On logics of formal inconsistency and fuzzy logics

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Paraconsistency is the study of logics (as deductive systems) having a negation operator $\neg$ such that not every contradiction $\{\varphi, \neg\varphi\}$ trivializes or explodes. In other words, a paraconsistent logic is a logic having at least a contradictory, non-trivial theory.

Among the plethora of paraconsistent logics proposed in the literature, the so-called *Logics of Formal Inconsistency* (LFIs), proposed in [3] (see also [2]), play an important role, since they internalize in the object language the very notions of consistency and inconsistency by means of specific connectives (either primitive or not). This generalizes the strategy of da Costa, which introduced in [5] the well-known hierarchy of systems C_n, for $n > 0$. Besides being able to distinguish between contradiction and inconsistency, on the one hand, and non-contradiction and consistency, on the other, LFIs are non-explosive logics, that is, paraconsistent: in general, a contradiction does not entail arbitrary statements, and so the Principle of Explosion $\varphi, \neg\varphi \vdash \psi$ does not hold. However, LFIs are *gently explosive*, in the sense that, adjoining the additional requirement of consistency, then contradictoriness does cause explosion: $\bigcirc(\varphi), \varphi, \neg\varphi \vdash \psi$ for every φ and ψ. Here, $\bigcirc(\varphi)$ denotes the consistency of φ. The general definition of LFIs we will adopt here, slightly modified from the original one proposed in [3] and [2], is the following:

Definition 1. Let $(\mathrm{L}, \vdash)$ be a logic defined in a language $\mathcal{L}$ containing a negation $\neg$, and let $\bigcirc(p)$ be a nonempty set of formulas of $\mathcal{L}$ depending exactly on the propositional variable p. Then L is an LFI (with respect to $\neg$ and $\bigcirc(p)$) if the following holds (here, $\bigcirc(\varphi) = \{\psi[p/\varphi] \ : \ \psi(p) \in \bigcirc(p)\}$):

(i) $\varphi, \neg\varphi \nvdash \psi$ for some φ and ψ, i.e. the logic is not explosive;

(ii) $\bigcirc(\varphi), \varphi \nvdash \psi$ for some φ and ψ;

(iii) $\bigcirc(\varphi), \neg\varphi \nvdash \psi$ for some φ and ψ; and

(iv) $\bigcirc(\varphi), \varphi, \neg\varphi \vdash \psi$ for every φ and ψ.

In many situations $\bigcirc(\varphi)$ is a singleton, whose element will we denoted by $\circ\varphi$, and $\circ$ is called a *consistency* operator in L with respect to $\neg$. It has to be noticed that in the frame of LFIs, the term *consistent* rather refers to formulas that basically exhibit a classical logic behaviour, so in particular an explosive behaviour. Such a consistency operator can be primitive (as in the case of most of the systems treated in [3] and [2]) or, on the contrary, it can be defined in terms of the other connectives of the language. For instance, in the

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well-known system C_1 by da Costa, consistency is defined by the formula $\circ\varphi = \neg(\varphi \wedge \neg\varphi)$ (see [5]).

Systems of mathematical fuzzy logic, understood as truth-preserving many-valued logics in the sense of [7, 4], are not paraconsistent. Indeed, in these systems, $\varphi\&\neg\varphi$ is always evaluated to 0, and hence any formula can be deduced from the set of premises $\{\varphi, \neg\varphi\}$. However, the situation is different if one considers, for each truth-preserving logic L, its companion $L^{\leq}$ that preserves degrees of truth as studied in [1]. In fact, in these systems $L^{\leq}$, a formula φ follows from a (finite) set of premises Γ when, for all evaluations e on a corresponding class of L-chains, $e(\varphi) \geq \min\{e(\psi) \mid \psi \in \Gamma\}$. Obviously, if L is not pseudo-complemented, there is always some evaluation e such that $e(\varphi \wedge \neg\varphi) > 0$. This says that $\{\varphi, \neg\varphi\}$ is not explosive in $L^{\leq}$ and thus, there are fuzzy logics preserving degrees of truth that are paraconsistent (see [6] for a preliminary study).

In this paper, given an axiomatic extension L of MTL that is not SMTL, we first study natural conditions a consistency operator $\circ$ on L-chains has to satisfy. These conditions are used then to define both a semilinear truth-preserving logic $L_\circ$, over the language of L expanded with a new unary connective $\circ$, as well as its paraconsistent companion $L_\circ^{\leq}$. Finally we consider several extensions of $L_\circ^{\leq}$, capturing several further properties one can ask to the consistency operator $\circ$. For instance, we introduce the logics $(L_\circ^{\neg\neg})^{\leq}$ (where the negation in the chains of the quasi-variety of L-algebras satisfies the condition $\neg\neg x = 1$ iff $x = 1$), the logic $(L_\circ^c)^{\leq}$ (where the operator $\circ$ is Boolean) and the logics $(L_\circ^{\min})^{\leq}$ or $(L_\circ^{\max})^{\leq}$ (where the consistency operators are the minimum and the maximum ones respectively).

Finally we study in the above logics the problem of recovering the classical reasoning by means of the consistency connective $\circ$, a very desirable property in the context of LFIs (see [2]), called DAT (Derivability Adjustment Theorem). When the operator $\circ$ enjoys a suitable propagation property in the logic L with respect to the classical connectives, then the DAT in $L_\circ^{\leq}$ assumes the following simplified form: for every finite set of formulas $\Gamma \cup \{\varphi\}$ in the language of classical propositional logic (**CPL**),

$$\text{(PDAT)} \quad \Gamma \vdash_{\mathbf{CPL}} \varphi \ \text{ iff } \ \{\circ p_1, \ldots, \circ p_n\} \cup \Gamma \vdash_{L_\circ^{\leq}} \varphi$$

where $\{p_1, \ldots, p_n\}$ is the set of propositional variables occurring in $\Gamma \cup \{\varphi\}$.

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