (the case $n = 1$ above), we extend the previous Definition 5.4 to the following one.

Definition 5.6. Let $\mathcal{A}$ be a canonical regular BL-chain, with $\mathcal{A} = \mathcal{A}_1 \oplus ... \oplus \mathcal{A}_n$, where $\mathcal{A}_i \in \{\mathbf{2}, \mathcal{G}, \mathcal{C}, \mathcal{C}^\infty, \Pi^\infty, \mathcal{L}\}$, for $i = 1, \ldots, n$. Let $\mathcal{B} \in Fin$. Then we define:

(i) We define a degree of "maximal embeddability" of $\mathcal{B}$ in $\mathcal{A}$ as follows:

$$g(\mathcal{B} \hookrightarrow \mathcal{A}) = \begin{cases} k, & \text{if } \mathcal{B} \in Fin(\mathcal{A}) \text{ and } \mathcal{B} \hookrightarrow \mathcal{A}_1 \oplus ... \oplus \mathcal{A}_k, \\ & \text{with } k \leq n \\ n+1, & \text{if } \mathcal{B} \notin Fin(\mathcal{A}) \end{cases}$$

(ii) We shall write $\mathcal{B} \hookrightarrow_{\min} \mathcal{A}$ to denote that $\mathcal{B} \hookrightarrow \mathcal{A}$ but $\mathcal{B}^- \not\hookrightarrow \mathcal{A}$, where $\mathcal{B}^-$ is the ordinal sum resulting from $\mathcal{B}$ by deleting its last component.

(iii) We shall write $g(\mathcal{B} \hookrightarrow_{\min} \mathcal{A}) = k$, with $k \leq n$, to denote that $g(\mathcal{B} \hookrightarrow \mathcal{A}) = k$ and $\mathcal{B} \hookrightarrow_{\min} \mathcal{A}_1 \oplus ... \oplus \mathcal{A}_k$.

With this definition, in the above example we would have $\mathcal{L} \oplus \mathbf{2} \hookrightarrow_{\min} \mathcal{L} \oplus \mathcal{G}$ but $\mathcal{L} \oplus \mathbf{2} \oplus \mathbf{2} \not\hookrightarrow_{\min} \mathcal{L} \oplus \mathcal{G}$, and $g(\mathcal{L} \oplus \mathbf{2} \hookrightarrow_{\min} \mathcal{L} \oplus \mathcal{G} \oplus \mathcal{L}) = 2$. On the other hand, since $\mathbf{2}, \mathcal{C} \in Fin'(\Pi^\infty)$, notice that we can perfectly have both $\mathcal{L} \oplus \mathbf{2} \hookrightarrow_{\min} \mathcal{L} \oplus \Pi^\infty$ and $\mathcal{L} \oplus \mathcal{C} \hookrightarrow_{\min} \mathcal{L} \oplus \Pi^\infty$.

Similar considerations to those exposed in the previous case, lead us to the finiteness of $Min(\mathcal{A}^\perp)$ for any canonical regular BL-chain $\mathcal{A}$.

Lemma 5.7. *Let $\mathcal{A} = \mathcal{A}_1 \oplus \ldots \oplus \mathcal{A}_n$ be a canonical regular BL-chain with finitely-many $\mathcal{L}$ components, and let $\mathcal{B} = \mathcal{B}_1 \oplus \ldots \oplus \mathcal{B}_k \in Fin$ such that $\mathcal{B} \in Min(\mathcal{A}^\perp)$. Then $k \leq n + 1$.*

Proof. First of all observe that $\mathcal{B}_1 \oplus \ldots \oplus \mathcal{B}_{k-1} \in Fin(\mathcal{A})$, otherwise $\mathcal{B}$ would not be minimal in $A^\perp$. To prove the lemma it suffices to show that for each $i = 1, \ldots, k-1$, there is j, with $i \leq j \leq n$, such that $\mathcal{B}_1 \oplus \ldots \oplus \mathcal{B}_i \hookrightarrow_{\min}$

$\mathcal{A}_1 \oplus \ldots \oplus \mathcal{A}_j$. We prove it by induction on i. For $i = 1$ it is clear since $\mathcal{B}_1 \hookrightarrow_{\min} \mathcal{A}_1$. Now assume $\mathcal{B}_1 \oplus \ldots \oplus \mathcal{B}_i \hookrightarrow_{\min} \mathcal{A}_1 \oplus \ldots \oplus \mathcal{A}_j$, with $1 < i < k-1$. Since $\mathcal{B}_1 \oplus \ldots \oplus \mathcal{B}_{i+1} \in Fin(\mathcal{A})$, there exists $j' > j$ such that $\mathcal{B}_1 \oplus \ldots \oplus \mathcal{B}_{i+1} \hookrightarrow \mathcal{A}_1 \oplus \ldots \oplus \mathcal{A}_{j'}$. Then it must be $\mathcal{B}_1 \oplus \ldots \oplus \mathcal{B}_{i+1} \hookrightarrow_{\min} \mathcal{A}_1 \oplus \ldots \oplus \mathcal{A}_{j'}$, since otherwise we would have $\mathcal{B}_1 \oplus \ldots \oplus \mathcal{B}_i \hookrightarrow \mathcal{A}_1 \oplus \ldots \oplus \mathcal{A}_{j'}$, in contradiction with the induction hypothesis. ■

As a corollary we have the following proposition, extending previous Proposition 5.5.

PROPOSITION 5.8. *Let $\mathcal{A}$ be a canonical regular BL-chain. Then $Min(\mathcal{A}^{\perp})$ is finite.*

Actually, the procedure $find\_Min^{\perp}$ presented in the previous case perfectly adapts to the present case, but we can improve it a bit for dealing with components Π^{∞} and $\mathcal{G}$. Namely, when a Π^{∞} component is between to $\mathcal{L}$ components, i.e. when $\mathcal{A}$ is of the form $\ldots \oplus \mathcal{L}_i \oplus \Pi^{\infty} \oplus \mathcal{L}_{i+1} \oplus \ldots$, we have to take into account that if an ordinal sum $\mathcal{U} \in$ *open_list* maximally embeds in $\ldots \oplus \mathcal{L}_i$, then it is worthless to consider any expansion of $\mathcal{U}$ with $\mathbf{2}$ and $\mathcal{C}$ components, since they will always embed in $\ldots \oplus \mathcal{L}_i \oplus \Pi^{\infty}$, only the expansion $\mathcal{U} \oplus \mathcal{L}$ makes sense to be considered (which will maximally embed in $\ldots \oplus \mathcal{L}_i \oplus \Pi^{\infty} \oplus \mathcal{L}_{i+1}$). A similar situation is encountered regarding $\mathcal{G}$ components. When $\mathcal{A} = \ldots \oplus \mathcal{W}_i \oplus \mathcal{G} \oplus \mathcal{W}_{i+1} \oplus \ldots$ and $\mathcal{U} \in$ *open_list* maximally embeds in $\ldots \oplus \mathcal{W}_i$, then it is not worth to consider the expansion $\mathcal{U} \oplus \mathbf{2}$, unless in the following situations: $\mathcal{W}_i = \mathcal{W}_{i+1} = \mathcal{C}^{\infty}$, $\mathcal{W}_i = \mathcal{C}^{\infty}$ and $\mathcal{W}_{i+1} = \mathcal{C}$, or $\mathcal{W}_i = \mathcal{C}$ and $\mathcal{W}_{i+1} = \mathcal{C}^{\infty}$.

These improvements amount to modify only one if-condition in the above procedure $find\_Min^{\perp}$, while the procedure *update*(*list*,$\mathcal{U}$) remains the same. Here below is the modified procedure where only the special treatment of Π^{∞} components is included.

procedure $find\_Min^{\perp}(\mathcal{A})$
% input: $\mathcal{A} = \mathcal{A}_1 \oplus ... \oplus A_n$ canonical BL-chain, hence $\mathcal{A}_1 \in \{\mathbf{2}, \mathcal{L}, \mathcal{G}, \Pi^{\infty}\}$ and
% $\mathcal{A}_i \in \{\mathbf{2}, \mathcal{G}, \mathcal{C}, \mathcal{C}^{\infty}, \Pi^{\infty}, \mathcal{L}\}$ for $i > 1$
% output: *minimal_list* – list in which minimal elements of $\mathcal{A}^{\perp}$ are stored
% auxiliary list: *open_list* – list containing nodes to be expanded
  $n = length(\mathcal{A})$;
  open_list = $[\mathcal{U}_{\emptyset}]$;
  minimal_list = [ ];
  do while *open_list* $\neq$ [ ]

$\mathcal{U} = first(open\_list)$;
$k = g(\mathcal{U} \looparrowright \mathcal{A})$;
if $k = 0$ then $expanded\_nodes = \{\mathcal{U} \oplus \mathbf{2}, \mathcal{U} \oplus \mathcal{L}\}$;
if $0 < k < n$ then $expanded\_nodes = \{\mathcal{U} \oplus \mathbf{2}, \mathcal{U} \oplus \mathcal{C}, \mathcal{U} \oplus \mathcal{L}\}$;
if $k = n$ then $expanded\_nodes = \{\mathcal{U} \oplus \mathbf{2}, \mathcal{U} \oplus \mathcal{C}\}$;
for all $\mathcal{U}' \in expanded\_nodes$ do
    $l = g(\mathcal{U}' \looparrowright \mathcal{A})$;
    if $l = 1$ or ($1 < l \leq n$ and $\mathcal{A}_l \neq \Pi^\infty$) then
        $open\_list = update(open\_list, \mathcal{U}')$;
    if $l = n + 1$ then $minimal\_list = update(minimal\_list, \mathcal{U}')$;
end for
$open\_list = remove(open\_list, \mathcal{U})$;
end do
end procedure

EXAMPLE 3. Consider the canonical regular BL-chain $\mathcal{A} = \Pi^\infty \oplus \mathcal{L} \oplus \mathcal{G} \oplus \mathcal{C}^\infty$. Using the above procedure, one builds the expanding tree depicted in Figure 2, where we can observe that $Min(\mathcal{A}^\perp) = \{\mathcal{L}, \mathbf{2} \oplus \mathcal{L} \oplus \mathcal{L}, \mathbf{2} \oplus \mathcal{L} \oplus \mathcal{C} \oplus \mathbf{2}\}$. □

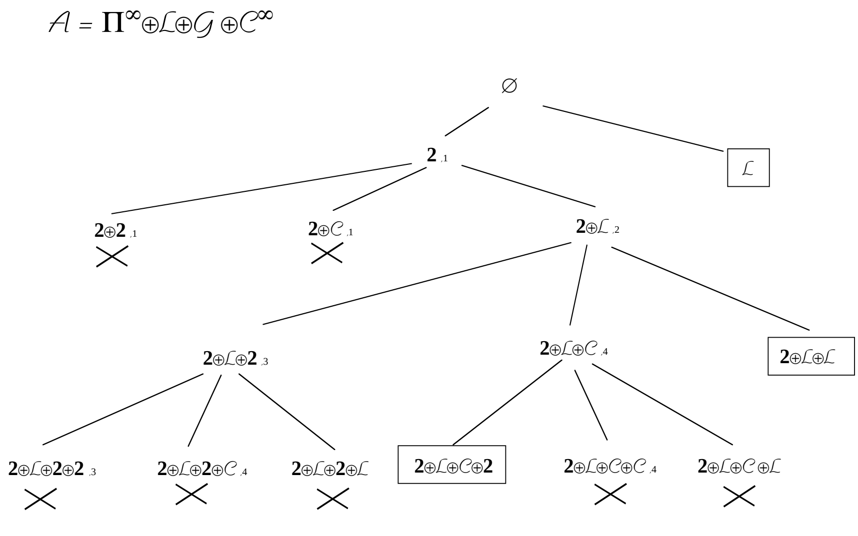

Figure 2. Expanded tree for $\mathcal{A} = \Pi^\infty \oplus \mathcal{L} \oplus \mathcal{G} \oplus \mathcal{C}^\infty$.

5.4. The case of canonical t-norm BL-chains

If we restrict ourselves to canonical t-norm BL-chains, then one can simplify a bit the procedure of finding $Min(\star^\perp)$, since, for instance, $\mathcal{C}^\infty$ components do not appear and **2** components only can appear preceding a $\mathcal{C}$ component. Then, in particular, the treatment of $\mathcal{G}$ components outlined in the previous section becomes simpler and easy to implement. The following modified $find\_Min^{\perp}$ procedure takes this into account.

procedure $find\_Min^{\perp}(\mathcal{A})$
% input: $\mathcal{A} = \mathcal{A}_1 \oplus ... \oplus A_n$, canonical t-norm BL-chain, where
% $\quad\quad \mathcal{A}_1 \in \{\mathbf{2}, \mathcal{L}, \mathcal{G}, \Pi^\infty\}$ and $\mathcal{A}_i \in \{\mathbf{2}, \mathcal{C}, \mathcal{L}, \mathcal{G}, \Pi^\infty\}$ for $i > 1$
% output: *minimal_list* – list in which minimal elements of $\mathcal{A}^\perp$ are stored
% auxiliary list: *open_list* – list containing nodes to be expanded
$\quad n = length(\mathcal{A})$;
$\quad open\_list = [\mathcal{U}_\emptyset]$;
$\quad minimal\_list = [\,]$;
$\quad$do while $open\_list \neq [\,]$
$\quad\quad \mathcal{U} = first(open\_list)$;
$\quad\quad k = g(\mathcal{U} \hookrightarrow \mathcal{A})$;
$\quad\quad$if $k = 0$ then $expanded\_nodes = \{\mathcal{U} \oplus \mathbf{2}, \mathcal{U} \oplus \mathcal{L}\}$;
$\quad\quad$if $0 < k < n$ then $expanded\_nodes = \{\mathcal{U} \oplus \mathbf{2}, \mathcal{U} \oplus \mathcal{C}, \mathcal{U} \oplus \mathcal{L}\}$;
$\quad\quad$if $k = n$ then $expanded\_nodes = \{\mathcal{U} \oplus \mathbf{2}, \mathcal{U} \oplus \mathcal{C}\}$;
$\quad\quad$for all $\mathcal{U}' \in expanded\_nodes$ do
$\quad\quad\quad\quad l = g(\mathcal{U}' \hookrightarrow \mathcal{A})$;
$\quad\quad\quad\quad$if $l = 1$ or ($1 < l \leq n$ and $\mathcal{A}_l \neq \mathcal{G}$ and $\mathcal{A}_l \neq \Pi^\infty$) then
$\quad\quad\quad\quad\quad open\_list = update(open\_list, \mathcal{U}')$;
$\quad\quad\quad\quad$if $l = n + 1$ then $minimal\_list = update(minimal\_list, \mathcal{U}')$;
$\quad\quad$end for
$\quad\quad open\_list = remove(open\_list, \mathcal{U})$;
$\quad$end do
end procedure

EXAMPLE 4. Let us consider a continuous t-norm $\star$ isomorphic to $\mathcal{G} \oplus \mathcal{L} \oplus \Pi^\infty \oplus \mathcal{L}$. The above procedure yields to the expanded tree of Figure 3, where one can get $Min(\star^\perp) = \{\mathcal{L}, \mathbf{2} \oplus \mathcal{C} \oplus \mathcal{L} \oplus \mathbf{2}, \mathbf{2} \oplus \mathcal{C} \oplus \mathcal{L} \oplus \mathcal{C}\}$. □

6. Conclusions and future work

In this paper we have solved the problem of finding equational definitions (axiomatic definitions resp.) of subvarieties of **BL** generated by single regular BL-chains (of schematic extensions of the logic BL that are complete with respect to a given regular BL-chain, resp.). In particular, we have solved the problem for standard BL-chains, i.e for BL-chains over [0,1] defined by a

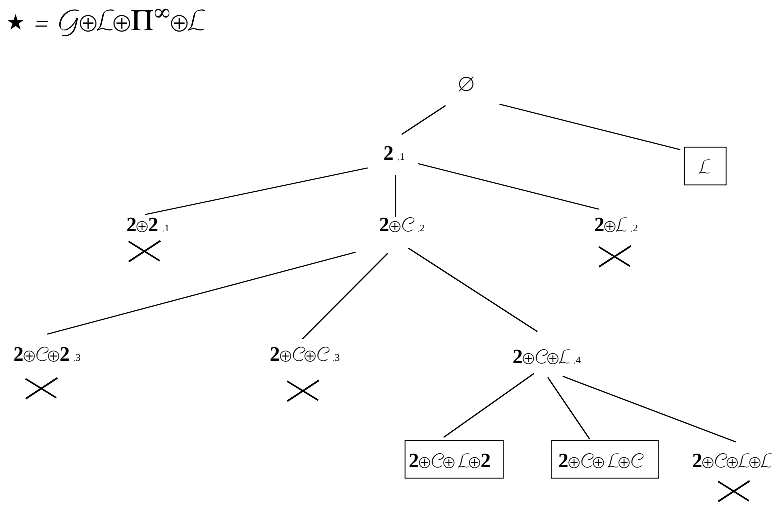

Figure 3. Expanded tree for $\star = \mathcal{G} \oplus \mathcal{L} \oplus \Pi^{\infty} \oplus \mathcal{L}$.

continuous t-norm. But these are not all the so called t-norm based subvarieties of **BL**, since they do not contain the subvarieties generated by families of t-norm BL-chains.

However, it is easy to generalize the first results of this paper to families of continuous t-norms. Let $T = \{*_i \mid i \in I\}$ be a family of continuous t-norms. If we define $Fin(T) = \cup_{i \in I} Fin(*_i)$ the following result holds.

LEMMA 6.1. $Var(T) = Var(Fin(T))$

PROOF. First it is clear that $Var(T) \supseteq Fin(*_i)$ for all $i \in I$ and thus $Var(T) \supseteq Var(\cup_{i \in I} Fin(*_i) = Var(Fin(T))$. On the other hand, from Lemma 3.1, $Var(Fin(*_i)) \supseteq \{*_i\}$ and thus $Var(Fin(T) \supseteq T$, which implies $Var(Fin(T)) \supseteq Var(T)$. The two inequalities prove the lemma. ■

PROPOSITION 6.2. *Let T and S be two families of continuous t-norms. Then $Var(T) \subseteq Var(S)$ iff $Fin(T) \subseteq Fin(S)$.*

The proof is completely analogous to the proof of Theorem 3.9.

The algorithm for obtaining the equations of the subvariety described in Section 5 can be obviously generalized to finite families of continuous t-norms, taking into account the above results and the definition of $Fin(T)$. Moreover, the algorithm is also applicable to any family of continuous t-norms whose decomposition as ordinal sums contains no $\mathcal{L}$ component, since in such a case the family defines the same subvariety than a finite subfamily of it, as the following proposition shows.

Proposition 6.3. *Let* $T = \{*_i \mid i \in I\}$ *be a family of continuous t-norms that have no* $\mathcal{L}$ *component, and let* n_i *be the number of components of the decomposition of* $*_i$ *as ordinal sum of the basic t-norms. Then there are two cases:*

1) *If* $\sup\{n_i \mid i \in I\} = n < \infty$*, then the set* T *itself is finite.*
2) *If* $\sup\{n_i \mid i \in I\} = \infty$*, then* $Var(T) = Var(\Pi^\infty)$*.*

Proof. The proof of 1) is evident since the full set of ordinal sums of t-norms with at most n t-norm components is finite. The proof of 2) is also easy since the set $Fin(T)$ is obviously equal to the set $Fin(\Pi^\infty)$. ■

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