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Fuzzy set-based Approximate Reasoning and Mathematical Fuzzy Logic

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1.1 Introduction

Zadeh proposed and developed the theory of approximate reasoning in a long series of papers in the 1970's (see e.g. [28, 29, 30, 31, 32, 34, 35]), at the same time when he introduced possibility theory [33] as a new approach to uncertainty modeling. His original approach is based on a fuzzy set-based representation of the contents of factual statements (expressing elastic restrictions on the possible values of some parameters) and of if-then rules relating such fuzzy statements. Zadeh himself wrote in [37] that *fuzzy logic in narrow sense*

> “ (...) is a logical system which aims a formalization of approximate reasoning. In this sense it is an extension of many-valued logic. However the agenda of fuzzy logic (FL) is quite different from that of traditional many-valued logic. Such key concepts in FL as the concept of linguistic variable, fuzzy if-then rule, fuzzy quantification and defuzzification, truth qualification, the extension principle, the compositional rule of inference and interpolative reasoning, among others, are not addressed in traditional systems. ”

Thus, according to Zadeh, fuzzy logic is something more than a system of many-valued logic, in particular it clearly departs at first glance from the standard view of (many-valued) logic where inference does not depend on the contents of propositions.

On the other hand, the study of the so-called *t-norm based fuzzy logics* corresponding to formal many-valued calculi with truth-values in the real unit interval $[0, 1]$ defined by a conjunction and an implication interpreted respectively by a (left-) continuous t-norm and its residuum[1], has had since the mid nineties a great development from many points of view (logical, algebraic, proof-theoretical, functional representation, and computational complexity). This has been witnessed by a number of important monographs that have appeared in the literature since then, see e.g. [18, 17, 25], and has very recently

[1] Thus, including e.g. the well-known Łukasiewicz and Gödel infinitely-valued logics, defined much before fuzzy logic was born, and corresponding to the calculi defined by Łukasiewicz and min t-norms respectively.

culminated with the handbook [3]. It is worth noticing that, although formal, all these systems originated as an attempt to provide sound logical foundations for fuzzy set theory as well as to address computational problems related to vagueness and imprecision. Indeed, Hájek, in the introduction of his celebrated monograph [18] makes the following comment to Zadeh's quotation:

> " Even if I agree with Zadeh's distinction (. . .) I consider formal calculi of many-valued logic to be the kernel of fuzzy logic in the narrow sense and the task of explaining things Zadeh mentions by means of this calculi to be a very promising task. "

Following this line of thought, a main part of our research efforts in the last years have been devoted to the study and definition of different t-norm based fuzzy logic systems (see e.g. [1, 9, 10, 11, 7, 12, 13, 8, 2]), but having in mind that a main task was to address as much as possible the different aspects of the agenda of the fuzzy logic in narrow sense not in principle directly covered by them, e.g. the approximate reasoning machinery (flexible constraints propagation, generalized modus ponens, compositional rule of inference, etc.).

In this short note, as our modest homage to Prof. Lotfi Zadeh and his great contributions, we revisit an approach (c.f. [18, 16, 4]) to understand some of Zadeh's approximate reasoning principles as sound deductions within a formal system of mathematical fuzzy logic, so trying to bridge the gap between both areas.

1.2 Propositional and predicate t-norm based fuzzy logics

T-norm based (propositional) logics correspond to logical calculi with the real interval $[0, 1]$ as set of truth-values and defined by a conjunction $\&$ and an implication $\rightarrow$ interpreted respectively by a left-continuous t-norm $*$ and its residuum $\Rightarrow$, and where negation is defined as $\neg\varphi = \varphi \rightarrow \overline{0}$, with $\overline{0}$ being the truth-constant for falsity. In this framework, each left continuous t-norm $*$ uniquely determines a semantical (propositional) calculus $PC(*)$ over formulas defined in the usual way from a countable set of propositional variables, connectives $\wedge$, $\&$ and $\rightarrow$ and truth-constant $\overline{0}$ [18]. Further connectives are defined as follows:

$\varphi \vee \psi$ is $((\varphi \rightarrow \psi) \rightarrow \psi) \wedge ((\psi \rightarrow \varphi) \rightarrow \varphi)$,
$\neg\varphi$ is $\varphi \rightarrow \bar{0}$,
$\varphi \equiv \psi$ is $(\varphi \rightarrow \psi)\&(\psi \rightarrow \varphi)$.

Evaluations of propositional variables are mappings e assigning to each propositional variable p a truth-value $e(p) \in [0, 1]$, which extend univocally to compound formulas as follows:

$$e(\overline{0}) = 0$$
$$e(\varphi \wedge \psi) = \min(e(\varphi), e(\psi))$$
$$e(\varphi \& \psi) = e(\varphi) * e(\psi)$$
$$e(\varphi \to \psi) = e(\varphi) \Rightarrow e(\psi)$$

Note that, from the above defintions, $e(\varphi \vee \psi) = \max(e(\varphi), e(\psi))$, $\neg\varphi = e(\varphi) \Rightarrow 0$ and $e(\varphi \equiv \psi) = e(\varphi \to \psi) * e(\psi \to \varphi)$. A formula φ is a said to be a 1-tautology of $PC(*)$ if $e(\varphi) = 1$ for each evaluation e. The set of all 1-tautologies of $PC(*)$ will be denoted as $TAUT(*)$.

Well-known axiomatic systems, like Łukasiewicz logic (Ł), Gödel logic (G), Product logic (Π), Basic Fuzzy logic (BL) and Monoidal t-norm logic (MTL) syntactically capture different sets of $TAUT(*)$ for different choices of the t-norm $*$, see e.g. [18, 17, 3]. In other words, the following condiitons hold true, where $*_{Ł}$, $*_G$ and $*_\Pi$ respectively denote the Łukasiewicz t-norm, the min t-norm and the product t-norm:

φ is provable in Ł iff $\varphi \in TAUT(*_{Ł})$
φ is provable in G iff $\varphi \in TAUT(*_G)$
φ is provable in Π iff $\varphi \in TAUT(*_\Pi)$
φ is provable in BL iff $\varphi \in TAUT(*)$ for all cont. t-norm $*$
φ is provable in MTL iff $\varphi \in TAUT(*)$ for all left-cont. t-norm $*$

These completeness results also extend to deductions from a finite set of premises but, in general, they do not extend to deductions from an infinite set of premises. Prominent exceptions are the case of Gödel logic and MTL.

Predicate logic versions of propositional t-norm based logics have also been defined and studied in the literature. Following [20] we provide below a general definition of the predicate logic $L_*\forall$ for any propositional logic L_* of a t-norm $*$. As usual, the propositional language of L_* is enlarged with a set of predicates $Pred$, a set of object variables Var and a set of object constants $Const$, together with the two classical quantifiers $\forall$ and $\exists$. An $[0, 1]$-valued L-interpretation for a predicate language $\mathcal{PL} = (Pred, Const)$ of $L_*\forall$ is a structure

$$\mathbf{M} = (M, (r_P)_{P \in Pred}, (m_c)_{c \in Const})$$

where $M \neq \emptyset$, $r_P : M^{ar(P)} \to [0, 1]$ and $m_c \in M$ for each $P \in Pred$ and $c \in Const$. For each evaluation of variables $v : Var \to M$, the truth-value $\|\varphi\|_{\mathbf{M},v}$ of a formula (where $v(x) \in M$ for each variable x) is defined inductively from

$$\|P(x, \cdots, c, \cdots)\|_{\mathbf{M},v} = r_P(v(x), \cdots, m_c \cdots),$$

taking into account that the value commutes with connectives (according to the above rules for the propositional case), and defining

$$\|(\forall x)\varphi\|_{\mathbf{M},v} = \inf\{\|\varphi\|_{\mathbf{M},v'} \mid v(y) = v'(y) \text{ for all variables, except } x\}$$
$$\|(\exists x)\varphi\|_{\mathbf{M},v} = \sup\{\|\varphi\|_{\mathbf{M},v'} \mid v(y) = v'(y) \text{ for all variables, except } x\}$$

From a syntactical point of view, the additional axioms on quantifiers for $L_*\forall$ are the following ones:

($\forall$1) $(\forall x)\varphi(x) \to \varphi(t)$ (t substitutable for x in $\varphi(x)$)
($\exists$1) $\varphi(t) \to (\exists x)\varphi(x)$ (t substitutable for x in $\varphi(x)$)
($\forall$2) $(\forall x)(\nu \to \varphi) \to (\nu \to (\forall x)\varphi)$ (x not free in ν)
($\exists$2) $(\forall x)(\varphi \to \nu) \to ((\exists x)\varphi \to \nu)$ (x not free in ν)
($\forall$3) $(\forall x)(\varphi \vee \nu) \to ((\forall x)\varphi \vee \nu)$ (x not free in ν)

Rules of inference of $L_*\forall$ are modus ponens and generalization: from φ infer $(\forall x)\varphi$.

The above mentioned propositional completeness results do not easily generalize to the first order case, MTL$\forall$ and G$\forall$ being remarkable exceptions. For more details on predicate fuzzy logics, including completeness and complexity results and model theory, the interested reader is referred to [20, 3].

1.3 T-norm based fuzzy logic modelling of approximate reasoning

In the literature one can find several approaches to cast main Zadeh's approximate reasoning constructs in a formal logical framework. In particular, Novák and colleagues have done much in this direction, using the model of fuzzy logic with evaluated syntax, fully elaborated in the monograph [25] (see the references therein and also [6]), and more recently he has developed a very powerful and sophisticated model of fuzzy type theory [21, 24]. In his monograph, Hájek [18] also has a part devoted to this task.

In what follows, we show a simple way of how to capture at a syntactical level, namely in a many-sorted version of predicate fuzzy logic calculus, say MTL$\forall$, some of the basic Zadeh's approximate reasoning patterns, basically from ideas in [18, 16]. It turns out that the logical structure becomes rather simple and the fact that fuzzy inference is in fact a (crisp) deduction becomes rather apparent.

Consider the simplest and most usual expressions in Zadeh's fuzzy logic of the form

$$\text{“}\mathbf{x} \text{ is } A\text{"},$$

with the intended meaning the variable x takes the value in A, represented by a fuzzy set μ_A on a certain domain U. The representation of this statement in the frame of possibility theory is the constraint

$$(\forall u)(\pi_{\mathbf{x}}(u) \leq \mu_A(u)),$$

where $\pi_{\mathbf{x}}$ stands for the possibility distribution for the variable $\mathbf{x}$. But such a constraint is very easy to represent in MTL$\forall$ as the formula[2]

[2] Caution: do not confuse the logical variable x in this logical expression from the linguistic (extra-logical) variable $\mathbf{x}$ in “$\mathbf{x}$ is A".

$$(\forall x)(X(x) \to A(x))$$

where A and X are many-valued *predicates* of the same sort in each particular model $\mathbf{M}$. Their interpretations (as fuzzy relations on their common domain) can be understood as the membership function $\mu_A : U \longrightarrow [0,1]$ and the possibility distribution $\pi_{\mathbf{x}}$ respectively. Indeed, one can easily observe that $\|(\forall x)(X(x) \to A(x))\|_M = 1$ if and only if $\|X(x)\|_{M,e} \leq \|A(x)\|_{M,e}$, for all x and any evaluation e. From now on, variables ranging over universes will be $\mathbf{x}, \mathbf{y}, \mathbf{z}$; "$\mathbf{x}$ is A" becomes $(\forall x)(X(x) \to A(x))$ or just $X \subseteq A$; if $\mathbf{z}$ is 2-dimensional variable $(\mathbf{x}, \mathbf{y})$, then an expression "$\mathbf{z}$ is R" becomes $(\forall x, y)(Z(x,y) \to R(x,y))$ or just $Z \subseteq R$.

In what follows, only two (linguistic) variables will be involved $\mathbf{x}, \mathbf{y}$ and $\mathbf{z} = (\mathbf{x}, \mathbf{y})$. Therefore we assume that X, Y (corresponding to the possibility distributions $\pi_{\mathbf{x}}$ and $\pi_{\mathbf{y}}$) are projections of a binary fuzzy predicate Z (corresponding to the joint possibility distribution $\pi_{\mathbf{x},\mathbf{y}}$). The axioms we need to state in order to formalize this assumption are:

$\Pi 1 : (\forall x, y)(Z(x,y) \to X(x))\ \&\ (\forall x, y)(Z(x,y) \to Y(y))$
$\Pi 2 : (\forall x)(X(x) \to (\exists y) Z(x,y))\ \&\ (\forall y)(Y(y) \to (\exists x) Z(x,y))$

Condition $\Pi 1$ expresses the monotonicity conditions $\pi_{\mathbf{x},\mathbf{y}}(u,v) \leq \pi_{\mathbf{x}}(u)$ and $\pi_{\mathbf{x},\mathbf{y}}(u,v) \leq \pi_{\mathbf{y}}(v)$, whereas both conditions $\Pi 1$ and $\Pi 2$ used together express the marginalization conditions $\pi_{\mathbf{x}}(u) = \sup_v \pi_{\mathbf{x},\mathbf{y}}(u,v)$ and $\pi_{\mathbf{y}}(v) = \sup_v \pi_{\mathbf{x},\mathbf{y}}(u,v)$. These can be equivalently presented as the only one condition $Proj$, as follows:

Proj: $(\forall x)(X(x) \equiv (\exists y) Z(x,y))\ \&\ (\forall y)(Y(y) \equiv (\exists x) Z(x,y))$

Next we shall consider several approximate reasoning patterns, and for each pattern we shall present a provable tautology (in MTL∀) and its corresponding derived deduction rule, which will automatically be sound.

1. *Entailment Principle:* From "$\mathbf{x}$ is A" infer "$\mathbf{x}$ is A^*", whenever $\mu_A(u) \leq \mu_{A^*}(u)$ for all u.

   Provable tautology:

   $$(A \subseteq A^*) \to (X \subseteq A \to X \subseteq A^*)$$

   Sound rule:

   $$\frac{A \subseteq A^*, X \subseteq A}{X \subseteq A^*}$$

2. *Truth-qualification:* From "$\mathbf{x}$ is A" infer that "($\mathbf{x}$ is A^*) is α-true", where $\alpha = \inf_u \mu_A(u) \Rightarrow \mu_{A^*}(u)$.

   Provable tautology:

   $$(X \subseteq A) \to (A \subseteq A^* \to X \subseteq A^*)$$

Sound rule:

$$\frac{X \subseteq A}{A \subseteq A^* \to X \subseteq A^*}$$

3. *Truth-modification:* From "($\mathbf{x}$ is A) is α-true" infer that "$\mathbf{x}$ is A^*", where $\mu_{A^*}(u) = \alpha \Rightarrow \mu_A(u)$.

   Provable tautology (where $\overline{\alpha}$ denotes a truth-constant):

$$(\overline{\alpha} \to (X \subseteq A)) \to (X \subseteq (\overline{\alpha} \to A)$$

   Sound rule:

$$\frac{(\overline{\alpha} \to (X \subseteq A)}{X \subseteq (\overline{\alpha} \to A)}$$

4. *Cylindrical extension:* From "$\mathbf{x}$ is A" infer "$(\mathbf{x}, \mathbf{y})$ is A^+", where $\mu_{A^+}(u, v) = \mu_A(u)$ for each v.

   Provable tautology:

$$\Pi 1 \to [(X \subseteq A) \to ((\forall xy)(A^+(x, y) \equiv A(x)) \to (Z \subseteq A^+))]$$

   Sound rule:

$$\frac{\Pi 1, X \subseteq A, (\forall xy)(A^+(x, y) \equiv A(x))}{Z \subseteq A^+}$$

5. *Projection*: From "$(\mathbf{x}, \mathbf{y})$ is R" infer "$\mathbf{y}$ is R_Y", where $\mu_{R_Y}(y) = \sup_u \mu_R(u, v)$ for each v.

   Provable tautology:

$$\Pi 2 \to ((Z \subseteq R) \to (\forall y)(Y(y) \to (\exists x)R(x, y)))$$

   Sound rule:

$$\frac{\Pi 2, Z \subseteq R}{(\forall y)(Y(y) \to (\exists x)R(x, y))}$$

6. *min–Combination*: From "$\mathbf{x}$ is A_1" and "$\mathbf{x}$ is A_2" infer "$\mathbf{x}$ is $A_1 \cap A_2$", where $\mu_{A_1 \cap A_2}(u) = \min(\mu_{A_1}(u), \mu_{A_2}(u))$.

   Provable tautology:

$$(X \subseteq A_1) \to ((X \subseteq A_2) \to (X \subseteq (A_1 \wedge A_2)))$$

   Sound rule:

$$\frac{X \subseteq A_1, X \subseteq A_2}{X \subseteq (A_1 \wedge A_2)}$$

   where $(A_1 \wedge A_2)(x)$ is an abbreviation for $A_1(x) \wedge A_2(x)$.

7. *Compositional rule of inference*: From "$(\mathbf{x}, \mathbf{y})$ is R_1" and "$(\mathbf{y}, \mathbf{z})$ is R_2" infer "$(\mathbf{x}, \mathbf{z})$ is $R_1 \circ R_2$", where $\mu_{R_1 \circ R_2}(u, w) = \sup_v \min(\mu_{R_1}(u, v), \mu_{R_2}(v, w))$.

   Provable tautology:

$$(Z_1 \subseteq R_1) \to ((Z_2 \subseteq R_2) \to (Z_3 \subseteq (R_1 \circ R_2)))$$

Sound rule:

$$\frac{Z_1 \subseteq R_1, Z_2 \subseteq R_2}{Z_3 \subseteq (R_1 \circ R_2)}$$

where $(R_1 \circ R_2)(x, z)$ is an abbreviation for $(\exists y)(R_1(x, y) \wedge R_2(y, z))$.

Note that the following rule

$$\frac{Cond, Proj,\ X \subseteq A,\ Z \subseteq R}{Y \subseteq B},$$

where $Cond$ is the formula $(\forall y)(B(y) \equiv (\exists x)(A(x) \wedge R(x, y)))$, formalizing the particular instance of *max–min composition rule*, from "$\mathbf{x}$ is A" and "$(\mathbf{x}, \mathbf{y})$ is R" infer "$\mathbf{y}$ is B", where $\mu_B(y) = \sup_u \min(\mu_A(u), \mu_R(u, v))$, is indeed a derived rule from the above ones.

More complex patterns like those related to inference with fuzzy if-then rules "if $\mathbf{x}$ is A then $\mathbf{y}$ is B" can also be formalized. As it has been discussed elsewhere (see e.g. [5, 4]), there are several semantics for the fuzzy if-then rules in terms of the different types constraints on the joint possibility distribution $\pi_{\mathbf{x},\mathbf{y}}$ it may induce. Each particular semantics will obviously have a different representation. We will describe just a couple of them.

Within the implicative interpretations of fuzzy rules, gradual rules are interpreted by the constraint $\pi_{\mathbf{x},\mathbf{y}}(u, v) \leq \mu_A(u) \Rightarrow \mu_B(v)$, for some residuated implication $\Rightarrow$. According to this interpretation, the folllowing is a derivable (sound) rule

$$\frac{Cond,\ Proj,\ X \subseteq A^*,\ Z \subseteq A \to B}{Y \subseteq B^*},$$

where $(A \to B)(x, y)$ stands for $A(x) \to B(y)$ and $Cond$ is $(\forall y)[B^*(y) \equiv (\exists x)(A^*(x) \wedge (A(x) \to B(y)))]$.

Finally, within the conjunctive model of fuzzy rules (i.e. Mamdani fuzzy rules), where a rule "if $\mathbf{x}$ is A then $\mathbf{y}$ is B" is interpreted by the constraint $\pi_{\mathbf{x},\mathbf{y}}(u, v) \geq \mu_A(u) \wedge \mu_B(v)$, and an observation "$\mathbf{x}$ is A^*" by a positive constraint $\pi_{\mathbf{x}}(u) \geq A^*(u)$, one can easily derive the Mamdani model (here with just one rule)

$$\frac{Cond,\ Proj,\ X \supseteq A^*,\ Z \supseteq A \wedge B}{Y \supseteq B^*},$$

where $Cond$ is $(\forall y)[B^*(y) \equiv (\exists x)(A^*(x) \wedge A(x)) \wedge B(y)]$.

1.4 Conclusions

In this short note we have put forward our thesis that Mathematical Fuzzy logic is not only the basic kernel of fuzzy logic in narrow sense (with which Zadeh and many fuzzy logicians agree) but also a logical framework where many of the well known concepts and approximate reasoning inference rules of fuzzy logic in narrow sense can be properly formalized. Obviously there are some others fuzzy concepts that are more difficult to be fully interpretable in Mathematical Fuzzy logic. Among them, we can cite:

- Linguistic modifiers. They have been partially interpreted as unary connectives in the logical framework, in particular the so-called fuzzy truth hedges (*very true*, *slightly true*, etc.). These are usually classified in two classes: truth-stressers, that modify the truth-value of an expression by decreasing it, and truth-depressers, that modify the truth-value of an expression by increasing it. The formalization in this kind of connectives has been within the framework of t-norm based fuzzy logics has been addressed in several papers, e.g. by Hájek [19], Vichodyl [26] and Esteva et al. [15].
- Fuzzy quantifiers. There is a nice chapter in Hájek's book [18] devoted to this topic where he axiomatizes the quantifier "*many*", but it is only a first step in the work needed to do (there are many non-answered questions). Also Nóvak has done very interesting work (see e.g. [22]) on formalizing linguistic quantifiers. In fact in first-order fuzzy logics like the ones mentioned in Section 1.2, the only formalized quantifiers are the classical ones $\forall$ and $\exists$, interpreted as inf and sup respectively.

Nevertheless we believet that, in the near future, new developments in mathematical fuzzy logic[3] will make possible the non-trivial task of defining formal systems of fuzzy logic closer and closer to the "logic" of human approximate reasoning as envisaged and proposed by Lotfi A. Zadeh since long ago.

[3] See e.g. [23] for a list of possible future tasks in the study of mathematical fuzzy logic and its applications.

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