

Towards a betting interpretation for necessity measures

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The aim of this work is to make a first step in the study of a betting interpretation in the style of de Finetti for necessity measures over many-valued events. In particular, we are interested in dealing with necessity measures over MV-algebras of functions $\mathcal{F}(X)$ of the form $\mathcal{F}(X) = \langle \mathcal{F}(X), \oplus, \neg, \mathbf{0}, \mathbf{1} \rangle$, where $\mathcal{F}(X)$ is the set of all functions from a finite set X into $[0, 1]$, \oplus and \neg result from the pointwise application of the functions $x \oplus y = \min\{1, x + y\}$ and $\neg : [0, 1] \rightarrow [0, 1]$ ($\neg x = 1 - x$), and $\mathbf{1}$ and $\mathbf{0}$ are constant functions.

A betting interpretation in the style of de Finetti for probability measures over MV-algebras in terms of states has already been deeply studied. Denote by \mathcal{H} the set of all the homomorphisms from $\mathcal{F}(X)$ into the standard MV-algebra $[0, 1]$ (cf. [1]).

Definition 1 ([6]). A map $s : \mathcal{F}(X) \rightarrow [0, 1]$ is said to be a *state* if:

- (s1) $s(\mathbf{1}) = 1$,
- (s2) whenever $\neg(\neg f \oplus \neg g) = \mathbf{0}$, $s(f \oplus g) = s(f) + s(g)$.

Let $\{f_1, \dots, f_m\}$ be a finite subset of $\mathcal{F}(X)$, and let

$$\mathbf{a} : f_i \mapsto \beta_i \in [0, 1], \text{ for } i = 1, \dots, m \quad (1)$$

be a mapping. We say that \mathbf{a} satisfies *de Finetti coherence criterion* ([2]) iff for every $\sigma_1, \dots, \sigma_m \in \mathbb{R}$, there is a $\mathbf{h} \in \mathcal{H}$ such that $\sum_{i=1}^m \sigma_i (\mathbf{a}(f_i) - \mathbf{h}(f_i)) \geq 0$.

The following result generalizes de Finetti's Theorem to the case of states over MV-algebras.

Theorem 1 ([7, 5]). *Let $\mathcal{F}(X)$, f_1, \dots, f_m , and \mathbf{a} be as above. Then the following are equivalent:*

- (i) \mathbf{a} satisfies de Finetti coherence criterion.
- (ii) The map \mathbf{a} extends to a state on $\mathcal{F}(X)$.
- (iii) There is a probability measure μ on \mathcal{H} such that for every $i = 1, \dots, m$,

$$\mathbf{a}(f_i) = \int_{\mathcal{H}} \mathbf{h}(f_i) d\mu.$$

(iv) \mathbf{a} extends to a convex combination of elements in \mathcal{H} .

The aim of this work is to find a suitable notion of coherence and a suitable betting interpretation, inspired by the above characterization, for necessity measures over MV-algebras of functions $\mathcal{F}(X)$.

Definition 2 ([4]). A map $N : \mathcal{F}(X) \rightarrow [0, 1]$ is said to be a *necessity measure* if the following conditions are satisfied:

(N1) $N(\mathbf{1}) = 1$ and $N(\mathbf{0}) = 0$.

(N2) $N(f \wedge g) = N(f) \wedge N(g)$.

(N3) for every real number r , denote by \mathbf{r} the function constantly equal to r . Then for every $r \in [0, 1]$, $N(\mathbf{r} \oplus f) = r \oplus N(f)$.

Let $\pi : X \rightarrow [0, 1]$ be normalized possibility distribution (i.e. $\bigvee_{x \in X} \pi(x) = 1$). Then we define the *generalized Sugeno integral given by π* , as the map $\oint d\pi : \mathcal{F}(X) \rightarrow [0, 1]$ such that, for every $f \in \mathcal{F}(X)$:

$$\oint f d\pi = \min_{x \in X} (-\pi(x) \oplus f(x)).$$

We are interested in answering the following question:

Can we find a characterization of necessity measures in the style of Theorem 1?

In order to find such a characterization, we need to borrow some concepts and techniques from Tropical Mathematics (and tropical algebraic geometry). Tropical mathematics is a rapidly growing area of modern mathematics that investigates the properties of the mathematical structure of the reals \mathbb{R} that arises when we replace in $\langle \mathbb{R}, +, \cdot, 0, 1 \rangle$ the product by the sum, and the sum by a idempotent operation, usually the *minimum* \wedge , or the *maximum* \vee . The structure $\mathbf{R}_\wedge = \langle \mathbb{R}, \min, + \rangle$ is called the min-plus semiring.

We are particularly interested in the notion of tropical convexity, whose study has quite a long tradition, and goes back to the earliest work of Vorobyev [8] and Zimmermann [9]. Take the tropical min-plus semiring \mathbf{R}_\wedge and extend $+$ and \min to any \mathbb{R}^n by the usual componentwise application of the operations in \mathbf{R}_\wedge . Fix a finite set $V = \{v_1, \dots, v_k\}$ of points in \mathbb{R}^n . Let x be a point in \mathbb{R}^n . Then we call x :

(i) a *tropical convex combination* of v_1, \dots, v_k , iff there are $\lambda_1, \dots, \lambda_k \in \mathbb{R}$ such that

$$x = \bigwedge_{i=1}^k \lambda_i + v_i.$$

We denote by $\text{tconv}(V)$ the set of points in \mathbb{R}^n that are tropical convex combinations of V .

(ii) a *bounded normalized tropical convex combination* iff there are $\lambda_1, \dots, \lambda_k \in \mathbb{R}$ such that $\bigwedge_{i=1}^k \lambda_i = 1$, and

$$x = \bigwedge_{i=1}^k (1 - \lambda_i) \oplus v_i.$$

We denote by $\text{b-n-tconv}(V)$ the set of points in \mathbb{R}^n that are tropical convex combinations of V .

Now we are ready to characterize those assessments \mathbf{a} defined on a finite subset of $\mathcal{F}(X)$ that extend to necessity measures.

Theorem 2. *Let $\{f_1, \dots, f_m\} \subseteq \mathcal{F}(X)$, and let \mathbf{a} be a mapping from $\{f_1, \dots, f_m\}$ into $[0, 1]$. Then the following are equivalent:*

- (i) *The map \mathbf{a} extends to a necessity measure on $\mathcal{F}(X)$.*
- (ii) *There exists a normalized distribution $\pi : X \rightarrow [0, 1]$, such that for every $i = 1, \dots, m$,*

$$\mathbf{a}(f_i) = 1 - \oint_X \neg f_i \, d\pi.$$

- (iii) *$\langle \beta_1, \dots, \beta_m \rangle \in b\text{-}n\text{-}tconv(\{\langle f_1(x), \dots, f_m(x) \rangle : x \in X\})$.*

Theorem 2 offers a characterization of the extension of necessity measures in terms generalized Sugeno integrals, and in terms of convex combinations. However, a coherence criterion in the style of de Finetti is still lacking. We plan to tackle this issue in our future work, and also to investigate extensions of this approach to different ways to define necessity and possibility measures over MV-algebras (see [4]).

References

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