Belief Functions on MV-algebras of Fuzzy Events Based on Fuzzy Evidence

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Abstract. Recently Kroupa has proposed a generalization of belief functions on MV-algebras, the latter being the chosen algebraic setting for fuzzy (or many-valued) events. However, Kroupa's belief functions evaluate the degree of belief in the occurrence of fuzzy events by taking into account (weighted) evidence on classical subsets. In other words, the focal elements, used in determining the degree of belief, are classical sets. Within the MV-algebraic setting, the aim of the present work is to introduce a generalization of Kroupa belief functions that allows to deal with fuzzy events supported by evidence on fuzzy subsets.

1 Introduction

Belief functions are measures of uncertainty that provide a degree of confidence in the occurrence of some event taking into account weighted bodies of evidence that support that event [15]. Such evidence plays a pivotal role in determining our belief, indeed the degree of belief is precisely determined by those weights assigned to the different bodies of evidence. In Dempster-Shafer theory, the evidence sets are called focal elements, and their weight is given by a mass function (a probability distribution over the focal elements).

In the classical setting (see [17]), both the events we are concerned with and the evidence behind them are precisely specified, i.e. we deal with Boolean two-valued events and focal elements. It is then interesting to define Belief functions for those cases in which the information at hand is not that precise.

In some early papers [4, 16, 19] several attempts to extend belief functions on fuzzy events and fuzzy evidence have been proposed. More recently Kroupa in [11] proposes to define belief functions on MV-algebras, the latter being the chosen algebraic setting for fuzzy (or many-valued) events. MV-algebras [2] generalize Boolean algebras and appear as the natural algebraic structures associated to the infinitely-valued Łukasiewicz logic. However, Kroupa's belief functions evaluate the degree of belief in the occurrence of fuzzy events by taking into account (weighted) evidence on classical subsets. In other words, the focal elements used in determining the degree of belief are still classical sets.

With this work, we want to take a step further and, within the MV-algebraic setting, introduce a generalization of Kroupa belief functions that allows to deal with fuzzy events supported by evidence on fuzzy subsets.

This paper is organized as follows. In the next section, we provide basic background information on belief functions and MV-algebras. In Section 3, we introduce our generalized notion of a belief function and compare it to Kroupa's definition. In Section 4, we give an integral representation of belief functions in terms of both Choquet and Sugeno integrals, and in Section 5, we briefly deal with the combination of two belief functions. We end with some final remarks.

2 Preliminary notions

2.1 Belief functions on Boolean algebras

In this section we are going to recall the basic framework of classical Dempster-Shafer theory and the basic definition and results about belief functions on Boolean algebras.

Consider a finite set X whose elements can be regarded as mutually exclusive (and exhaustive) propositions of interest, and whose powerset $\mathcal{P}(X)$ represents all the propositions of interest. The set X is usually called the *frame of discernment*, and every element $x \in X$ represents the lowest level of discernible information we can deal with.

Consider a frame of discernment X. A map $m : \mathcal{P}(X) \to [0, 1]$ is said to be a *basic belief assignment*, or a *mass assignment* provided that $m(\emptyset) = 0$, and $\sum_{A \in \mathcal{P}(X)} m(A) = 1$. Given a set X and a mass assignment m on $\mathcal{P}(X)$, for every $A \in \mathcal{P}(X)$, the *belief of* A is defined as

$$\mathbf{b}_m(A) = \textstyle\sum_{B \subseteq A} m(B).$$

Every mass assignment m on $\mathcal{P}(X)$ is in fact a probability distribution on $\mathcal{P}(X)$ naturally induces a probability measure P_m on $\mathcal{P}(\mathcal{P}(X))$, and hence the belief function $\mathbf{b}_m$ defined from m, can be equivalently described as follows: for every $A \in \mathcal{P}(X)$ it holds $\mathbf{b}_m(A) = P_m(\{B \in \mathcal{P}(X) : B \subseteq A\})$. Therefore, identifying the set $\{B \in \mathcal{P}(X) : B \subseteq A\}$ with its characteristic function on $\mathcal{P}(\mathcal{P}(X))$ defined by

$$\beta_A : B \in \mathcal{P}(X) \mapsto \begin{cases} 1 & \text{if } B \subseteq A \\ 0 & \text{otherwise,} \end{cases} \tag{1}$$

it is easy to see that, for every $A \in \mathcal{P}(X)$, and for every mass assignment $m : \mathcal{P}(X) \to [0, 1]$, we have $\mathbf{b}_m(A) = P_m(\beta_A)$. This easy characterization will be important when we will discuss the extensions of belief functions on MV-algebras. A trivial observation about the map β_A that can be useful to understand our generalization is the following: for every $A \in \mathcal{P}(A)$, β_A can be regarded as a map evaluating the (strict) inclusion of B into A, for every subset B of X.

A subset A of X such that $m(A) > 0$ is said to be a *focal elements*. Every belief function is characterized by the value that m takes over its focal elements, and therefore the focal elements of a belief function $\mathbf{b}_m$, contain the pieces of evidence that characterize $\mathbf{b}_m$ itself. For every set X and for every mass assignment m, call $\mathfrak{F}_m$ the set of focal elements of $\mathcal{P}(X)$ with respect to m. It is well known that several subclasses of belief functions can be characterized just by the structure of their focal elements. In particular, when $\mathfrak{F}_m = \{\{x\} : x \in X\}$, then it is clear that $\mathbf{b}_m$ is a probability measure. Moreover if the focal elements are nested subsets of X, i.e. $\mathfrak{F}_m$ is a chain with respect to the order of set inclusionship, then $\mathbf{b}_m$ is a *necessity measure* [4, 15]; this means that $\mathbf{b}_m(X) = 1$, $\mathbf{b}_m(\emptyset) = 0$, and $\mathbf{b}_m(A_1 \cap A_2) = \min\{\mathbf{b}_m(A_1), \mathbf{b}_m(A_2)\}$ (we refer the reader to [3, 18] for a basic introduction to necessity measures and their dual possibility measures on Boolean algebras).

2.2 MV-algebras and states

An MV-algebra is a system $M = (M, \oplus, \neg, 0)$ of type $(2, 1, 0)$ where M is a non-empty set, the reduct $(M, \oplus, 0)$ is an abelian monoid, and the further equations are satisfied: $\neg\neg x = x$, $x \oplus \neg 0 = \neg 0$, $\neg(\neg x \oplus y) = \neg(\neg y \oplus x)$.

The class of MV-algebras forms a variety that we denote as usual by $\mathbb{MV}$. In every MV-algebra M, we define as usual the following operations: for all $x, y \in M$, $x \odot y = \neg(\neg x \oplus \neg y)$, $x \Rightarrow y = \neg x \oplus y$, $x \vee y = (x \Rightarrow y) \Rightarrow y$, $x \wedge y = \neg(\neg x \vee \neg y)$, and $1 = \neg 0$.

For every $x, y \in M$, we write that $x \leq y$ provided that $x \Rightarrow y = 1$ holds in M. As a matter of facts $\leq$ is a partial order on M, and M is said to be linearly ordered (or an MV-chain) whenever $\leq$ is a linear order.

Example 1. (1) Every Boolean algebra A is an MV-algebra and in every MV-algebra M the set $B(M) = \{x : x \oplus x = x\}$ of its idempotent elements is the domain of the largest Boolean subalgebra of M, called the *Boolean skeleton* of M.

(2) Endow the real unit interval $[0, 1]$ with operations so defined: $x \oplus y = \min\{1, x+y\}$ and $\neg x = 1-x$. Then the structure $[0, 1]_{MV} = ([0, 1], \oplus, \neg, 0)$ is an MV-algebra called the *standard* MV-algebra. In this algebra $x \odot y = \max(0, x + y - 1)$, $x \Rightarrow y = \min(1, 1 - x + y)$, $x \wedge y = \min(x, y)$ and $x \vee y = \max(x, y)$. Chang Theorem (cf. [2]) shows that the algebra $[0, 1]_{MV}$ generates $\mathbb{MV}$.

For every finite set X, consider the MV-algebra $[0, 1]^X$ of all functions from X into $[0, 1]$ and whose operations are defined by a pointwise application of those of $[0, 1]_{MV}$. These MV-algebras will be the algebraic framework where we are going to define belief measures over. Adopting the same notation of [7] we will call them *finite domain MV-clans*. Notice that finite domain MV-clans are described, in algebraic terms, as those MV-algebras which are finite direct product of the standard MV-algebra $[0, 1]_{MV}$.

If not otherwise specified, it will be henceforth assumed that in any structure $M = [0, 1]^X$, the set X to be finite.

Normalized and additive maps on MV-algebras have been introduced by Kôpka and Chovanec in [9], and then by Mundici under the name of MV-algebraic states (or simply states) in [13]. By a state on an MV-algebra M we then mean a map $\mathbf{s} : M \to [0, 1]$ satisfying the following properties:

(i) $\mathbf{s}(1) = 1$,
(ii) whenever $x \odot y = 0$, $\mathbf{s}(x \oplus y) = \mathbf{s}(x) + \mathbf{s}(y)$.

As it is shown in [13], every state $\mathbf{s}$ also satisfies $\mathbf{s}(x \vee y) = \mathbf{s}(x) + \mathbf{s}(y) - \mathbf{s}(x \wedge y)$.

It is worth noticing that the restriction of every state $\mathbf{s}$ on M, to its Boolean skeleton $B(M)$, is a finitely additive probability. Moreover there exists a one-to-one correspondence between the states on an MV-algebra M and the class of regular Borel probability measures on $\mathcal{B}(Max(M))$: the σ-algebra of Borel subsets of the class of the maximal filters of M. As a matter of fact, by Kroupa-Panti's Theorem (cf. [10, 14]), for every state $\mathbf{s}$ on M there exists a unique Borel probability measure μ on $\mathcal{B}(Max(M))$ such that $\mathbf{s}$ coincides with the Lebesgue integral with respect to μ. The following result is a particular case of Kroupa-Panti's Theorem.

Theorem 1. *Let be X be a non-empty (possibly infinite) set. For every state $\mathbf{s}$ on the MV-algebra of functions $M = [0,1]^X$ there exists a finitely additive probability measure μ on $\mathcal{P}(X)$ such that for each $a \in M$,*

$$\mathbf{s}(a) = \int_X a \, \mathrm{d}\mu.$$

3 Belief functions on finite domain MV-clans

In [11], Kroupa provides a generalization of belief functions to be defined on finite domain MV-clans as follows. Let $M = [0,1]^X$ be a finite domain MV-clan, denote by $\mathcal{P}(X)$ the powerset of X, and consider, for every $a : X \to [0,1]$ the map $\hat{\rho}_a : \mathcal{P}(X) \to [0,1]$ defined as follows: for every $B \subseteq A$,

$$\hat{\rho}_a(B) = \min\{a(x) : x \in B\}. \tag{2}$$

Remark 1. Notice that $\hat{\rho}_a$ generalizes β_A in the following sense: whenever $A \in B(M) = \mathcal{P}(X)$, then $\hat{\rho}_A = \beta_A$. Namely, for every $A \in B(M)$, $\hat{\rho}_A(B) = 1$ if $B \subseteq A$, and $\hat{\rho}_A(B) = 0$ otherwise.

Definition 1. *We call a map $\hat{\mathbf{b}} : M \to [0,1]$ to be a* Kroupa belief function *provided that there is a state $\hat{\mathbf{s}} : [0,1]^{\mathcal{P}(X)} \to [0,1]$ such that for every $a \in M$, $\hat{\mathbf{b}}(a) = \hat{\mathbf{s}}(\hat{\rho}_a)$.*

The state $\hat{\mathbf{s}}$ needed in the definition of $\hat{\mathbf{b}}$ is called the *state assignment* in [11]. Although $\hat{\mathbf{b}}$ has been directly introduced as a combination of $\hat{\rho}$ with the state assignment $\hat{\mathbf{s}}$, one can introduce a notion of *mass assignment* even for this generalized case. Indeed, since X is finite, it turns out that one can equivalently write

$$\hat{\mathbf{b}}(a) = \textstyle\sum_{B \subseteq X} \hat{\rho}_a(B) \cdot \hat{\mathbf{s}}(B).$$

In particular, since $1 = \hat{\mathbf{b}}(X) = \sum_{B \subseteq X} \hat{\mathbf{s}}(B)$, the restriction of the state $\hat{\mathbf{s}}$ to $\mathcal{P}(X)$, call it $\hat{m}$, is a classical mass asignment, and hence we can claim without loss of generality that every (classical) mass assignment on $\mathcal{P}(X)$ is a mass assignment even for this general case. Now we are allowed to speak about *focal elements* of $\hat{\mathbf{b}}$ as those elements in $\mathcal{P}(X)$ that the mass assignment $\hat{m}$ maps into a non-zero value.

Notice that, although the arguments of Kroupa's definition of belief function are fuzzy sets, the mass assignments that characterize each of these belief functions are defined on crisp (i.e. Boolean) sets, and therefore the focal elements associated to every Kroupa belief function, are crisp sets. In other words every $\hat{\mathbf{b}}$ is defined over a crisp, and not fuzzy, pieces of evidence.

Now we are going to introduce a further generalization of belief functions on a finite domain MV-clan that allows the focal elements to be fuzzy sets.

Although the necessary modification in using a state instead of a probability measure as additive map to define $\hat{\mathbf{b}}$, Kroupa's definition of belief function makes use, for every $a \in M$, of the map $\hat{\rho}_a$ which evaluates the degree of inclusion $\hat{\rho}_a(B)$ of a classical (i.e. crisp, Boolean) subset B of X, into the fuzzy set a. The definition that we introduce below generalizes Kroupa's definition by introducing, for every $a \in M$, a map ρ_a sending every fuzzy set $b \in M$, into a degree of inclusion of the fuzzy set b into a. To be more precise, let $M = [0,1]^X$ be a finite domain MV-clan, and consider, for every $a \in M$ a map $\rho_a : M \to [0,1]$ defined as follows: for every $b \in M$,

$$\rho_a(b) = \min\{b(x) \Rightarrow a(x) : x \in X\}. \tag{3}$$

where $\Rightarrow$ denotes the Łukasiewicz implication function ($x \Rightarrow y = \min(1, 1 - x + y)$)[1].

Remark 2. In a sense, for every $a \in M$, ρ_a can be identified as the membership function of the fuzzy set of elements of M (and hence the fuzzy subsets of X) that are *included* in a. In particular one has $\rho_a(b) = 1$ whenever $b \leq a$ (pointwisely). Also notice that the Boolean skeleton $B(M)$ of any finite domain MV-clan $M = [0,1]^X$ coincides with $\mathcal{P}(X)$ and hence, as also the following result shows in further details, for every $a \in M$ the map ρ_a extends $\hat{\rho}_a$ in the domain.

Proposition 1. *(i) For all $a, a' \in M$, $\rho_{a \wedge a'} = \min\{\rho_a, \rho_{a'}\}$, and $\rho_{a \vee a'} \geq \max\{\rho_a, \rho_{a'}\}$.*

(ii) For every $a \in M$, the restriction of ρ_a to $B(M)$ coincides with the transformation $\hat{\rho}_a$ of equation (2).

(iii) For every $A \in B(M)$, the restriction of ρ_A to $B(M)$ coincides with the transformation β_A of equation (1)

Proof. (i) In every MV-chain, and in particular in the standard chain $[0,1]_{MV}$ it holds the equation $\neg\gamma \oplus (\alpha \wedge \beta) = (\neg\gamma \oplus \alpha) \wedge (\neg\gamma \oplus \beta)$, that is to say $(\gamma \Rightarrow$

[1] The choice here of $\Rightarrow$ is due to the MV-algebraic setting, but other choices could be made in other settings.

$(\alpha \wedge \beta)) = (\gamma \Rightarrow \alpha) \wedge (\gamma \Rightarrow \beta)$ holds. Therefore, for every $a, a', b \in M$,

$$\begin{aligned}\rho_{a\wedge a'}(b) &= \min\{b(x) \Rightarrow (a \wedge a')(x) : x \in X\} \\ &= \min\{b(x) \Rightarrow (a(x) \wedge a'(x)) : x \in X\} \\ &= \min\{(b(x) \Rightarrow a(x)) \wedge (b(x) \Rightarrow a'(x)) : x \in X\} \\ &= \min\{\rho_a(b), \rho_{a'}(b)\}.\end{aligned}$$

An easy computation shows that $\rho_{a\vee a'} \geq \max\{\rho_a, \rho_{a'}\}$.

(ii) For every $B \in B(M)$, $\rho_a(B) = \min\{B(x) \Rightarrow a(x) : x \in X\}$. Whenever $x \notin B$, $B(x) = 0$, and hence $B(x) \Rightarrow a(x) = 1$ for all those $x \notin B$. On the other hand for all $x \in B$, $B(x) = 1$, and hence $B(x) \Rightarrow a(x) = 1 \Rightarrow a(x) = a(x)$ for all $x \in B$. Therefore $\rho_a(B) = \min\{a(x) : x \in B\}$ and our claim is settled.

(iii) Trivially follows from (ii) and Remark 1. □

Definition 2. *Let X be a finite set and let $M = [0,1]^X$. A map $\mathbf{b} : M \to [0,1]$ will be called a* belief function *on the finite domain MV-clan M provided there exists a state $\mathbf{s} : [0,1]^M \to [0,1]$ such that for every $a \in M$,*

$$\mathbf{b}(a) = \mathbf{s}(\rho_a). \tag{4}$$

We will denote by $Bel(M)$ the class of all the belief functions over a finite domain MV-clan M.

It is clear from the definition that $Bel(M)$ is a convex set, since states are closed by convex combinations. Moreover, due to Theorem 1, it holds that for any belief function $\mathbf{b} : M \to [0,1]$ there exists a finitely additive probability measure μ on $\mathcal{P}(M)$ such that

$$\mathbf{b}(a) = \int_M \rho_a \, \mathrm{d}\mu.$$

Proposition 2. *For every finite domain MV-clan M, and for every $\mathbf{b} \in Bel(M)$, $\mathbf{b}$ is totally monotone, i.e. $\mathbf{b}$ is monotone, and it satisfies: for all $a_1, \ldots, a_n \in M$,*

$$\mathbf{b}\left(\bigvee_{i=1}^{n} a_i\right) \geq \sum_{j=1}^{n} (-1)^{j+1} \cdot \mathbf{b}\left(\bigwedge_{k=1}^{j} a_i\right).$$

Proof. Since for every $a \in M$, ρ_a is monotone, and every state $\mathbf{s}$ is monotone, also $\mathbf{b}$ is monotone. Moreover, for every n and for every $a_1, \ldots, a_n \in M$, from (4) and Proposition 1 (i), one has:

$$\begin{aligned}\mathbf{b}\left(\bigvee_{i=1}^{n} a_i\right) &= \mathbf{s}(\rho_{a_1\vee\ldots\vee a_n}) \\ &\geq \mathbf{s}(\rho_{a_1} \vee \ldots \vee \rho_{a_n}) \\ &= \textstyle\sum_{j=1}^{n} (-1)^{j+1} \cdot s\left(\bigwedge_{k=1}^{j} \rho_{a_i}\right) \\ &= \textstyle\sum_{j=1}^{n} (-1)^{j+1} \cdot s\left(\rho_{a_1\wedge\ldots\wedge a_j}\right) \\ &= \textstyle\sum_{j=1}^{n} (-1)^{j+1} \cdot \mathbf{b}\left(\bigwedge_{k=1}^{j} a_i\right).\end{aligned}$$

Therefore our claim is settled. □

In contrast with the case where total monotonicity is a property that characterizes belief functions on Boolean algebras, in the case of MV-algebras the problem of showing that total monotonicity characterizes belief functions is open.

For every belief function $\mathbf{b} : M \to [0,1]$ given by a state $\mathbf{s}$ on $[0,1]^M$, whenever $Supp(\mathbf{s}) = \{a \in M : \mathbf{s}(\{a\}) > 0\}$ is countable, we can introduce a notion of *mass assignment* that fully characterizes $\mathbf{b}$. Indeed define $m : M \to [0,1]$ such that, for every $a \in M$, $m(a) = \mathbf{s}(\{a\})$. Notice that $\sum_{a \in M} m(a) = 1$. Then it is well known that m defined as above characterizes $\mathbf{s}$ as follows: for every $f \in [0,1]^M$, $\mathbf{s}(f) = \sum_{a \in Supp(\mathbf{s})} f(a) \cdot m(a)$. Let us call *countably supported* those belief functions $\mathbf{b}$ given by a state $\mathbf{s}$ satisfying that $Supp(\mathbf{s}) = \{a \in M : \mathbf{s}(\{a\}) > 0\}$ is countable. Notice that, whenever X is finite (as it is in our case), every Kroupa belief function is countably supported.

The focal elements arising from our definition of a countably supported belief function, are elements of the MV-algebra $M = [0,1]^X$, and hence are not crisp sets, in general. This supports the interpretation that the belief functions defined as in (4) differ from Kroupa definition by providing a more general account of evidence theory. Indeed the evidence that in our approach can be represented is not just limited to be over crisp subsets, but rather we can now deal with evidence on fuzzy information within this framework.

Example 2. Let us revisit Smets' well-known story of the murder of Mrs. Jones [17]. There are 3 suspects of being her murderer: Peter, Paul andMary. Consider the information provided the janitor of the building where Mrs. Jones lives. He heard the victim yelling and saw *a small man* running. It turns out that Paul and Mary are not tall while Peter is taller ((Paul is 1.65 m. tall, Mary is 1.60 m tall and Peter is 1.85 m.). So, actually the subset of small suspects of $X = \{Peter, Paul, Mary\}$ can be considered as a fuzzy set, with membership function, say,

$$\mu_{small}(Peter) = 0, \mu_{small}(Paul) = 0.7, \mu_{small}(Mary) = 0.9.$$

On the other hand, Mary has short hair, so she may be mistaken as a man at first sight, and hence, the subset of suspects looking like a man can be considered fuzzy as well, with membership function:

$$\mu_{man\text{-}like}(Peter) = 1, \mu_{man\text{-}like}(Paul) = 1, \mu_{man\text{-}like}(Mary) = 0.5.$$

The evidence supplied by the janitor may be represented by a mass assignment $m : [0,1]^X \to [0,1]$ such that $m(small \wedge man\text{-}like) = \alpha > 0$, $m(X) = 1 - \alpha$ and $m(f) = 0$ for any other $f \in [0,1]^X$. Here we interpret $\wedge$ by the min operator, so we have

$$\mu_{small \wedge man\text{-}like}(Peter) = 0, \mu_{small \wedge man\text{-}like}(Paul) = 0.7, \mu_{small \wedge man\text{-}like}(Mary) = 0.5$$

Assume we interested in computing the belief that the suspect be Paul. We need then to compute

$$\rho_{\{Paul\}}(small \wedge man\text{-}like) = \min_{x \in X} \ \mu_{small \wedge man\text{-}like}(x) \Rightarrow \mu_{Paul}(x)$$

$$= \min\{0 \Rightarrow 0, 1 \Rightarrow 0.7, 0.5 \Rightarrow 0\}$$
$$= \min\{0.7, 0.5\} = 0.5$$

and $\rho_{\{Paul\}}(X) = 0$. Then we finally have

$$\begin{aligned}\mathbf{b}(\{Paul\}) &= \sum_{f \in [0,1]^X} \rho_{\{Paul\}}(f) \cdot m(\{f\}) \\ &= \rho_{\{Paul\}}(\mathit{small} \wedge \mathit{man\text{-}like}) \cdot m(\mathit{small} \wedge \mathit{man\text{-}like}) \\ &= 0.5 \cdot \alpha > 0\end{aligned}$$

Hence we get a positive belief degree on Paul being the murderer. This is in contrast with the results we would obtain with both the classical and Kroupa's models, where focal elements are only allowed to be classical subsets of X, in case we assume Mary can be mistaken as a man. Indeed, in that case, we would be forced to take as focal element, besides X itself, the set $\mathit{small} \wedge \mathit{man\text{-}like} = \{Paul, Mary\}$, and since there would be no focal element included in $\{Paul\}$, we woud get $\mathbf{b}(\{Paul\}) = 0$.

4 Belief functions and their integral representations

The map $\rho_c : M \to [0,1]$ that we defined in (3) can be represented in two ways as an integral. This representation, in turns, enable us to provide an integral description of belief functions on MV-algebras. In this section we are going to address this issue.

Let us start recalling that, given a set X, a map $\pi : X \to [0,1]$ is called a *possibility distribution*, and π is said to be *normalized* if there is a $x \in X$ such that $\pi(x) = 1$. Given a possibility distribution π, the Sugeno integral of a function $f : X \to [0,1]$ with respect to π is defined as the value $\max_{x \in X}(\min(\pi(x), f(x)))$. When we replace the min operation by Łukasiewicz t-norm (or even more in general by an arbitrary t-norm T), we obtain the so called *generalized Sugeno integral*: for every $f : X \to [0,1]$,

$$\oint_X f \, \mathrm{d}\pi = \max_{x \in X}(\pi(x) \odot f(x)).$$

The dual of the generalized Sugeno integral is defined as follows: for all $f : X \to [0,1]$,

$$\oint_X' f \, \mathrm{d}\pi = 1 - \oint_X (1 - f) \, \mathrm{d}\pi = \min_{x \in X}(\pi(x) \Rightarrow f(x)). \tag{5}$$

Following [11], consider a function $a \in M = [0,1]^X$, and a monotone set function $\beta : \mathcal{P}(X) \to [0,1]$ such that $\beta(\emptyset) = 0$ and $\beta(X) = 1$ (also called capacity). The *Choquet integral* of a with respect to β is defined as

$$\oint a \, \mathrm{d}\beta = \int_0^1 \beta(a^{-1}([t,1])) \, \mathrm{d}t.$$

Since we are only concerning finite domain MV-clans, for every $a \in M$, $\oint a \, \mathrm{d}\beta$ exists, and letting $X = \{x_1, \ldots, x_n\}$ indexed in a way that $y_1 \geq y_2 \geq \ldots \geq y_n$ where $y_i = a(x_i)$, and letting $y_{n+1} = 0$ and for every $i = 1, \ldots, n$, $S_i = \{x_1, \ldots, x_i\}$, $\oint a \, \mathrm{d}\beta = \sum_{i=1}^{n} (y_i - y_{i+1})\beta(S_i)$.

Theorem 2. *For every finite domain MV-clan $M = [0,1]^X$, and for every $\mathbf{b}$: $Bel(M)$, the following hold:*

1. *There exists a fin. additive probability measure μ on $\mathcal{P}(M)$ such that, for every $c \in M$,*

$$\mathbf{b}(c) = \int_M \left(\oint'_X J \, \mathrm{d}c \right) \mathrm{d}\mu.$$

   where $J : M \to M$ is the identity function
2. *There exists a fin. additive probability measure μ on $\mathcal{P}(M)$ such that, for every $c \in M$,*

$$\mathbf{b}(c) = \int_M \left(\oint I_c \, \mathrm{d}\chi_X \right) \mathrm{d}\mu.$$

   where $I_c : M \to M$ is defined by $I_c(a) = a \Rightarrow c$, and χ_X is the characteristic function of X over $\mathcal{P}(X)$.

Proof. From (4), for every $c \in M$, there exists a state $\mathbf{s} : [0,1]^M \to [0,1]$ such that for all $c \in M$ $\mathbf{b}(c) = \mathbf{s}(\rho_c)$ and therefore, from Theorem 1, there exists a unique probability measure $\mu : \mathcal{P}(M) \to [0,1]$ such that

$$\mathbf{b}(c) = \int_M \rho_c \, \mathrm{d}\mu. \tag{6}$$

Then 1, and 2 follow from the description of ρ_c.

1. For every $c, a \in M$ recall that $\rho_c(a) = \min\{a(x) \Rightarrow c(x) : x \in X\}$. Then we have:

$$\rho_c(a) = \oint'_X a \, \mathrm{d}\pi_c = \oint'_X J(a) \, \mathrm{d}\pi_c.$$

2 Since for all $S \in \mathcal{P}(X)$, $\chi_X(S) = 1$ if $S = X$, and 0 if $S \neq X$, it is easy to observe that

$$\rho_c(a) = \oint (a \Rightarrow c) \, \mathrm{d}\chi_X = \oint I_c(a) \, \mathrm{d}\chi_X,$$

□

Notice that for the case of countably supported belief functions, the above integral representations can be simplified as follows.

Corollary 1. *For every finite domain MV-clan* $M = [0,1]^X$ *with* X*, and every countably supported belief function* $\mathbf{b}$ *on* M*, there exists a mass assignment* $m : M \to [0,1]$ *such that the following hold:*

1. *for every* $c \in M$, $\mathbf{b}(c) = \sum_{a \in Supp(\mathbf{s})} (\min_{x \in X}(a(x_i) \Rightarrow c(x_i))) \cdot m(a)$;
2. *for every* $c \in M$, $\mathbf{b}(c) = \sum_{a \in Supp(\mathbf{s})} (\sum_{i=1}^{n}(y_i - y_{i+1}) \cdot \chi_X(S_i)) \cdot m(a)$*, where* $X = \{x_1, \ldots, x_n\}$ *such that* $y_i = a(x_i) \Rightarrow c(x_i)$*,* $y_{n+1} = 0$*, with* $y_1 \geq \ldots \geq y_n$*, and* $S_i = \{x_1, \ldots, x_i\}$ *for all* $i = 1, \ldots, n$*.*

5 Combining belief functions

In this section we will present a way to generalize the well-known Dempster rule to combine the information brought by two belief functions $\mathbf{b}_1, \mathbf{b}_2 \in Bel(M)$, into a third $\mathbf{b}_{1,2} \in Bel(M)$.

First of all let us introduce an easy result about the definition of states in a product space.

Proposition 3. *For every MV-algebra* $M = [0,1]^X$*, and for every pair of states* $s_1, s_2 : M \to [0,1]$*, there exists a state* $s_{1,2} : M \times M \to [0,1]$ *such that for every* $(b,c) \in M \times M$*,* $s_{1,2}(b,c) = s_1(b) \cdot s_2(c)$*.*

Let $\mathbf{s}_1, \mathbf{s}_2$ be two states on $[0,1]^M$ such that $\mathbf{b}_1(a) = \mathbf{s}_1(\rho_a)$ and $\mathbf{b}_2(a) = \mathbf{s}_2(\rho_a)$ for all $a \in M$. Further let $\mu_1, \mu_2 : \mathcal{P}(M) \to [0,1]$ be two probabilities such that for $i = 1, 2$, $\mathbf{s}_i(f) = \int_M f \, \mathrm{d}\mu_i$ as ensured by Theorem 1.

Consider the mapping $\mu_{1,2} : \mathcal{P}(M \times M) \to [0,1]$ to be, as in the proof of Proposition 3, the product measure on the product space generated by $M \times M$ such that $\mu_{1,2}(b,c) = \mu_1(b) \cdot \mu_2(c)$ for all $(b,c) \in M \times M$. Then call $\mathbf{s}_{1,2}$ that unique state on $[0,1]^{M \times M}$ defined by integrating on $\mu_{1,2}$. Since every $f \in [0,1]^{M \times M}$ is measurable in the product space generated by $M \times M$ with measure $\mu_{1,2}$, $s_{1,2}$ exists, and moreover notice that, if there exists $g, h : M \to [0,1]$ such that $f : (\overline{x}, \overline{y}) \mapsto g(\overline{x}) \cdot h(\overline{y})$, then by Proposition 3, $s_{1,2}(f) = s_1(g) \cdot s_2(h)$.

Finally, for every $a \in M$, consider the map $\rho_a^\wedge : M \times M \to [0,1]$ defined by $\rho_a^\wedge(b,c) = \rho_a(b \wedge c)$. Then we are ready to define the following combination of belief functions.

Definition 3. *(Generalized Dempster rule) Given* $\mathbf{b}_1, \mathbf{b}_2 \in Bel(M)$ *as above, define its combination* $\mathbf{b}_{1,2} : M \to [0,1]$ *as follows: for all* $a \in M$*,*

$$\mathbf{b}_{1,2}(a) = \mathbf{s}_{1,2}(\rho_a^\wedge). \tag{7}$$

From (7) we then obtain: for all $a \in M$,

$$\begin{aligned} \mathbf{b}_{1,2}(a) &= \int_{M \times M} \rho_a^\wedge \, \mathrm{d}\mu_{1,2} \\ &= \int_{M \times M} \rho_a(b \wedge c) \, \mathrm{d}\mu_1(b) \, \mathrm{d}\mu_2(c) \end{aligned}$$

and in the case of countable support belief functions, this yields

$$\mathbf{b}_{1,2}(a) = \sum_{b,c \in M} \rho_a(b \wedge c) \cdot \mu_1(\{b\}) \cdot \mu_2(\{c\}).$$

Notice that the above expression reduces to $\mathbf{b}_{1,2}(a) = \sum_{d \in M} \sum_{b,c \in M, b \wedge c = d} \rho_a(d) \cdot (\mu_1(\{b\}) \cdot \mu_2(\{c\})) = \sum_{d \in M} \rho_a(d) \cdot \mu^*(\{d\})$, where $\mu^*(\{d\}) = \sum_{b,c \in M, b \wedge c = d} \mu_1(\{b\}) \cdot \mu_2(\{c\})$ is indeed a mass assignment and hence $\mathbf{b}_{1,2} \in Bel(M)$.

6 Conclusion and future work

In this paper we have introduced a generalization of belief functions on MV-algebras of fuzzy sets that further extends Kroupa definition (cf. [11]) by allowing focal elements to be fuzzy sets, and not just classical sets. Indeed focal elements play a central role in the (classical) theory of belief functions because they can be interpreted as those basic pieces of information that are probabilistically evaluated by the mass assignment to define the belief function we are considering. More than the foundational aspects, another important role of focal elements regards the fact that several particular belief functions (as like probability measures, necessity and possibility measures) can be characterized by the fact that their focal elements satisfy a certain structural property. In particular, for the classical case, it is well known that probability measures are those belief functions whose focal elements are singletons, while necessity measures coincide with those belief functions whose focal elements are nested (with respect to inclusion between sets). When we extend belief functions from Boolean algebras, to MV-algebras, although it can be easily proved that *states* on MV-algebras coincides with that particular belief functions whose focal elements are singletons, the belief functions having nested focal elements no longer satisfies the usual property of a necessity measure: for all $a, a' \in M$, $\mathbf{b}(a \wedge a') = \min\{\mathbf{b}(a), \mathbf{b}(a')\}$ (cf. [6] for an axiomatic approach of possibility and necessity measures on MV-algebras). This fact was observed in [4] and it has been recently stressed in [11]. Of course, since our definition of belief functions extends somehow Kroupa's, we cannot expect to recover within our framework a characterization of necessity measures on MV-algebras via nested structure for focal elements.

In our future work we plan to investigate which further properties should a nested class of focal elements satisfy in order to characterize necessity and possibility measures on MV-algebras. Following the line of [12], we also plan to deepen the study on belief functions on more general MV-algebras than the ones considered in this paper where the notion of state is well developed and enjoy particularly nice properties (like the class of semi-simple MV-algebras, that can be represented as certain class of continuous real-valued functions), as well as investigating their algebraic and geometrical properties, and axiomatic characterization. Moreover, following the line of [8], we also plan to introduce a multi-modal expansion of Łukasiewicz logic that could allow to treat both our as well as Kroupa definition of belief function on finite MV-algebras. Indeed, as a belief function on an MV-algebra is defined by combining a state $\mathbf{s}$ with the map $\rho : f \in [0,1]^X \mapsto \rho_f \in [0,1]^{[0,1]^X}$ (in our case, and the map $\hat{\rho} : f \in [0,1]^X \mapsto [0,1]^{\mathcal{P}(X)}$ in the case of Kroupa definition) that behaves like a necessity measure on $[0,1]^X$, we argue that a belief function on a finite MV-algebra can be axiomatized by combining the axioms of a state (cf. [5]) with the axioms for the two possible extensions of the modal logic K on finite MV-algebras as provided in [1] (i.e. the one relative to those Kripke frames with many-valued accessibility relation, and the one that is complete with respect to those particular frames whose accessibility relation is two-valued) to respectively characterize ρ and $\hat{\rho}$.

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