Quasi-bisimulation: a new paradigm for the strict implication language

Félix Bou
Universitat de Barcelona
Gran Via 585, 08007 Barcelona (Spain)
E-mail: bou@mat.ub.es

The purpose of this contribution is to introduce the notion of quasi-bisimulation that characterizes the strict implication fragment of modal propositional language in the same way as bisimulation characterizes this one.

Fix a set AtProp of *atomic propositions*. The set $\mathcal{L}^{MOD}$ of *modal formulas* is the smallest set X such that i) $\mathsf{AtProp} \subseteq X$, ii) X contains the logical constant $\bot$ (*falsum*), iii) X is closed under boolean connectives $\wedge, \vee, \sim, \supset$ (*conjunction, disjunction, negation, implication*) as well as under the unary *necessity operator* $\Box$. A modal formula φ can be seen as the first order formula $ST_{x_0}(\varphi)$ where ST is the well known *standard translation*.

The *strict implication (si)* fragment is the set $\mathcal{L}^{SI} \subseteq \mathcal{L}^{MOD}$ defined as the smallest set X such that i) $\mathsf{AtProp} \cup \{\bot\} \subseteq X$, ii) X is closed under $\wedge, \vee$, iii) if $\varphi, \psi \in X$ then $\Box(\varphi \supset \psi) \in X$. The semantic condition inherited by the binary new connective of strict implication $\Box(\varphi \supset \psi)$ on Kripke models is the same as the condition for the implication of intuitionistic logic (and superintuitionistic logics) in its standard Kripke semantic.

Now we single out several relations (proper classes) between pointed Kripke models. We write $\langle \mathcal{M}, m \rangle \leftrightsquigarrow \langle \mathcal{N}, n \rangle$ in the case that $\forall \varphi \in \mathcal{L}^{MOD}$, $\langle \mathcal{M}, m \rangle \models \varphi$ iff $\langle \mathcal{N}, n \rangle \models \varphi$; and we write $\langle \mathcal{M}, m \rangle \rightsquigarrow \langle \mathcal{N}, n \rangle$ if $\forall \varphi \in \mathcal{L}^{SI}$, $\langle \mathcal{M}, m \rangle \models \varphi$ implies $\langle \mathcal{N}, n \rangle \models \varphi$. As usual a *bisimulation* between $\mathcal{M}$ and $\mathcal{N}$ is a set $Z \subseteq M \times N$ satisfying i) 'atomic invariance' at Z-corresponding states, ii) $Z \circ R^{\mathcal{N}} \subseteq R^{\mathcal{M}} \circ Z$, iii) $Z^{-1} \circ R^{\mathcal{M}} \subseteq R^{\mathcal{N}} \circ Z^{-1}$. We write $\langle \mathcal{M}, m \rangle \equiv \langle \mathcal{N}, n \rangle$ if exists Z a bisimulation between $\mathcal{M}$ and $\mathcal{N}$ such that $\langle m, n \rangle \in Z$. A *quasi-bisimulation* between $\mathcal{M}$ and $\mathcal{N}$ is a pair $\langle U, Z \rangle$ such that i) $Z \subseteq U \subseteq M \times N$, ii) Z is a bisimulation between $\mathcal{M}$ and $\mathcal{N}$, iii) 'atomic preservation' at U-corresponding states, iv) $U \circ R^{\mathcal{N}} \subseteq R^{\mathcal{M}} \circ Z$ (i.e., if $\langle m, n \rangle \in U$ and $\langle n, n' \rangle \in R^{\mathcal{N}}$ then exists $m' \in M$ such that $\langle m, m' \rangle \in R^{\mathcal{M}}$ and $\langle m', n' \rangle \in Z$). We write $\langle \mathcal{M}, m \rangle \preceq \langle \mathcal{N}, n \rangle$ if exists $\langle U, Z \rangle$ a quasi-bisimulation between $\mathcal{M}$ and $\mathcal{N}$ such that $\langle m, n \rangle \in U$. Using the relation $\preceq$ we introduce what it does mean that a first order formula $\alpha(x_0)$ is *preserved for quasi-bisimulations*.

Theorem 1 *A first order formula $\alpha(x_0)$ is preserved for quasi-bisimulations iff it is equivalent to $ST_{x_0}(\varphi)$ for a certain $\varphi \in \mathcal{L}^{SI}$.*

Finally, it is interesting to emphasize that some results suggest that the si-fragment and the notion of quasi-bisimulation are not only interesting by themselves, they also appear in a natural way if we want to understand the full modal language and the notion of bisimulation. This is illustrated here with two results.

Theorem 2 $\langle \mathcal{M}, m \rangle \leftrightsquigarrow \langle \mathcal{N}, n \rangle$ *iff* $\langle \mathcal{M}, m \rangle \rightsquigarrow \langle \mathcal{N}, n \rangle$ *and* $\langle \mathcal{N}, n \rangle \rightsquigarrow \langle \mathcal{M}, m \rangle$.

Theorem 3 $\langle \mathcal{M}, m \rangle \equiv \langle \mathcal{N}, n \rangle$ *iff* $\langle \mathcal{M}, m \rangle \preceq \langle \mathcal{N}, n \rangle$ *and* $\langle \mathcal{N}, n \rangle \preceq \langle \mathcal{M}, m \rangle$.

Thus, the relation $\preceq$ is a quasi-order (i.e., reflexive and transitive) that generates the equivalence relation $\equiv$. This is why we have called quasi-bisimulation to our new concept, it induces a quasi-order whose equivalence relation is the bisimilarity relation.