Chapter XI: Arithmetical Complexity of First-Order Fuzzy Logics

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1 Introduction

This chapter presents several results on the complexity of predicate fuzzy logics, understood as first-order versions of ($\triangle$-)core propositional fuzzy logics (see the previous chapter). We will discuss several semantics for them, and for each semantics we will try to classify the complexity (in the sense of arithmetical hierarchy) of the sets of tautologies (formulae which are always evaluated into 1), of the positive tautologies (formulae which are always evaluated into a strictly positive value), of satisfiable formulae (formulae which are evaluated into 1 by some evaluation) and of positively satisfiable formulae (formulae which are evaluated into a strictly positive number by some evaluation).

The most important semantics we will discuss are the *general semantics* (given by all chains for the logic), the *real semantics* (given by all the chains for the logic having as lattice reduct $[0, 1]$) and the *standard semantics* (that is, the intended semantics for the logic, in some case coinciding with the real semantics, but in general a proper subclass of the real semantics; we will be more precise in Section 3). We will also consider the *rational semantics* (given by the rational valued chains for the logic), the *finite semantics* (given by all finite chains for the logic), the *complete chain semantics* (given by all complete chains for the logic), and the *witnessed semantics* (given by all models in which the truth value of each universally quantified formula is the minimum of truth values of instances and analogously for existential quantifier and maximum). Finally we will also discuss fragments of predicate logics, like the falsum-free fragment, the fragment with negation, implication and quantifiers and the monadic fragment.

The results, collected in tables present throughout the chapter, show that our predicate logics, with a very few exceptions (like the monadic fragment of classical logic), turn out to be undecidable (we will prove a quite general undecidability result in Section 2). Hence, the main problem we will address in this chapter is not whether a given predicate logic is decidable or not, but rather how undecidable it is, i.e. what is its undecidability degree.

For the general semantics, the undecidability degrees are low ($\mathbf{\Sigma_1}$ for tautologicity and $\mathbf{\Pi_1}$ for satisfiability). For the standard semantics, it depends: in the cases where we have standard completeness, like MTL or IMTL, the undecidability degrees are trivially as in the general semantics, in other cases, like Łukasiewicz first-order logic, the undecidability degrees are higher but still in the arithmetical hierarchy, while in product logic or in BL logic both tautologicity and satisfiability for the standard semantics fall outside the arithmetical hierarchy.

In this chapter, a basic knowledge of first-order fuzzy logics and arithmetic is assumed. One can find the necessary background in the former chapters of this handbook. In particular, for the arithmetical hierarchy, see the notes in Chapter X, Section 2.2. Recall classical logic with its deductive system and its models: a formula is provable if and only if it is a tautology (true in all models—completeness of classical logic); the set of all such formulae is recursively enumerable (i.e. in $\mathbf{\Sigma_1}$ in the sense of the arithmetical hierarchy) and, if its language has at least one predicate whose arity is at least binary, then it is $\mathbf{\Sigma_1}$-complete.[1] Similarly for theories like Peano arithmetic PA: the set of its provable formulae (equal to the set of formulae true in all its models) is $\mathbf{\Sigma_1}$-complete. PA has its *standard model:* the structure $\mathbf{N}$ of natural numbers with addition and multiplication. And the set of all formulae true in this standard model is extremely undecidable, it is outside the arithmetical hierarchy.

As regards to axiomatic systems for arithmetic, we will mainly use Robinson's arithmetic Q^+. Its axioms are those of equality plus the following ones:

$(\forall x)(S(x) \neq 0)$
$(\forall x)(\forall y)(S(x) = S(y) \to (x = y))$
$(\forall x)(\neg x = 0 \to (\exists y)(S(y) = x))$
$(\forall x)(x + 0 = x)$
$(\forall x)(\forall y)(x + S(y) = S(x + y))$
$(\forall x)(x \cdot 0 = 0)$
$(\forall x)(\forall y)(x \cdot S(y) = x \cdot y + x)$
$(\forall x)(\forall y)(x \leq y \leftrightarrow \exists z(y = x + z))$
$(\forall x)(\forall y)(x \leq y \vee y \leq x)$
$(\forall x)(\forall y)(\forall z)((x \leq y \wedge y \leq z) \to x \leq z)$
$(\forall x)(\forall y)((x \leq y \wedge y \leq x) \to x = y)$
$(\forall x)(\forall y)((x \leq y \wedge (x \neq y)) \leftrightarrow S(x) \leq y)$.

For every natural number n, $\overline{n}$ is defined by $\overline{0} = 0$ and $\overline{n+1} = S(\overline{n})$. Peano arithmetic PA is obtained from Q^+ by adding for any formula $\phi(x_1, \ldots, x_n, y)$ the following induction schema: $\forall x_1 \ldots \forall x_n((\phi(x_1, \ldots, x_n, 0) \wedge \forall y(\phi(x_1, \ldots, x_n, y) \to \phi(x_1, \ldots, x_n, S(y)))) \to \forall y \phi(x_1, \ldots, x_n, y))$.

This chapter is organized as follows: Section 2 contains abstract results on any semantics given by a class of linearly ordered algebras and in particular results on the general semantics of fuzzy predicate logics. Section 3 is devoted to the complexity of standard semantics of logics extending the basic fuzzy logic $\mathrm{BL}\forall$, particularly Łukasiewicz, Gödel and product logic, $\mathrm{SBL}\forall$ and $\mathrm{BL}\forall$ itself. Section 4 deals with the complexity of semantics given by finite and rational chains and their relations to the real-valued semantics. Section 5 completes the picture by presenting several further results on arithmetical hierarchy of first-order fuzzy logics: complexity of semantics of witnessed models, complexity of semantics of completely ordered models, results on some fragments of fuzzy

[1] Please do not confuse the completeness of a theory in a logic with the $\mathbf{\Sigma_n}$-completeness or $\mathbf{\Pi_n}$-completeness of a set of formulae in the sense of arithmetical hierarchy. Such set is $\mathbf{\Sigma_n}$-complete if it is in $\mathbf{\Sigma_n}$ and is $\mathbf{\Sigma_n}$-hard; similarly for $\mathbf{\Pi_n}$.

logics (with restricted set of connectives or only with monadic predicates), and results on axiomatic extensions of Łukasiewicz. We conclude the section with a list of open problems. Finally, Section 6 completes the chapter with some historical and bibliographical notes for further reading.

2 General results and general semantics

This section considers equality-free first-order fuzzy logics in the full vocabulary $\mathcal{P}$, i.e. containing functional and relational symbols of all arities (monadic fragments are addressed in Section 5). Moreover, we will work with arbitrary classes of linearly ordered MTL-algebras or their expansions corresponding to ($\triangle$-)core fuzzy logics in richer languages. These classes will be usually denoted by $\mathbb{K}$, and we will always assume (to avoid dealing with non-interesting trivial cases) that they are not empty and do not contain the trivial algebra. When $\mathbb{K}$ is a class of (expansions of) MTL-chains and no further condition is assumed, we just say for simplicity that it is *a class of chains*.

DEFINITION 2.0.1. *Given a class* $\mathbb{K}$ *of chains we define the following sets of sentences:*

$$\mathrm{TAUT}(\mathbb{K}) = \{\varphi \in \mathrm{Sent}_{\mathcal{P}} \mid \textit{for every } \boldsymbol{A} \in \mathbb{K} \textit{ and every safe } \boldsymbol{A}\textit{-structure } \mathbf{M}, \; \|\varphi\|^{\boldsymbol{A}}_{\mathbf{M}} = \bar{1}^{\boldsymbol{A}}\}.$$

$$\mathrm{TAUT}_{\mathrm{pos}}(\mathbb{K}) = \{\varphi \in \mathrm{Sent}_{\mathcal{P}} \mid \textit{for every } \boldsymbol{A} \in \mathbb{K} \textit{ and every safe } \boldsymbol{A}\textit{-structure } \mathbf{M}, \; \|\varphi\|^{\boldsymbol{A}}_{\mathbf{M}} > \bar{0}^{\boldsymbol{A}}\}.$$

$$\mathrm{SAT}(\mathbb{K}) = \{\varphi \in \mathrm{Sent}_{\mathcal{P}} \mid \textit{there exist } \boldsymbol{A} \in \mathbb{K} \textit{ and a safe } \boldsymbol{A}\textit{-structure } \mathbf{M} \textit{ such that } \|\varphi\|^{\boldsymbol{A}}_{\mathbf{M}} = \bar{1}^{\boldsymbol{A}}\}.$$

$$\mathrm{SAT}_{\mathrm{pos}}(\mathbb{K}) = \{\varphi \in \mathrm{Sent}_{\mathcal{P}} \mid \textit{there exist } \boldsymbol{A} \in \mathbb{K} \textit{ and a safe } \boldsymbol{A}\textit{-structure } \mathbf{M} \textit{ such that } \|\varphi\|^{\boldsymbol{A}}_{\mathbf{M}} > \bar{0}^{\boldsymbol{A}}\}.$$

DEFINITION 2.0.2. *Let* L *be a ($\triangle$-)core fuzzy logic. Instead of* $\mathrm{TAUT}(\mathbb{K})$ *we write*

- $\mathrm{genTAUT}(\mathrm{L}\forall)$ *if* $\mathbb{K}$ *is the class of all* L*-chains (the* general semantics*).*
- $\mathrm{realTAUT}(\mathrm{L}\forall)$ *if* $\mathbb{K}$ *is the class of all* real L*-chains, i.e. whose lattice reduct is the real unit interval* $[0,1]$.
- $\mathrm{stTAUT}(\mathrm{L}\forall)$ *if* $\mathbb{K}$ *is a subclass of real* L*-chains which are considered the* intended real semantics *(or* standard semantics*) of* L.
- $\mathrm{ratTAUT}(\mathrm{L}\forall)$ *if* $\mathbb{K}$ *is the class of all* rational L*-chains, i.e. whose lattice reduct is the rational unit interval* $[0,1]^{\mathbb{Q}}$.
- $\mathrm{intratTAUT}(\mathrm{L}\forall)$ *if* $\mathbb{K}$ *consists of a single rational* L*-chain which is considered the* intended rational L*-chain.*
- $\mathrm{finTAUT}(\mathrm{L}\forall)$ *if* $\mathbb{K}$ *is the class of all* finite L*-chains.*

We define analogous notations for the sets $\mathrm{TAUT}_{\mathrm{pos}}(\mathbb{K})$*,* $\mathrm{SAT}(\mathbb{K})$ *and* $\mathrm{SAT}_{\mathrm{pos}}(\mathbb{K})$ *in all the cases.*

Given a ($\triangle$-)core fuzzy logic L, we write $\Sigma \models_{\mathrm{real(L\forall)}} \varphi$ meaning that $\Sigma \models_{\mathbb{K}} \varphi$ when $\mathbb{K}$ is the class consisting of all real L-chains; moreover $\mathrm{realCons}(\mathrm{L}\forall, \Sigma)$ denotes the set $\{\varphi \in \mathrm{Sent}_{\mathcal{P}} \mid \Sigma \models_{\mathrm{real(L\forall)}} \varphi\}$, and the analogous definitions for the other semantics.

LEMMA 2.0.3 ([5]). *Let* L *be a ($\triangle$-)core fuzzy logic. If* $\boldsymbol{A}$ *and* $\boldsymbol{B}$ *are* L*-chains such that there is a σ-embedding (i.e. an embedding preserving all existing infima and suprema) from* $\boldsymbol{A}$ *into* $\boldsymbol{B}$*, then:*

1. $\mathrm{TAUT}(\boldsymbol{B}) \subseteq \mathrm{TAUT}(\boldsymbol{A})$,
2. $\mathrm{TAUT}_{\mathrm{pos}}(\boldsymbol{B}) \subseteq \mathrm{TAUT}_{\mathrm{pos}}(\boldsymbol{A})$,
3. $\mathrm{SAT}(\boldsymbol{A}) \subseteq \mathrm{SAT}(\boldsymbol{B})$,
4. $\mathrm{SAT}_{\mathrm{pos}}(\boldsymbol{A}) \subseteq \mathrm{SAT}_{\mathrm{pos}}(\boldsymbol{B})$.

The negation operation $\neg a = a \to \overline{0}$ allows us to obtain several easy but useful relations between sets of tautologies and satisfiable sentences. We choose a rather general formulation to cope with other possible negations in logics expanded with extra connectives as those presented in Chapter VIII.

LEMMA 2.0.4. *Let* $\mathbb{K}$ *be a class of chains and let* $\sim$ *be an operation present in all members of* $\mathbb{K}$ *such that for every* x*,* $\sim x = 1$ *iff* $x = 0$*. Then for every* $\varphi \in \mathrm{Sent}_{\mathcal{P}}$*:*

1. $\varphi \in \mathrm{TAUT}_{\mathrm{pos}}(\mathbb{K})$ *iff* $\sim\varphi \notin \mathrm{SAT}(\mathbb{K})$,
2. $\varphi \in \mathrm{SAT}_{\mathrm{pos}}(\mathbb{K})$ *iff* $\sim\varphi \notin \mathrm{TAUT}(\mathbb{K})$.

LEMMA 2.0.5. *Let* $\mathbb{K}$ *be a class and let* $\sim$ *be an operation present in all members of* $\mathbb{K}$ *such that for every* x*,* $\sim x = 0$ *iff* $x = 1$*. Then for every* $\varphi \in \mathrm{Sent}_{\mathcal{P}}$*:*

1. $\varphi \in \mathrm{SAT}(\mathbb{K})$ *iff* $\sim\varphi \notin \mathrm{TAUT}_{\mathrm{pos}}(\mathbb{K})$,
2. $\varphi \in \mathrm{TAUT}(\mathbb{K})$ *iff* $\sim\varphi \notin \mathrm{SAT}_{\mathrm{pos}}(\mathbb{K})$.

The previous lemma applies, in particular, when $\sim$ is an involutive negation (i.e. $\sim\sim x = x$ for every x), e.g. when $\mathbb{K}$ can is a class of expansions of IMTL-chains.

LEMMA 2.0.6. *Let* $\mathbb{K}$ *be a class of chains and let* $\sim$ *be an operation present in all members of* $\mathbb{K}$ *such that for every* x*,* $\sim\sim x = 1$ *iff* $x > 0$*. Then for every* $\varphi \in \mathrm{Sent}_{\mathcal{P}}$*:*

1. $\varphi \in \mathrm{TAUT}_{\mathrm{pos}}(\mathbb{K})$ *iff* $\sim\sim\varphi \in \mathrm{TAUT}(\mathbb{K})$,
2. $\varphi \in \mathrm{SAT}_{\mathrm{pos}}(\mathbb{K})$ *iff* $\sim\sim\varphi \in \mathrm{SAT}(\mathbb{K})$.

The previous lemma applies, in particular, when $\sim$ is a strict negation (i.e. $\sim x = \overline{0}$ for every $x \neq \overline{0}$), e.g. when $\mathbb{K}$ can is a class of expansions of SMTL-chains. For chains with $\triangle$ we have the following:

LEMMA 2.0.7. *Let $\mathbb{K}$ be a class of chains with the $\triangle$ operation. Then for every sentence φ we have the following relations:*

1. $\varphi \in \mathrm{SAT}(\mathbb{K})$ *iff* $\neg\triangle\varphi \notin \mathrm{TAUT}(\mathbb{K})$,
2. $\varphi \in \mathrm{TAUT}(\mathbb{K})$ *iff* $\neg\triangle\varphi \notin \mathrm{SAT}(\mathbb{K})$ *iff* $\neg\triangle\varphi \notin \mathrm{SAT}_{\mathrm{pos}}(\mathbb{K})$,
3. $\varphi \in \mathrm{TAUT}_{\mathrm{pos}}(\mathbb{K})$ *iff* $\neg\triangle(\neg\varphi) \in \mathrm{TAUT}(\mathbb{K})$,
4. $\varphi \in \mathrm{SAT}_{\mathrm{pos}}(\mathbb{K})$ *iff* $\neg\triangle(\neg\varphi) \in \mathrm{SAT}(\mathbb{K})$.

We can obtain some lower bounds for the complexity of some of these problems. In case of the sets of satisfiable sentences, it is easy that they are always $\mathbf{\Pi_1}$-hard.

PROPOSITION 2.0.8. *For every class $\mathbb{K}$ of chains, $\mathrm{SAT}(\mathbb{K})$ is $\mathbf{\Pi_1}$-hard.*

Proof. Recall that we assume $\mathbb{K}$ to be non-empty. If φ is a sentence and $\{P_i \mid 1 \leq i \leq n\}$ are the predicate symbols from $\mathcal{P}$ appearing in φ, we define the sentence $Crisp(\varphi) = \bigwedge_{1\leq i\leq n}(\forall\overrightarrow{x})(P_i(\overrightarrow{x}) \vee \neg P_i(\overrightarrow{x}))$. Recall that $\boldsymbol{B}_2$ denotes the Boolean algebra of two elements. Now just observe that for every $\varphi \in \mathrm{Sent}_{\mathcal{P}}$, $\varphi \in \mathrm{SAT}(\boldsymbol{B}_2)$ iff $Crisp(\varphi)\&\varphi \in \mathrm{SAT}(\mathbb{K})$, and since the satisfiability problem in classical logic is $\mathbf{\Pi_1}$-hard so it must be $\mathrm{SAT}(\mathbb{K})$. □

Now we consider the TAUT problems. In the sequel, for every sentence φ, 2φ denotes the sentence $\neg((\neg\varphi)\ \&\ (\neg\varphi))$.

LEMMA 2.0.9. *Let* L *be any ($\triangle$-)core fuzzy logic. For every sentence φ, we have that $2\varphi \vee 2(\neg\varphi) \in \mathrm{genTAUT}(\mathrm{L}\forall)$.*

Proof. Let $\boldsymbol{A}$ be an L-chain and $\mathbf{M}$ an $\boldsymbol{A}$-model. If $\|\varphi\|^{\boldsymbol{A}}_{\mathbf{M}} \leq \|\neg\varphi\|^{\boldsymbol{A}}_{\mathbf{M}}$, then we have $\|(\neg\neg\varphi)^2\|^{\boldsymbol{A}}_{\mathbf{M}} = \overline{0}^{\boldsymbol{A}}$. If $\|\varphi\|^{\boldsymbol{A}}_{\mathbf{M}} > \|\neg\varphi\|^{\boldsymbol{A}}_{\mathbf{M}}$, then $\|(\neg\varphi)^2\|^{\boldsymbol{A}}_{\mathbf{M}} = \overline{0}^{\boldsymbol{A}}$. (Observe that $\neg x \leq x$ implies $\neg x \leq \neg\neg x$, thus $\neg x \to (\neg x \to \overline{0}) = \overline{1}$, hence $(\neg x)^2 \to \overline{0} = \overline{1}$.) In either case we have $\|\neg(\neg\varphi)^2 \vee \neg(\neg\neg\varphi)^2\|^{\boldsymbol{A}}_{\mathbf{M}} = \overline{1}^{\boldsymbol{A}}$. □

DEFINITION 2.0.10. *Let φ be a classical sentence. Consider its prenex normal form in classical logic, $Q_1x_1 \ldots Q_nx_n\ \psi(x_1,\ldots,x_n)$, where ψ is a lattice combination of literals. We define a formula $\varphi^\star$ by induction as follows: if φ is a literal, then $\varphi^\star = 2\varphi$; * commutes with quantifiers, $\wedge$ and $\vee$.*

LEMMA 2.0.11. *Let φ be a lattice combination of literals,* L *be a ($\triangle$-)core fuzzy logic and $\mathbb{K}$ a class of* L*-chains. The following are equivalent:*

(1) φ is a classical propositional tautology,

(2) $\varphi^\star$ is an L*-tautology,*

(3) $\varphi^\star$ is a tautology for every chain in $\mathbb{K}$,

(4) $\varphi^\star$ is a positive tautology for every chain in $\mathbb{K}$.

Proof. Recall that by our standing assumption $\mathbb{K}$ is non-empty. $(2) \Rightarrow (3)$ and $(3) \Rightarrow (4)$ are obvious. We prove $(1) \Rightarrow (2)$. By distributivity, φ can be equivalently written as $\bigwedge_{i=1}^{n} \bigvee_{j=1}^{n_i} \alpha_{i,j}$, where $\alpha_{i,j}$ are literals. Thus, φ is a classical tautology iff for every $i \in \{1, \ldots, n\}$, $\bigvee_{j=1}^{n_i} \alpha_{i,j}$ is a classical tautology. This is the case if for every $i \in \{1, \ldots, n\}$ there are $j_1, j_2 \in \{1, \ldots, n_i\}$ such that $\alpha_{i,j_1} = \neg\alpha_{i,j_2}$. Hence, $2\alpha_{i,j_1} \vee 2\alpha_{i,j_2}$ is an L-tautology by previous lemma and, since this formula implies $\bigvee_{j=1}^{n_i} 2\alpha_{i,j}$, we have that $\varphi^\star$ is an L-tautology. We finally prove $(4) \Rightarrow (1)$ by contraposition. If φ is not a classical propositional tautology, then there is an evaluation e on $\boldsymbol{B}_2$ such that $e(\varphi) = 0$. Since $\varphi^\star$ and φ are equivalent in classical logic, we also have $e(\varphi^\star) = 0$. Now, given any $\boldsymbol{A} \in \mathbb{K}$, it is clear that e can also be seen as an evaluation on $\boldsymbol{A}$ and $e(\varphi^\star) = \overline{0}^{\boldsymbol{A}}$. □

LEMMA 2.0.12. *Let $\varphi = (\exists x_1) \ldots (\exists x_n)\ \psi(x_1, \ldots, x_n)$, where ψ is a lattice combination of literals, be a purely existential formula,* L *be a ($\triangle$-)core fuzzy logic and $\mathbb{K}$ a class of* L*-chains. The following are equivalent:*

(1) $\varphi \in \mathrm{TAUT}(\boldsymbol{B}_2)$,

(2) $\varphi^\star \in \mathrm{genTAUT}(\mathrm{L}\forall)$,

(3) $\varphi^\star \in \mathrm{TAUT}(\mathbb{K})$,

(4) $\varphi^\star \in \mathrm{TAUT}_{\mathrm{pos}}(\mathbb{K})$.

Proof. Again, $(2) \Rightarrow (3)$ and $(3) \Rightarrow (4)$ are obvious. $(4) \Rightarrow (1)$ is proved as in the previous lemma. We prove $(1) \Rightarrow (2)$. Suppose that φ is a classical tautology. By Herbrand's Theorem, there is a classical propositional tautology of the form $\bigvee_{i=1}^{m} \psi(t_1^i, \ldots, t_n^i)$, where the t_j^i's are closed terms. By the previous lemma, recalling that * commutes with $\vee$, we have that $\bigvee_{i=1}^{m} \psi^\star(t_1^i, \ldots, t_n^i) \in \mathrm{genTAUT}(\mathrm{L}\forall)$. By an easy proof in L$\forall$, we can derive $\varphi^\star = (\exists x_1) \ldots (\exists x_n)\ \psi^\star(x_1, \ldots, x_n)$, and hence we have proved (2). □

THEOREM 2.0.13. *For every class $\mathbb{K}$ of chains, the sets* $\mathrm{TAUT}(\mathbb{K})$ *and* $\mathrm{TAUT}_{\mathrm{pos}}(\mathbb{K})$ *are $\boldsymbol{\Sigma_1}$-hard.*

Proof. The set of provable existential formulae of first-order classical logic is $\boldsymbol{\Sigma_1}$-hard (observe that here we are using our general hypothesis that assumes that our first-order logics have a full vocabulary). Indeed, the Herbrand form φ^H of any sentence φ is purely existential, and φ is provable iff φ^H is provable. The claim now follows from the previous lemma. □

In order to prove that the $\mathrm{SAT}_{\mathrm{pos}}(\mathbb{K})$ problems are $\boldsymbol{\Pi_1}$-hard, we will deal with their complement, the sets of $\mathbb{K}$-contradictions:

- $\mathrm{TAUT}_0(\mathbb{K}) = \{\varphi \in \mathrm{Sent}_{\mathcal{P}} \mid$ for every $\boldsymbol{A} \in \mathbb{K}$ and every safe $\boldsymbol{A}$-structure $\mathbf{M}$, $\|\varphi\|_{\mathbf{M}}^{\boldsymbol{A}} = \overline{0}^{\boldsymbol{A}}\}$.

An adaptation of the proof of the $\boldsymbol{\Sigma_1}$-hardness of the TAUT and $\mathrm{TAUT}_{\mathrm{pos}}$ problems allows to obtain the same result for the newly defined set. We do it in the following lemmata and their consequences.

LEMMA 2.0.14. *Let* L *be any ($\triangle$-)core fuzzy logic. For every sentence* φ *we have* $\varphi^2 \wedge (\neg\varphi)^2 \in \mathrm{genTAUT}_0(\mathrm{L}\forall)$.

Proof. Let $\boldsymbol{A}$ be an L-chain and $\mathbf{M}$ an $\boldsymbol{A}$-model. Otherwise, if $\|\varphi\|^{\boldsymbol{A}}_{\mathbf{M}} \leq \|\neg\varphi\|^{\boldsymbol{A}}_{\mathbf{M}}$, then $\|\varphi^2\|^{\boldsymbol{A}}_{\mathbf{M}} = \overline{0}^{\boldsymbol{A}}$. If $\|\varphi\|^{\boldsymbol{A}}_{\mathbf{M}} > \|\neg\varphi\|^{\boldsymbol{A}}_{\mathbf{M}}$, then $\|(\neg\varphi)^2\|^{\boldsymbol{A}}_{\mathbf{M}} = \overline{0}^{\boldsymbol{A}}$. In either case we obtain $\|\varphi^2 \wedge (\neg\varphi)^2\|^{\boldsymbol{A}}_{\mathbf{M}} = \overline{0}^{\boldsymbol{A}}$. □

This yields a definition analogous to the previous one:

DEFINITION 2.0.15. *Let* φ *be a classical sentence. Consider its prenex normal form in classical logic,* $Q_1x_1 \dots Q_nx_n\ \psi(x_1,\dots,x_n)$*, where* ψ *is a lattice combination of literals. We define a formula* φ° *by induction as follows: if* φ *is a literal, then* $\varphi^\circ = \varphi^2$*;* $^\circ$ *commutes with quantifiers,* $\wedge$ *and* $\vee$*.*

LEMMA 2.0.16. *Let* φ *be a lattice combination of literals,* L *be a ($\triangle$-)core fuzzy logic and* $\mathbb{K}$ *a class of* L*-chains. The following are equivalent:*

(1) φ *is a classical propositional contradiction,*

(2) φ° *is an* L*-contradiction,*

(3) φ° *is a contradiction for every chain in* $\mathbb{K}$*.*

Proof. $(2) \Rightarrow (3)$ is trivial. We show $(1) \Rightarrow (2)$. By distributivity, φ can be equivalently written as $\bigvee_{i=1}^{n} \bigwedge_{j=1}^{n_i} \alpha_{i,j}$, where $\alpha_{i,j}$ are literals. Thus, φ is a classical contradiction iff for every $i \in \{1,\dots,n\}$, $\bigwedge_{j=1}^{n_i} \alpha_{i,j}$ is a classical contradiction. Hence, for every $i \in \{1,\dots,n\}$ there are $j_1, j_2 \in \{1,\dots,n_i\}$ such that $\alpha_{i,j_1} = \neg\alpha_{i,j_2}$. Therefore, $\alpha^2_{i,j_1} \wedge \alpha^2_{i,j_2}$ is an L-contradiction by the previous lemma and, since this formula is implied by $\bigwedge_{j=1}^{n_i} \alpha^2_{i,j}$, we have that φ° is an L-contradiction too. $(3) \Rightarrow (1)$ can be easily proved by contraposition. If φ is not a classical propositional contradiction, then there is an evaluation e on $\mathcal{B}_2$ such that $e(\varphi) = 1$. Since φ° and φ are equivalent in classical logic, we also have $e(\varphi^\circ) = 1$. Now, given any $\boldsymbol{A} \in \mathbb{K}$, it is clear that e can also be seen as an evaluation on $\boldsymbol{A}$ and $e(\varphi^\circ) = \overline{1}^{\boldsymbol{A}}$. □

LEMMA 2.0.17 (Dual Herbrand's Theorem). *Let* $\varphi = (\forall x_1)\dots(\forall x_n)\ \psi(x_1,\dots,x_n)$ *be a purely universal sentence.* φ *is a classical contradiction if, and only if, there exists* m *and closed terms* $\{t^i_1,\dots,t^i_n \mid i = 1,\dots,m\}$ *such that* $\bigwedge_{i=1}^{m} \psi(t^i_1,\dots,t^i_n)$ *is a classical propositional contradiction.*

Proof. It is a trivial consequence of the usual Herbrand's Theorem. □

LEMMA 2.0.18. *Let* $\varphi = (\forall x_1)\dots(\forall x_n)\ \psi(x_1,\dots,x_n)$*, where* ψ *is a lattice combination of literals, be a purely universal formula,* L *be a ($\triangle$-)core fuzzy logic and* $\mathbb{K}$ *a class of* L*-chains. The following are equivalent:*

(1) $\varphi \in \mathrm{TAUT}_0(\mathcal{B}_2)$,

(2) $\varphi^\circ \in \mathrm{genTAUT}_0(\mathrm{L}\forall)$,

(3) $\varphi^\circ \in \mathrm{TAUT}_0(\mathbb{K})$.

Proof. Again, $(2) \Rightarrow (3)$ is trivial and $(3) \Rightarrow (1)$ is proved as in Lemma 2.0.16. Let us show $(1) \Rightarrow (2)$. Suppose that φ is a classical contradiction. By the dual Herbrand's Theorem, there are closed terms t_j^i such that $\bigwedge_{i=1}^m \psi(t_1^i, \ldots, t_n^i)$ is a classical propositional contradiction. By Lemma 2.0.16, recalling that $^\circ$ commutes with $\wedge$, we have that $\bigwedge_{i=1}^m \psi^\circ(t_1^i, \ldots, t_n^i) \in \mathrm{genTAUT}_0(\mathrm{L}\forall)$. Therefore, we obtain $\varphi^\circ = (\forall x_1) \ldots (\forall x_n)\, \psi^\circ(x_1, \ldots, x_n) \in \mathrm{genTAUT}_0(\mathrm{L}\forall)$. □

LEMMA 2.0.19. *The set of classical purely universal first-order contradictions is* $\mathbf{\Sigma_1}$*-hard.*

Proof. First observe that the set all contradictions is $\mathbf{\Sigma_1}$-hard (again because we are in the full vocabulary). Indeed, the set of all tautologies is $\mathbf{\Sigma_1}$-hard and we have that for any sentence φ, φ is a contradiction iff $\neg\varphi$ is a tautology. Now given any sentence φ we can write the following chain of equivalencies: φ is a contradiction iff $\neg\varphi$ is a tautology iff its Herbrand form (purely existential) $(\neg\varphi)^H$ is a tautology iff $\neg(\neg\varphi)^H$ is a contradiction. The latter is a purely universal formula, so we are done. □

THEOREM 2.0.20. *For every (non-empty) class* $\mathbb{K}$ *of chains, the set* $\mathrm{TAUT}_0(\mathbb{K})$ *is* $\mathbf{\Sigma_1}$*-hard and thus* $\mathrm{SAT}_{\mathrm{pos}}(\mathbb{K})$ *is* $\mathbf{\Pi_1}$*-hard.*

Proof. It follows from the previous two lemmata and the fact that $\mathrm{SAT}_{\mathrm{pos}}(\mathbb{K})$ is the complementary set of $\mathrm{TAUT}_0(\mathbb{K})$. □

On the other hand, completeness with respect to a Hilbert-style calculus gives upper bounds for the complexity:

PROPOSITION 2.0.21. *Let* L *be a recursively axiomatizable ($\triangle$-)core fuzzy logic and* $\mathbb{K}$ *be a class of* L*-chains. If* L$\forall$ *has the* FS$\mathbb{K}$C*, then* $\mathrm{TAUT}(\mathbb{K})$ *and* $\mathrm{TAUT}_{\mathrm{pos}}(\mathbb{K})$ *are* $\mathbf{\Sigma_1}$*, while* $\mathrm{SAT}(\mathbb{K})$ *and* $\mathrm{SAT}_{\mathrm{pos}}(\mathbb{K})$ *are* $\mathbf{\Pi_1}$.

Proof. $\mathrm{TAUT}(\mathbb{K})$ is $\mathbf{\Sigma_1}$ because it is the set of theorems of a recursively axiomatizable logic. Using Lemma 2.0.4 ($\varphi \in \mathrm{SAT}_{\mathrm{pos}}(\mathbb{K})$ iff $\neg\varphi \notin \mathrm{TAUT}(\mathbb{K})$) we obtain that $\mathrm{SAT}_{\mathrm{pos}}(\mathbb{K})$ is $\mathbf{\Pi_1}$. As regards to $\mathrm{SAT}(\mathbb{K})$, notice that for every $\varphi \in \mathrm{Sent}_{\mathcal{P}}$ we have: $\varphi \in \mathrm{SAT}(\mathbb{K})$ iff $\varphi \not\models_{\mathbb{K}} \overline{0}$ iff $\varphi \nvdash_{\mathrm{L}\forall} \overline{0}$. Using again Lemma 2.0.4 (now $\varphi \in \mathrm{TAUT}_{\mathrm{pos}}(\mathbb{K})$ iff $\neg\varphi \notin \mathrm{SAT}(\mathbb{K})$) we obtain that $\mathrm{TAUT}_{\mathrm{pos}}(\mathbb{K})$ is $\mathbf{\Sigma_1}$. □

In particular, since a first-order logic is always complete with respect to the semantics of all chains, we obtain:

COROLLARY 2.0.22. *For every recursively axiomatizable first-order ($\triangle$-)core fuzzy logic* L$\forall$*,* $\mathrm{genTAUT}(\mathrm{L}\forall)$ *and* $\mathrm{genTAUT}_{\mathrm{pos}}(\mathrm{L}\forall)$ *are* $\mathbf{\Sigma_1}$*-complete,* $\mathrm{genSAT}(\mathrm{L}\forall)$ *and* $\mathrm{genSAT}_{\mathrm{pos}}(\mathrm{L}\forall)$ *are* $\mathbf{\Pi_1}$*-complete.*

Moreover, it yields the following general undecidability result:

COROLLARY 2.0.23. *For every ($\triangle$-)core fuzzy logic, the first-order logic* L$\forall$ *is undecidable.*

See all the results for the general semantics in Table 1.

Problem	Complexity
$\mathrm{genTAUT}(\mathrm{L}\forall)$	$\mathbf{\Sigma_1}$-complete
$\mathrm{genSAT}(\mathrm{L}\forall)$	$\mathbf{\Pi_1}$-complete
$\mathrm{genTAUT}_{\mathrm{pos}}(\mathrm{L}\forall)$	$\mathbf{\Sigma_1}$-complete
$\mathrm{genSAT}_{\mathrm{pos}}(\mathrm{L}\forall)$	$\mathbf{\Pi_1}$-complete

Table 1. Complexity results for the general semantics.

3 Complexity of standard semantics

Let L be a ($\triangle$-)core fuzzy logic. Recall that the *standard semantics* of L is given by a subclass class $\mathbb{K}$ of the class of all real L-chains which are considered to be the *intended real* L*-chains*. What are the intended real L-chains? This can possibly have several meanings but for t-norm based logics we may understand it as follows: if the L is introduced as the logic of a (left-)continuous t-norm $*$ then the intended chain is just $[0, 1]_*$ (like Łukasiewicz logic, etc.); if it is introduced as the logic of a set of t-norms (like BL) then $\mathbb{K}$ is the set of the corresponding algebras $[0, 1]_*$. Note that each real BL-chain is given by a continuous t-norm, thus for BL all real BL-chains are intended, and hence in the case of BL standard chains coincide with real chains.

We shall discuss particular prominent logics: first logics extending $\mathrm{BL}\forall$ and then logics extending $\mathrm{MTL}\forall$ (but not $\mathrm{BL}\forall$). The main results we will obtain are collected in Table 2, where $(Ł\oplus)\forall$ stands for any logic given by a continuous t-norm which is an ordinal sum of Łukasiewicz t-norm with any continuous t-norm (i.e. Łukasiewicz t-norm is its first component in the ordinal sum representation with possibly infinitely many components); analogously for $(\mathrm{G}\oplus)\forall$ and $(\Pi\oplus)\forall$.

We will often identify a standard BL-algebra $\boldsymbol{A}$ with its corresponding continuous t-norm, and hence we identify the components of the t-norm with the corresponding algebras, which will be called *t-norm components of* $\boldsymbol{A}$. Thus, when speaking e.g. of a product component of a t-norm we may indifferently mean either the isomorphic copy of the product t-norm or the isomorphic copy of the standard product algebra.

We start with Gödel logic $\mathrm{G}\forall$. It is clear that each countable G-chain $\boldsymbol{A}$ embeds into the standard G-chain $[0, 1]_{\mathrm{G}}$ by an isomorphism preserving all infinite suprema and infima existing in $\boldsymbol{A}$. This gives standard completeness. Therefore:

$$\mathrm{stTAUT}(\mathrm{G}\forall) = \mathrm{genTAUT}(\mathrm{G}\forall) \qquad \mathrm{stTAUT}_{\mathrm{pos}}(\mathrm{G}\forall) = \mathrm{genTAUT}_{\mathrm{pos}}(\mathrm{G}\forall)$$
$$\mathrm{stSAT}(\mathrm{G}\forall) = \mathrm{genSAT}(\mathrm{G}\forall) \qquad \mathrm{stSAT}_{\mathrm{pos}}(\mathrm{G}\forall) = \mathrm{genSAT}_{\mathrm{pos}}(\mathrm{G}\forall).$$

LEMMA 3.0.1. *The sets* $\mathrm{stTAUT}(\mathrm{G}\forall)$ *and* $\mathrm{stTAUT}_{\mathrm{pos}}(\mathrm{G}\forall)$ *are* $\mathbf{\Sigma_1}$*-complete, while the sets* $\mathrm{stSAT}_{\mathrm{pos}}(\mathrm{G}\forall)$ *and* $\mathrm{stSAT}(\mathrm{G}\forall)$ *are* $\mathbf{\Pi_1}$*-complete.*

Now let us turn to first-order Łukasiewicz logic. First, it is easy to show that $\mathrm{stTAUT}(Ł\forall)$ is in $\mathbf{\Pi_2}$. This follows immediately from the Pavelka-style completeness of $\mathrm{RPL}\forall$ discussed in Chapter VIII: $\varphi \in \mathrm{stTAUT}(Ł\forall)$ iff the provability degree $|\varphi|_{\mathrm{RPL}\forall}$ of φ equals 1, i.e. $(\forall r < 1$ rational$)(\exists d)(d$ is an RPL-proof of $(\overline{r} \to \varphi))$. Clearly, this condition is Π_2. The proof of $\mathbf{\Pi_2}$-hardness is much harder. From now on, our logic is $Ł\forall$.

Logic	stTAUT	stSAT	$\text{stTAUT}_{\text{pos}}$	$\text{stSAT}_{\text{pos}}$
MTL∀	$\mathbf{\Sigma_1}$-complete	$\mathbf{\Pi_1}$-complete	$\mathbf{\Sigma_1}$-complete	$\mathbf{\Pi_1}$-complete
IMTL∀	$\mathbf{\Sigma_1}$-complete	$\mathbf{\Pi_1}$-complete	$\mathbf{\Sigma_1}$-complete	$\mathbf{\Pi_1}$-complete
SMTL∀	$\mathbf{\Sigma_1}$-complete	$\mathbf{\Pi_1}$-complete	$\mathbf{\Sigma_1}$-complete	$\mathbf{\Pi_1}$-complete
WCMTL∀	$\mathbf{\Sigma_1}$-hard	$\mathbf{\Pi_1}$-hard	$\mathbf{\Sigma_1}$-hard	$\mathbf{\Pi_1}$-hard
ΠMTL∀	$\mathbf{\Sigma_1}$-hard	$\mathbf{\Pi_1}$-hard	$\mathbf{\Sigma_1}$-hard	$\mathbf{\Pi_1}$-hard
BL∀	Non-arithmetic	Non-arithmetic	Non-arithmetic	Non-arithmetic
SBL∀	Non-arithmetic	Non-arithmetic	Non-arithmetic	Non-arithmetic
Ł∀	$\mathbf{\Pi_2}$-complete	$\mathbf{\Pi_1}$-complete	$\mathbf{\Sigma_1}$-complete	$\mathbf{\Sigma_2}$-complete
G∀	$\mathbf{\Sigma_1}$-complete	$\mathbf{\Pi_1}$-complete	$\mathbf{\Sigma_1}$-complete	$\mathbf{\Pi_1}$-complete
Π∀	Non-arithmetic	Non-arithmetic	Non-arithmetic	Non-arithmetic
(Ł⊕)∀	$\mathbf{\Pi_2}$-hard	$\mathbf{\Pi_1}$-complete	$\mathbf{\Sigma_1}$-complete	$\mathbf{\Sigma_2}$-complete
(G⊕)∀	$\mathbf{\Sigma_1}$-hard	$\mathbf{\Pi_1}$-complete	$\mathbf{\Sigma_1}$-complete	$\mathbf{\Pi_1}$-complete
(Π⊕)∀	Non-arithmetic	Non-arithmetic	Non-arithmetic	Non-arithmetic
C_nMTL∀	$\mathbf{\Sigma_1}$-complete	$\mathbf{\Pi_1}$-complete	$\mathbf{\Sigma_1}$-complete	$\mathbf{\Pi_1}$-complete
C_nIMTL∀	$\mathbf{\Sigma_1}$-complete	$\mathbf{\Pi_1}$-complete	$\mathbf{\Sigma_1}$-complete	$\mathbf{\Pi_1}$-complete
WNM∀	$\mathbf{\Sigma_1}$-complete	$\mathbf{\Pi_1}$-complete	$\mathbf{\Sigma_1}$-complete	$\mathbf{\Pi_1}$-complete
NM∀	$\mathbf{\Sigma_1}$-complete	$\mathbf{\Pi_1}$-complete	$\mathbf{\Sigma_1}$-complete	$\mathbf{\Pi_1}$-complete

Table 2. Complexity results for the standard semantics of prominent first-order fuzzy logics.

DEFINITION 3.0.2.

(1) Call a formula classical *if all connectives it contains are among* $\wedge, \vee, \neg$.

(2) A standard model $\mathbf{M}$ *(of* Ł∀*) is* predefinite *if for each classical formula* φ *and each evaluation* v, $\|\varphi\|_{\mathbf{M},\mathrm{v}} \neq \frac{1}{2}$.

(3) For an n*-ary predicate* P, $\delta(P)$ *is the formula* $[(\forall x_1, \ldots, x_n)(P(x_1, \ldots, x_n) \vee \neg P(x_1, \ldots, x_n)]^2$.

It is easy to verify that a model $\mathbf{M}$ is predefinite if, and only if, $\|\delta(P)\|_{\mathbf{M}} > 0$ for each predicate P.

DEFINITION 3.0.3. *For each predefinite structure* $\mathbf{M} = \langle M, \langle P_{\mathbf{M}} \rangle_P, \langle f_{\mathbf{M}} \rangle_f \rangle$ *the corresponding Boolean structure is* $\mathbf{M}_{/01} = \langle M, \langle P'_{\mathbf{M}} \rangle_P, \langle f_{\mathbf{M}} \rangle_f \rangle$ *where* $P'_{\mathbf{M}}(\overrightarrow{a}) = 1$ *iff* $P_{\mathbf{M}}(\overrightarrow{a}) > \frac{1}{2}$, *otherwise* $P'_{\mathbf{M}}(\overrightarrow{a}) = 0$.

LEMMA 3.0.4. *Let* $\mathbf{M}$ *be predefinite and* φ *a classical formula. Then*

$$\|\varphi\|_{\mathbf{M}_{/01}} = 1 \textit{ iff } \|\varphi\|_{\mathbf{M}} > \tfrac{1}{2} \textit{ iff } \|\varphi^2\|_{\mathbf{M}} > 0.$$

Proof. This is clear for atomic formulae and follows by an easy induction for all classical formulae (elaborate the induction step for $\neg, \wedge, \forall$ using the preceding lemma). □

We shall need the following fact from recursion theory:

LEMMA 3.0.5. *There is a $\mathbf{\Sigma_1}$-relation $C \subseteq \mathsf{N}^2$ such that, if we define $C_m = \{n \mid \langle m, n\rangle \in C\}$ then the set $Fin = \{m \mid C_m$ is finite$\}$ is a $\mathbf{\Sigma_2}$-complete set (thus $\mathsf{N} \setminus Fin$ is a $\mathbf{\Pi_2}$-complete set).*

DEFINITION 3.0.6. *Assume that $\gamma(x, y)$ is a $\mathbf{\Sigma_1}$-formula defining a binary relation C in the standard model of arithmetics $\mathbf{N}$, i.e. such that for each $m, n \in \mathsf{N}$,*

$$\langle m, n\rangle \in C \quad \textit{iff} \quad \|\gamma(\overline{m}, \overline{n})\|_{\mathbf{N}} = 1.$$

Let Predef *stand for $\delta(=) \wedge \delta(\leq)$ and let, for each m, γ^*_m stand for the formula* Predef $\wedge (Q^+)^2 \wedge (\forall x, y)(2\gamma(\overline{m}, x) \wedge 2\gamma(\overline{m}, y) \wedge 2(x \neq y) \to \neg(U(x) \leftrightarrow U(y)))$ *where U is a new unary predicate (see page 854 for a presentation of Q^+).*

LEMMA 3.0.7. *Under the above relation, C_m is finite iff $\gamma^*_m \in \mathrm{stSAT_{pos}}(\text{Ł}\forall)$.*

Proof. First assume that C_m is finite, say $C_m = \{n_1, \ldots, n_k\}$. Take $\mathbf{N}$ and expand it to a model $\mathbf{M}$ by defining $U_{\mathbf{M}}(n_i) = i/k$, $U_{\mathbf{M}}(j) = 0$ for j distinct from $n_1, \ldots, n_k$. Verify easily that $\|\gamma^*_m\|_{\mathbf{M}} \geq \frac{1}{k}$. Indeed, $\|\mathrm{Predef}\|_{\mathbf{M}} = 1,\ \|Q^+\|_{\mathbf{M}} = 1$. Take $a, b \in \mathsf{N}$ and assume that the value $\|2\gamma(m, a) \wedge 2\gamma(m, b) \wedge 2(a \neq b)\|_{\mathbf{M}}$ is positive (otherwise there is nothing to prove). This means that the values of $\gamma(m, a), \gamma(m, b), a \neq b$ are $> \frac{1}{2}$ and hence $= 1$ (since in $\mathbf{M}$ everything except $U_{\mathbf{M}}$ is crisp). But then $a \neq b$ and $a, b \in C_m$; thus for some $i, j \leq k$ we have $a = n_i, b = n_j, i \neq j$ and $\|\neg(U(a) \leftrightarrow U(b))\|_{\mathbf{M}} = |\frac{i}{k} - \frac{j}{k}| \geq \frac{1}{k}$. Thus $\|\gamma^*_m\|_{\mathbf{M}} \geq \frac{1}{k}$.

Conversely, let C_m be infinite, $C_m = \{n_i\}_{i=1}^{\infty}$. We show that γ^*_m is not positively satisfiable. Assume it is, let $\|\gamma^*_m\|_{\mathbf{M}} = t > 0$. Delete $U_{\mathbf{M}}$ from $\mathbf{M}$; we obtain a predefinite model $\mathbf{M}'$ of the language of Q^+, hence $\mathbf{M}'' = \mathbf{M}'_{/01}$ is a model of Q^+. We may assume that $\mathbf{N}$ is an initial segment of $\mathbf{M}''$. Since $\|\gamma(m, n_i)\|_{\mathbf{N}} = 1$ for $i = 1, 2, \ldots$, we have $\|\gamma(m, n_i)\|_{\mathbf{M}''} = 1$, thus $\|\gamma(m, n_i)\|_{\mathbf{M}'} > \frac{1}{2}$. Hence $\|2\gamma(m, n_i)\|_{\mathbf{M}'} = 1$. For $i \neq j$ we obtain $\|2(n_i \neq n_j)\|_{\mathbf{M}'} = 1$. Come back to $\mathbf{M}$ (returning $U_{\mathbf{M}}$). Since $\|\gamma^*_m\|_{\mathbf{M}} = t$ we obtain $\|\neg(U(n_i) \leftrightarrow U(n_j))\|_{\mathbf{M}} \geq t$, thus putting $\|U(n_i)\|_{\mathbf{M}} = t_i$ we obtain $|t_i - t_j| \geq t$ for $i \neq j$. But this is a clear contradiction, because i, j run over all natural numbers and t_i, t_j, t are positive reals. This completes the proof. □

THEOREM 3.0.8. *The set* $\mathrm{stSAT_{pos}}(\text{Ł}\forall)$ *is $\mathbf{\Sigma_2}$-complete and the set* $\mathrm{stTAUT}(\text{Ł}\forall)$ *is $\mathbf{\Pi_2}$-complete.*

Proof. The mapping associating to each natural m the formula γ^*_m reduces the $\mathbf{\Sigma_2}$-complete set Fin from the Lemma 3.0.5 to the set $\mathrm{stSAT_{pos}}(\text{Ł}\forall)$. □

Now let us discuss $\mathrm{stTAUT_{pos}}(\text{Ł}\forall)$ and $\mathrm{stSAT}(\text{Ł}\forall)$ (it is easy to check the same result could be proved for RPL$\forall$ as well).

THEOREM 3.0.9. $\mathrm{stTAUT_{pos}}(\text{Ł}\forall)$ *is a $\mathbf{\Sigma_1}$-complete set and hence* $\mathrm{stSAT}(\text{Ł}\forall)$ *is a $\mathbf{\Pi_1}$-complete set.*

Proof. If $\varphi \in \mathrm{stTAUT_{pos}}(\text{Ł}\forall)$ then $\neg\varphi$ has no model over $[0, 1]_{\text{Ł}}$ (no standard model) and therefore is contradictory; recall that each consistent theory over Ł$\forall$ has a standard model (see e.g. [12, 13]); thus $\neg\varphi \vdash_{\text{Ł}\forall} \overline{0}$. Conversely, if $\neg\varphi \vdash_{\text{Ł}\forall} \overline{0}$ then $\neg\varphi$ has no model and hence $\varphi \in \mathrm{stTAUT_{pos}}(\text{Ł}\forall)$. Clearly $\neg\varphi \vdash_{\text{Ł}\forall} \overline{0}$ is a $\mathbf{\Sigma_1}$ condition; thus $\mathrm{stTAUT_{pos}}(\text{Ł}\forall)$ is in $\mathbf{\Sigma_1}$. It is $\mathbf{\Sigma_1}$-hard by Theorem 2.0.13. □

Now let us consider the logics (Ł$\oplus$)$\forall$ and (G$\oplus$)$\forall$ and their standard semantics. Take an arbitrary continuous t-norm which is an ordinal sum whose first summand is $\boldsymbol{C}$, $\boldsymbol{C}$ being Łukasiewicz, Gödel or product t-norm. Denote this t-norm by $\boldsymbol{C}\oplus$ and for simplicity assume that the first positive idempotent is $\frac{1}{2}$. This can always be always be achieved up to an isomorphism. Furthermore, we may assume without loss of generality that the isomorphism from the restriction of $\boldsymbol{C}\oplus$ to $[0, \frac{1}{2}]$ to $\boldsymbol{C}$ defined on $[0, 1]$ is just the mapping sending x to $2x$. Let us say that our t-norm *begins well* with $\boldsymbol{C}$. $\|\varphi\|^{\boldsymbol{C}}_{\mathbf{M}}$ denotes the truth value of a sentence φ in a $\boldsymbol{C}$-model $\mathbf{M}$; similarly for $\|\varphi\|^{\boldsymbol{C}\oplus}_{\mathbf{M}}$. $(\boldsymbol{C}\oplus)\forall$ is the predicate logic given by $\boldsymbol{C}\oplus$.

DEFINITION 3.0.10. *Let h be the following mapping of $[0,1]$ onto itself: $h(x) = 2x$ for $x \leq \frac{1}{2}$, $h(x) = 1$ for $x \in [\frac{1}{2}, 1]$. Let $\mathbf{M} = \langle M, \langle P_{\mathbf{M}}\rangle_{P\in\mathbf{P}}, \langle f_{\mathbf{M}}\rangle_{f\in\mathbf{F}}\rangle$ be a $[0,1]$-structure of the language in question. We define a structure $h(\mathbf{M})$ of the form $\langle M, \langle P'_{\mathbf{M}}\rangle_{P\in\mathbf{P}}, \langle f_{\mathbf{M}}\rangle_{f\in\mathbf{F}}\rangle$ where for each P (n-ary) and each tuple $a_1, \dots, a_n \in M$, $P'_{\mathbf{M}}(a_1, \dots, a_n) = h(P_{\mathbf{M}}(a_1, \dots, a_n))$. Furthermore, we define another structure $\mathbf{M}/2$ as $\langle M, \langle P_{\mathbf{M}/2}\rangle_{P\in\mathbf{P}}, \langle f_{\mathbf{M}}\rangle_{f\in\mathbf{F}}\rangle$ where $(P_{\mathbf{M}/2})(a_1, \dots, a_n) = P_{\mathbf{M}}(a_1, \dots, a_n)/2$.*

LEMMA 3.0.11. *Let $\boldsymbol{C}\oplus$ begin well with $\boldsymbol{C}$ and let h be the mapping in the previous definition. Then h is an algebraic homomorphism of $\langle [0,1], \boldsymbol{C}\oplus, \to^{\boldsymbol{C}\oplus}, 0, 1\rangle$ onto $\langle [0,1], \boldsymbol{C}, \to^{\boldsymbol{C}}, 0, 1\rangle$ preserving infinite joins and meets. Consequently, for each sentence φ,*

(1) $h(\|\varphi\|^{\boldsymbol{C}\oplus}_{\mathbf{M}}) = \|\varphi\|^{\boldsymbol{C}}_{h(\mathbf{M})}$,

(2) $\|\varphi\|^{\boldsymbol{C}}_{\mathbf{M}} = h(\|\varphi\|^{\boldsymbol{C}\oplus}_{\mathbf{M}/2})$.

The proof is easy.

THEOREM 3.0.12. *A sentence φ is a standard positive tautology of $\boldsymbol{C}\oplus$ iff it is a standard positive tautology of $\boldsymbol{C}$. Moreover, φ is standardly positively $\boldsymbol{C}\oplus$-satisfiable iff φ is standardly positively $\boldsymbol{C}$-satisfiable. In symbols we have:* $\mathrm{stTAUT}_{\mathrm{pos}}((\boldsymbol{C}\oplus)\forall) = \mathrm{stTAUT}_{\mathrm{pos}}(\boldsymbol{C}\forall)$ *and* $\mathrm{stSAT}_{\mathrm{pos}}((\boldsymbol{C}\oplus)\forall) = \mathrm{stSAT}_{\mathrm{pos}}(\boldsymbol{C}\forall)$.

Proof. This is immediate from the preceding lemma which shows that there is an $\mathbf{M}$ with $\|\varphi\|^{\boldsymbol{C}\oplus}_{\mathbf{M}} = 0$ iff there is an $\mathbf{M}$ with $\|\varphi\|^{\boldsymbol{C}}_{\mathbf{M}} = 0$, and the same for $\neq 0$. □

LEMMA 3.0.13. *Let $\boldsymbol{C}$ be* Ł *or* G *and let φ be a sentence such that $\varphi \in \mathrm{stSAT}(\boldsymbol{C}\forall)$. Then there exists a Henkin theory T proving φ. For $\boldsymbol{C}$ being* Ł*, T can be assumed to be maximally consistent.*

Proof. It follows from the properties of Henkin theories, see Section 4 of Chapter II. □

Call a model $\mathbf{M}$ *fully named* if each element of $\mathbf{M}$ is the interpretation of a constant. Recall that the canonical model $\mathbf{CM}(T)$ of a Henkin theory T is fully named (see Chapter II). In particular, for each formula $(\exists x)\alpha(x, y, \dots)$ and each $a, \dots \in \mathbf{CM}(T)$, if $(\exists x)\alpha(x, a, \dots)$ is 1-true in $\mathbf{CM}(T)$ then there is a $b \in \mathbf{CM}(T)$ such that $\alpha(b, a, \dots)$ is 1-true there. This is used in the proof of the next theorem.

LEMMA 3.0.14. *Let $\boldsymbol{C}$ be* Ł *or* G *and let $\boldsymbol{C}\oplus$ be a t-norm beginning by $\boldsymbol{C}$. For an arbitrary sentence φ, if $\varphi \in \mathrm{stSAT}(\boldsymbol{C})$ then $\varphi \in \mathrm{stSAT}(\boldsymbol{C}\oplus)$.*

Proof. Given φ, let $T_0 = \{\varphi\}$, let T be its Henkin extension over the logic $\boldsymbol{C}\forall$ and let $\mathbf{M} = \mathbf{CM}(T)$. For $\boldsymbol{C}$ being Gödel use the fact that each (countable) model over G∀ can be understood as a standard model by embedding the G-algebra of truth functions into [0,1] by a 1-1 mapping of truth values preserving all infinite joins and meets (easy). Thus $\mathbf{M}$ is a standard fully named Henkin model of φ. Now let $\boldsymbol{C}$ be Łukasiewicz. One can show that the Lindenbaum algebra $Lind$ used to construct the canonical model $\mathbf{CM}(T)$ is Archimedean, i.e. for each its element $x < 1$ there is a natural n such that $x^n = 0$. Furthermore, each (countable) Archimedean MV-chain embeds to the standard algebra $[0,1]_{Ł}$ by an isomorphic embedding i preserving all infinite joins and meets existing in this chain. Hence the Ł-model $\mathbf{M} = \mathbf{CM}(T)$ is made standard by replacing each value from $Lind$ by its i-image. (If necessary consult [12] and some references thereof.)

Thus in both cases we have a standard fully named model of Henkin theory T. Let $f(x) = \frac{x}{2}$ for $x < 1$, $f(1) = 1$. Make $\mathbf{M}$ to a $\boldsymbol{C}\oplus$-structure $\mathbf{M}'$ (with the $\boldsymbol{C}$-component on $[0, \frac{1}{2}]$) and with the same domain as $\mathbf{M}$ by defining $P_{\mathbf{M}'}(a_1, \dots) = f(P_{\mathbf{M}}(a_1, \dots))$ for all P and $a_1, \dots$ We show by induction on the complexity of closed $\overline{T}$-formulae α, $\|\alpha\|^{\boldsymbol{C}\oplus}_{\mathbf{M}'} = f(\|\alpha\|^{\boldsymbol{C}}_{\mathbf{M}})$.

This is evident for atoms and connectives (since $[0, \frac{1}{2}) \cup \{1\}$ is a $\boldsymbol{C}$-subalgebra of $[0,1]_*$) and for $\forall$ (since f preserves infinite meets); similarly for $\|(\exists x)\beta\|^{\boldsymbol{C}}_{\mathbf{M}_1} < 1$. For $\|(\exists x)\beta\|^{\boldsymbol{C}}_{\mathbf{M}_1} = 1$ use witnessing: there is a c such that $\|\beta(c)\|^{\boldsymbol{C}}_{\mathbf{M}_1} = 1$. In particular, since $\|\varphi\|^{\boldsymbol{C}}_{\mathbf{M}} = 1$ we obtain $\|\varphi\|^{\boldsymbol{C}}_{\mathbf{M}_1} = 1$ and $\|\varphi\|^{\boldsymbol{C}\oplus}_{\mathbf{M}'} = 1$. □

THEOREM 3.0.15.

(1) $\mathrm{stSAT_{pos}}(\mathrm{G}\forall) = \mathrm{stSAT}(\mathrm{G}\forall)$.

(2) $\mathrm{stSAT}((\mathrm{G}\oplus)\forall) = \mathrm{stSAT}(\mathrm{G}\forall)$.

(3) $\mathrm{stSAT}((Ł\oplus)\forall) = \mathrm{stSAT}(Ł\forall)$.

Proof. (1) Clearly each sentence standardly satisfiable in G∀ is positively satisfiable there. Conversely if $0 < r = \|\varphi\|^{\mathrm{G}}_{\mathbf{M}} < 1$ for some standard $\mathbf{M}$ then taking a one-one increasing mapping of $[0,1]$ onto itself produce an isomorphic copy $\mathbf{M}'$ of $\mathbf{M}$ such that $\|\varphi\|^{\mathrm{G}}_{\mathbf{M}'} = \frac{1}{2}$. Then apply the homomorphism h from Definition 3.0.10 and observe that it is a homomorphism of the G-structure $\mathbf{M}'$ to the G-structure $h(\mathbf{M}')$ sending $\frac{1}{2}$ to 1. Thus $\|\varphi\|^{\mathrm{G}}_{h(\mathbf{M}')} = 1$.

(2) Each sentence standardly satisfiable in $(\mathrm{G}\oplus)\forall$ is standardly satisfiable in G∀ by Lemma 3.0.11; the converse inclusion follows from Lemma 3.0.14. A similar line of reasoning proves (3). □

COROLLARY 3.0.16.

$$\mathrm{stSAT_{pos}}((\mathrm{G}\oplus)\forall) = \mathrm{stSAT_{pos}}(\mathrm{G}\forall) = \mathrm{stSAT}(\mathrm{G}\forall) = \mathrm{stSAT}((\mathrm{G}\oplus)\forall).$$

LEMMA 3.0.17. *If $*$ begins with* Ł *then for each φ, φ is a standard tautology of* Ł∀ *iff $\neg\neg\varphi$ is a tautology of $*$; similarly for satisfiability.*

Proof. By Lemma 3.0.11, φ is a standard tautology of Ł∀ iff φ is a $[\frac{1}{2}, 1]$-tautology of $*$ (for each $\mathbf{M}$, $\|\varphi\|^{\boldsymbol{C}\oplus}_{\mathbf{M}} \in [\frac{1}{2}, 1]$) iff $\neg\neg\varphi$ is a tautology of $*$. Similarly for satisfiability. □

The following theorem collects results of arithmetical complexity not stated till now.

THEOREM 3.0.18.

(1) $\mathrm{stSAT}((\mathrm{G}\oplus)\forall) = \mathrm{stSAT}_{\mathrm{pos}}((\mathrm{G}\oplus)\forall)$ *is* $\mathbf{\Pi_1}$*-complete,* $\mathrm{stTAUT}_{\mathrm{pos}}((\mathrm{G}\oplus)\forall)$ *is* $\mathbf{\Sigma_1}$*-complete and* $\mathrm{stTAUT}((\mathrm{G}\oplus)\forall)$ *is* $\mathbf{\Sigma_1}$*-hard.*

(2) $\mathrm{stTAUT}_{\mathrm{pos}}((\text{Ł}\oplus)\forall)$ *and* $\mathrm{stSAT}((\text{Ł}\oplus)\forall)$ *are* $\mathbf{\Sigma_1}$*-complete,* $\mathrm{stSAT}_{\mathrm{pos}}((\text{Ł}\oplus)\forall)$ *is* $\mathbf{\Sigma_2}$*-complete, and* $\mathrm{stTAUT}((\text{Ł}\oplus)\forall)$ *is* $\mathbf{\Pi_2}$*-hard.*

Proof. (1) By Corollary 3.0.16, and Theorems 3.0.12 and 2.0.20. (2) By Theorem 3.0.12, Lemma 3.0.13, Theorem 3.0.15, Lemma 3.0.17 and Theorem 3.0.9. □

We now characterize the classes $\mathbb{K}$ of standard BL-algebras such that $\mathrm{TAUT}(\mathbb{K})$ is recursively axiomatizable. More precisely, we will prove that $\mathrm{TAUT}(\mathbb{K})$ is recursively axiomatizable iff $\mathbb{K}$ is the singleton of $[0,1]_{\mathrm{G}}$.

In the sequel, if the free variables of ψ are among $x_1, \dots, x_n$ and $a_1, \dots, a_n \in M$, we write $\|\psi(a_1, \dots, a_n)\|^{\boldsymbol{A}}_{\mathbf{M}}$ to mean $\|\psi\|^{\boldsymbol{A}}_{\mathbf{M},\mathrm{v}}$ where $\mathrm{v}(x_i) = a_i$ $(i = 1, \dots, n)$.

We warn the reader that even if we identify the t-norm components with their associated algebras, t-norm components of a standard BL-algebra are in general different from its Wajsberg components. Here below we list some basic differences.

(1) The top of a Wajsberg component is 1, and its supremum or its infimum may fail to be in the component. To the contrary, 1 need not belong to a Łukasiewicz or product or Gödel component, and such components contain their supremum and their infimum.

(2) An infinite Wajsberg component can never be a product algebra, and product components are ordinal sums of a two-element Wajsberg algebra and a cancellative hoop (possibly with 1 replaced by the supremum of the component).

(3) A Gödel component is considered as the ordinal sum of uncountably many two-element Wajsberg hoops, again, possibly with 1 replaced by the supremum of the component.

(4) In an ordinal sum of Wajsberg components there must be a first component, whereas the same is not true of an ordinal sum of t-norm components.

(5) In an ordinal sum of Wajsberg components, each component is a subhoop of the whole algebra $\boldsymbol{A}$, whereas the same is not true of t-norm components (considered as BL-algebras), because if $x \leq y$ are in a t-norm component $\boldsymbol{A}_i$ of $\boldsymbol{A}$ such that $\max(A_i) < 1$ and if $\to$ and $\to_i$ denote the residual in $\boldsymbol{A}$ and in $\boldsymbol{A}_i$ respectively, then $x \to_i y = \max(A_i)$, and $x \to y = 1$.

For any two formulae λ and υ, we write $\lambda \uparrow \upsilon$ for $(\lambda \to \upsilon) \to \upsilon$, and we adopt a similar notation for elements of a BL-chain.

LEMMA 3.0.19. *Let* $\boldsymbol{A}$ *be a* BL*-chain,* $\mathbf{M}$ *a first-order* $\boldsymbol{A}$*-structure and* λ, υ *sentences, possibly with parameters from* M*. Then,* $\|\lambda \uparrow \upsilon\|^{\boldsymbol{A}}_{\mathbf{M}} = \overline{1}^{\boldsymbol{A}}$ *if either* $\|\upsilon\|^{\boldsymbol{A}}_{\mathbf{M}} = \overline{1}^{\boldsymbol{A}}$ *or* $\|\lambda\|^{\boldsymbol{A}}_{\mathbf{M}} = \overline{1}^{\boldsymbol{A}}$*, or* $\|\lambda\|^{\boldsymbol{A}}_{\mathbf{M}} > \|\upsilon\|^{\boldsymbol{A}}_{\mathbf{M}}$ *and* $\|\lambda\|^{\boldsymbol{A}}_{\mathbf{M}}$ *and* $\|\upsilon\|^{\boldsymbol{A}}_{\mathbf{M}}$ *are not in the same Wajsberg component. Otherwise,* $\|\lambda \uparrow \upsilon\|^{\boldsymbol{A}}_{\mathbf{M}} = \max\{\|\lambda\|^{\boldsymbol{A}}_{\mathbf{M}}, \|\upsilon\|^{\boldsymbol{A}}_{\mathbf{M}}\}$.

Proof. The claim is trivial if either $\|\upsilon\|^{\boldsymbol{A}}_{\mathbf{M}} = \overline{1}^{\boldsymbol{A}}$ or $\|\lambda\|^{\boldsymbol{A}}_{\mathbf{M}} = \overline{1}^{\boldsymbol{A}}$. Thus, suppose $\|\upsilon\|^{\boldsymbol{A}}_{\mathbf{M}} < \overline{1}^{\boldsymbol{A}}$ and $\|\lambda\|^{\boldsymbol{A}}_{\mathbf{M}} < \overline{1}^{\boldsymbol{A}}$. If $\|\lambda\|^{\boldsymbol{A}}_{\mathbf{M}}$ and $\|\upsilon\|^{\boldsymbol{A}}_{\mathbf{M}}$ are in the same Wajsberg component, then clearly $\|\lambda \uparrow \upsilon\|^{\boldsymbol{A}}_{\mathbf{M}} = \|\lambda \vee \upsilon\|^{\boldsymbol{A}}_{\mathbf{M}}$, and the claim follows.

If $\|\lambda\|^{\boldsymbol{A}}_{\mathbf{M}} \leq \|\upsilon\|^{\boldsymbol{A}}_{\mathbf{M}}$, then $\|\lambda \uparrow \upsilon\|^{\boldsymbol{A}}_{\mathbf{M}} = \|\upsilon\|^{\boldsymbol{A}}_{\mathbf{M}}$, and again the claim follows. Finally, if $\|\lambda\|^{\boldsymbol{A}}_{\mathbf{M}} > \|\upsilon\|^{\boldsymbol{A}}_{\mathbf{M}}$ and these two elements are not in the same Wajsberg component, then $\|\lambda \to \upsilon\|^{\boldsymbol{A}}_{\mathbf{M}} = \|\upsilon\|^{\boldsymbol{A}}_{\mathbf{M}}$, and $\|\lambda \uparrow \upsilon\|^{\boldsymbol{A}}_{\mathbf{M}} = \overline{1}^{\boldsymbol{A}}$. □

Let ϕ be a sentence. We define for every sentence ψ, possibly with parameters from M, the sentence ψ^{ϕ} in the following inductive way:

- $\psi^{\phi} = \psi \vee \phi$ if ψ is atomic and different from $\overline{0}$ and $\overline{1}$; $\overline{0}^{\phi} = \phi$ and $\overline{1}^{\phi} = \overline{1}$.
- $^{\phi}$ commutes with $\to$, $\exists$ and $\forall$, i.e. $(\psi \to \gamma)^{\phi} = \psi^{\phi} \to \gamma^{\phi}$, $((\exists x)\psi)^{\phi} = (\exists x)(\psi^{\phi})$ and $((\forall x)\psi)^{\phi} = (\forall x)(\psi^{\phi})$.
- $(\psi \,\&\, \gamma)^{\phi} = (\psi^{\phi} \,\&\, \gamma^{\phi}) \vee \phi$.

Note that $\vdash_{\mathrm{BL}\forall} \phi \to \psi^{\phi}$ for every formula ψ.

In the sequel, given a Łukasiewicz or Gödel or product component $\boldsymbol{C}$ of a standard BL-algebra $\boldsymbol{A}$, given a formula ϕ and an $\boldsymbol{A}$-structure $\mathbf{M}$ such that $\|\phi\|^{\boldsymbol{A}}_{\mathbf{M}} \in \boldsymbol{C}$, but $\|\phi\|^{\boldsymbol{A}}_{\mathbf{M}} \neq \max(\boldsymbol{C})$, $\boldsymbol{C}^{\phi}$ denotes the algebra whose domain is $\{c \in C \mid \|\phi\|^{\boldsymbol{A}}_{\mathbf{M}} \leq c\}$, whose bottom is $\|\phi\|^{\boldsymbol{A}}_{\mathbf{M}}$ and whose operations are $x \star^{\phi} y = \max\{(x \star y), \|\phi\|^{\boldsymbol{A}}_{\mathbf{M}}\}$, and $x \to^{\phi} y = \min\{(x \to y), \max(\boldsymbol{C})\}$. Note that $\boldsymbol{C}^{\phi}$ is isomorphic to $\boldsymbol{C}$ if $\boldsymbol{C}$ is either a Łukasiewicz or a Gödel component, or if $\boldsymbol{C}$ is a product component and $\|\phi\|^{\boldsymbol{A}}_{\mathbf{M}} = \min(\boldsymbol{C})$. On the other hand, if $\boldsymbol{C}$ is a product component and $\|\phi\|^{\boldsymbol{A}}_{\mathbf{M}} > \min(\boldsymbol{C})$, then $\boldsymbol{C}^{\phi}$ is isomorphic to $[0,1]_{\text{Ł}}$.

Suppose now that $\boldsymbol{C}$ is a Łukasiewicz or Gödel or product component of a standard BL-algebra $\boldsymbol{A}$, let m be its maximum, and let $\mathbf{M}$ be an $\boldsymbol{A}$-structure. Let ϕ be a sentence such that $\|\phi\|^{\boldsymbol{A}}_{\mathbf{M}} \in \boldsymbol{C} \setminus \{m\}$, and let $\boldsymbol{C}^{\phi}$ be as shown above. Let $S = \{a \in A \mid \forall c \in C(c \leq a)\}$. Note that $m \in S$, and that S is closed under $\star$. Indeed, if $a, b \in S$, then assuming without loss of generality $a \leq b$, either a is an idempotent, and then $a \star b = a \in S$, or a belongs to a Wajsberg component $\boldsymbol{W}$ whose intersection with $\boldsymbol{C}$ is either empty or reduced to the minimum of S. In any case $W \subseteq S$, therefore $a \star b \in W \subseteq S$. Now define a $\boldsymbol{C}^{\phi}$-structure $\mathbf{M}^{\phi}$ such that constants and function symbols are interpreted as in $\mathbf{M}$ and for every n-ary predicate P and every $a_1, \ldots, a_n \in M$, $P_{\mathbf{M}^{\phi}}(a_1, \ldots, a_n) = (P_{\mathbf{M}}(a_1, \ldots, a_n) \vee \|\phi\|^{\boldsymbol{A}}_{\mathbf{M}}) \wedge m$. The interpretation is then extended to a map $\|\ldots\|^{\boldsymbol{C}^{\phi}}_{\mathbf{M}^{\phi}}$ from the set of all formulae into $\boldsymbol{C}^{\phi}$ in the usual way. For simplicity, let us write $\|\psi\|_{\phi}$ instead of $\|\psi\|^{\boldsymbol{C}^{\phi}}_{\mathbf{M}^{\phi}}$, and $\|\psi\|$ instead of $\|\psi\|^{\boldsymbol{A}}_{\mathbf{M}}$.

LEMMA 3.0.20. *For every formula ξ one has:*

(i) $\|\xi^{\phi}\| \in S$ *iff* $\|\xi\|_{\phi} = m$.

(ii) If $\|\xi^{\phi}\| \notin S$, *then* $\|\xi\|_{\phi} = \|\xi^{\phi}\|$.

Proof. First of all, note that if $\|\xi^\phi\| \notin S$, then $\|\xi^\phi\| \in C^\phi \setminus \{m\}$, because $\|\xi^\phi\| \geq \|\phi\| = \min(\boldsymbol{C}^\phi)$. We prove the first claim by induction on ξ.

The claim is obvious if ξ is atomic (in particular, if $\xi = \overline{0}$ or $\xi = \overline{1}$). If $\xi = \lambda \mathbin{\&} \upsilon$, then (i) follows immediately from the induction hypothesis and from the fact that S is closed under $\star$. As regards to (ii), if $\|\xi^\phi\| \notin S$, then at least one of $\|\xi^\phi\|$, $\|\upsilon^\phi\|$ is not in S, since S is closed under $\star$. If they are both in $C^\phi \setminus \{m\}$, then by the induction hypothesis, $\|\lambda \mathbin{\&} \upsilon\|_\phi = \|\lambda\|_\phi \star^\phi \|\upsilon\|_\phi = \|\lambda^\phi\| \star^\phi \|\upsilon^\phi\| = \|(\lambda^\phi \mathbin{\&} \upsilon^\phi) \vee \phi\| = \|\xi^\phi\|$. If, say, $\|\lambda^\phi\| \in S$ and $\|\upsilon^\phi\| \in C^\phi \setminus \{m\}$, then $\|\xi^\phi\| = \|\upsilon^\phi\|$, and by the induction hypothesis, $\|\xi^\phi\| = \|\upsilon^\phi\| = \|\upsilon\|_\phi = m \star_\phi \|\upsilon\|_\phi = \|\lambda\|_\phi \star_\phi \|\upsilon\|_\phi = \|\xi\|_\phi$.

Suppose now that $\xi = \lambda \to \upsilon$; as regards to (i), $\|\xi^\phi\| \in S$ iff either $\|\upsilon^\phi\| \in S$ or $\|\lambda^\phi\| \leq \|\upsilon^\phi\|$ and $\|\upsilon^\phi\| \in C^\phi \setminus \{m\}$. In the former case, $\|\upsilon\|_\phi = m$, and $\|\xi^\phi\| \in S$. In the latter case, by the induction hypothesis, $\|\xi\|_\phi = \|\lambda\|_\phi \to^\phi \|\upsilon\|_\phi = \|\lambda^\phi\| \to^\phi \|\upsilon^\phi\| = m$.

Conversely, if $\|\xi\|_\phi = m$, then $\|\lambda\|_\phi \leq \|\upsilon\|_\phi$, therefore either $\|\upsilon\|_\phi = m$ and then, by the induction hypothesis, $\|\upsilon^\phi\| \in S$ and $\|\xi^\phi\| \in S$, or $\|\lambda^\phi\| \leq \|\upsilon^\phi\|$, and then $\|\xi^\phi\| = 1$. In any case, $\|\xi^\phi\| \in S$.

As regards to (ii), if $\|\xi^\phi\| \in C^\phi \setminus \{m\}$, then we must have $\|\upsilon^\phi\| \in C^\phi \setminus \{m\}$, and $\|\upsilon^\phi\| < \|\lambda^\phi\|$. If $\|\lambda^\phi\| \in S$, then $\|\xi^\phi\| = \|\upsilon^\phi\|$, and by the induction hypothesis $\|\xi\|_\phi = \|\upsilon\|_\phi = \|\upsilon^\phi\| = \|\xi^\phi\|$. If $\|\lambda^\phi\| \in C^\phi \setminus \{m\}$, then $\|\xi\|_\phi = \|\lambda\|_\phi \to^\phi \|\upsilon\|_\phi = \|\lambda^\phi\| \to \|\upsilon^\phi\| = \|\xi^\phi\|$.

Next, suppose $\xi = (\forall x)\lambda(x)$. If $\|\xi^\phi\| \in S$, then for all $d \in M$, $\|\lambda(d)^\phi\| \in S$. Hence claim (i) follows from the induction hypothesis. If $\|\xi^\phi\| \in C^\phi \setminus \{m\}$, then since $\inf(S) = m \in S$, there is a $d \in M$ such that $\|\lambda(d)^\phi\| \in C^\phi \setminus \{m\}$, and claim (ii) follows from the induction hypothesis.

Finally, suppose that $\xi = (\exists x)\lambda(x)$. If for some $d \in M$, $\|\lambda(d)^\phi\| \in S$, then claim (i) follows from the induction hypothesis. If $\|\xi^\phi\| \in S$, but for all $d \in M$, $\|\lambda(d)^\phi\| \in C^F \setminus \{m\}$, then $\|\xi^\phi\| = m$. Thus, by the induction hypothesis, $\|\xi\|_\phi = \sup\{\|\lambda(d)\|_\phi \mid d \in M\} = \sup\{\|\lambda(d)^\phi\| \mid d \in M\} = m$, as desired. □

THEOREM 3.0.21. *Let* $\mathbb{K}$ *be a class of standard* BL*-algebras that contains a* BL*-algebra* $\boldsymbol{A}$ *which is not isomorphic to* $[0,1]_{\mathrm{G}}$*. Then* $\mathrm{TAUT}(\mathbb{K})$ *is* $\boldsymbol{\Pi_2}$*-hard. Therefore, the only logic* L *which is complete with respect to a class* $\mathbb{K}$ *of standard* BL*-algebras such that* $\mathrm{realTAUT}(\mathrm{L}\forall)$ *is recursively axiomatizable is Gödel logic.*

Proof. We will recursively reduce the set $\mathrm{stTAUT}(\text{Ł}\forall)$, which is known to be $\boldsymbol{\Pi_2}$-complete, to $\mathrm{TAUT}(\mathbb{K})$. Let us extend the language by adding of a new unary predicate symbol U. Let $\phi = (\forall x)U(x)$, let $\gamma = (\forall x)((U(x) \to \phi) \vee (U(x) \uparrow \phi)$, and let, for every sentence ψ of L, $\psi^\star$ be the sentence

$$\psi^\star = \psi^\phi \vee (\exists x)(\psi^\phi \uparrow U(x)) \vee \gamma.$$

LEMMA 3.0.22. *Let* $\boldsymbol{A} \in \mathbb{K}$*, let* $\star$ *and* $\to$ *be its monoid operation and its residual respectively, and let* $\mathbf{M}$ *be an* $\boldsymbol{A}$*-structure such that* $\|\psi^\star\|^{\boldsymbol{A}}_{\mathbf{M}} \neq 1$*. Then:*

(i) There is $d \in M$ *such that* $\|U(d)\|^{\boldsymbol{A}}_{\mathbf{M}}$ *and* $\|\phi\|^{\boldsymbol{A}}_{\mathbf{M}}$ *are in the same Wajsberg component* $\boldsymbol{W}$*, and* $\|\phi\|^{\boldsymbol{A}}_{\mathbf{M}} < \|U(d)\|^{\boldsymbol{A}}_{\mathbf{M}} < 1$*.*

(ii) $\|\psi^\phi\|^{\boldsymbol{A}}_{\mathbf{M}} \in \boldsymbol{W} \setminus \{1\}$*.*

Proof. Throughout the whole proof we write $\|\dots\|$ for $\|\dots\|^{\boldsymbol{A}}_{\mathbf{M}}$.

(i) If for all $d \in M$ either $\|\phi\| = \|U(d)\|$, or $\|U(d)\| = 1$, or $\|\phi\|$ and $\|U(d)\|$ do not belong to the same component, then by Lemma 3.0.19 for all $d \in M$ we would have either $\|U(d) \to \phi\| = 1$, or $\|U(d) \uparrow \phi\| = 1$. Hence, $\|\gamma\| = 1$, and $\|\psi^\star\| = 1$, a contradiction.

(ii) Let $\boldsymbol{W}$ be the Wajsberg component which $\|\phi\|$ belongs to. If $\|\psi^\phi\| \notin W \setminus \{1\}$, then since $\|\psi^\phi\| \geq \|\phi\| = \inf\{\|U(d)\| \mid d \in M\}$, we have that for some $d \in M$, $\|U(d)\| < \|\psi^\phi\|$, and either $\|\psi^\phi\| = 1$ or $\|U(d)\|$ and $\|\psi^\phi\|$ are not in the same component. In the first case $\|\psi^\star\| = 1$, and in the second one by Lemma 3.0.19 we would have $\|(\exists x)(\psi^\phi \uparrow U(x))\| \geq \|\psi^\phi \uparrow U(d)\| = 1$, and once again $\|\psi^\star\| = 1$, which is impossible. □

The rest of the proof of Theorem 3.0.21 Now let $\boldsymbol{A}$, $\mathbf{M}$, ϕ, $\boldsymbol{W}$, etc. be as in the lemma, and let $m = \sup(W)$ and $c = \inf(W)$. Let us write once again $\|\dots\|$ for $\|\dots\|^{\boldsymbol{A}}_{\mathbf{M}}$. By Lemma 3.0.22 (i), $\boldsymbol{W}$ is a Wajsberg component of $\boldsymbol{A}$ with more than two elements. Let $C = (W \setminus \{1\}) \cup \{c, m\}$. Then since $\boldsymbol{A}$ is a standard BL-algebra and since $\boldsymbol{W}$ has more than two elements, C is the domain of a t-norm component $\boldsymbol{C}$ of $\boldsymbol{A}$, which is either a Łukasiewicz or a product component (remember that a Gödel t-norm component is the ordinal sum of Wajsberg components of two elements, and hence if it has more than two elements it is not a Wajsberg component). Moreover, by Lemma 3.0.22 (i), $\|\phi\| < m$. Finally, if $\boldsymbol{C}$ is a product component, then $c < \|\phi\|$, because $\|\phi\| \in W$, $\boldsymbol{W}$ is cancellative (hence unbounded), and $c = \inf(\boldsymbol{W}) \notin W$. Now let $\boldsymbol{C}^\phi$ be defined from $\boldsymbol{C}$ as in Lemma 3.0.20. Then $\boldsymbol{C}^\phi$ is an isomorphic copy of $[0,1]_{\text{Ł}}$. Let S and $\|\dots\|_\phi$ be defined as in Lemma 3.0.20. Then by Lemma 3.0.20 for every sentence λ we have:

(i) If $\|\lambda^\phi\| \in S$, then $\|\lambda\|_\phi = m$.

(ii) If $\|\lambda^\phi\| \notin S$, then $\|\lambda\|_\phi = \|\lambda^\phi\|$.

We conclude the proof of Theorem 3.0.21 by demonstrating that $\psi \in \mathrm{stTAUT}(\text{Ł}\forall)$ iff $\psi^\star \in \mathrm{TAUT}(\mathbb{K})$. Suppose $\psi^\star \notin \mathrm{TAUT}(\mathbb{K})$. Let $\boldsymbol{A}$ be a standard BL-chain and $\mathbf{M}$ be an $\boldsymbol{A}$-structure such that $\|\psi^\star\|^{\boldsymbol{A}}_{\mathbf{M}} < 1$. Let $\boldsymbol{W}$, $\boldsymbol{C}$ and $\boldsymbol{C}^\phi$ be as in Lemma 3.0.22. Then by Lemma 3.0.22 $\boldsymbol{C}^\phi$ is isomorphic to $[0,1]_{\text{Ł}}$, and by Lemma 3.0.20 we obtain an interpretation $\|\dots\|_\phi$ into $\boldsymbol{C}^\phi$ such that $\|\psi\|_\phi < 1$. Hence $\psi \notin \mathrm{stTAUT}(\text{Ł}\forall)$.

Conversely, suppose $\psi \notin \mathrm{stTAUT}(\text{Ł}\forall)$. Let $\boldsymbol{A} \in \mathbb{K}$ be not isomorphic to $[0,1]_{\mathrm{G}}$ and $\mathbf{M}$ be a $[0,1]_{\text{Ł}}$-structure such that $\|\psi\|^{[0,1]_{\text{Ł}}}_{\mathbf{M}} < 1$. Then $\boldsymbol{A}$ has either a Łukasiewicz component or a product component, $\boldsymbol{C}$ say. We define an $\boldsymbol{A}$-structure $\mathbf{M}'$ as follows:

- The domain of $\mathbf{M}'$ is the domain M of $\mathbf{M}$, and the interpretation of all constant symbols and function symbols is as in $\mathbf{M}$.
- For all $d_0 \in M$, let $U_{\mathbf{M}'}(d_0)$ be such that $U_{\mathbf{M}'}(d_0) \in C$, and $\overline{0}^{\boldsymbol{C}} < \inf\{U_{\mathbf{M}'}(d) \mid d \in M\} < U_{\mathbf{M}'}(d_0) < \sup\{U_{\mathbf{M}'}(d) \mid d \in M\} < \overline{1}^{\boldsymbol{C}}$.
- Before defining the interpretation of the other predicate symbols, we note that for all $d \in M$, $\overline{0}^{\boldsymbol{C}} < \|\phi\|^{\boldsymbol{A}}_{\mathbf{M}'} < \|U(d)\|^{\boldsymbol{A}}_{\mathbf{M}'} < \overline{1}^{\boldsymbol{C}}$, and hence the algebra $\boldsymbol{C}^\phi$ defined from $\boldsymbol{C}$ as in Lemma 3.0.20, is isomorphic to $[0,1]_{\text{Ł}}$ via an isomorphism h.

Then define for every n-ary predicate P and for every $d_1, \dots, d_n \in M$ we have: $P_{\mathbf{M}'}(d_1, \dots, d_n) = h(P_{\mathbf{M}}(d_1, \dots, d_m))$.

Now let us write $\|\dots\|$ for $\|\dots\|^{\boldsymbol{A}}_{\mathbf{M}'}$, and let us define an interpretation $\|\dots\|_\phi$ from $\|\dots\|$ as in Lemma 3.0.20. Then by Lemma 3.0.20 we obtain $1^{\boldsymbol{C}} > \|\psi\|_\phi = \|\psi^\phi\|$. Moreover, since for all $d \in M$, $\|\phi\| < \|U(d)\|$ and $\|\phi\|, \|U(d)\| \in \boldsymbol{C}^\phi \setminus \{1\}$, by Lemma 3.0.19, $\|\gamma\| < \overline{1}^{\boldsymbol{C}}$. Finally, again by Lemma 3.0.19, $\|(\exists x)(\psi^\phi \uparrow U(x))\| = \max\{\sup\{\|U(d)\| \mid d \in M\}, \|\psi^\phi\|\} < \overline{1}^{\boldsymbol{C}}$. Thus, we have $\|\psi^\star\| < \overline{1}^{\boldsymbol{C}} \leq 1$, and $\psi^\star \notin \mathrm{TAUT}(\mathbb{K})$. This concludes the proof of Theorem 3.0.21. □

We now prove the non-arithmeticity of sets of the form $\mathrm{TAUT}(\mathbb{K})$, $\mathrm{TAUT}_{\mathrm{pos}}(\mathbb{K})$, $\mathrm{SAT}(\mathbb{K})$, and $\mathrm{SAT}_{\mathrm{pos}}(\mathbb{K})$, where $\mathbb{K}$ is a set of standard BL-algebras which contains a BL-algebra which is either isomorphic to $[0,1]_\Pi$ or begins with Π. See e.g. [22]. We start with the SAT classes.

THEOREM 3.0.23. *Suppose that* $\mathbb{K}$ *contains a standard* BL*-algebra which begins with* Π *or is isomorphic to* $[0,1]_\Pi$*. Then* $\mathrm{SAT}(\mathbb{K})$ *and* $\mathrm{SAT}_{\mathrm{pos}}(\mathbb{K})$ *are not arithmetical.*

Proof. We work in a language containing the language of Peano Arithmetic PA, including a binary predicate symbol $\leq$ for order, plus an additional unary predicate U. For every formula ψ of PA, we denote by $\psi^{\neg\neg}$ the formula obtained replacing every atomic subformula γ of ψ by $\neg\neg\gamma$. Note that if a BL-chain either begins with Π or is isomorphic to $[0,1]_\Pi$, then $\psi^{\neg\neg}$ is interpreted either as 0 or as 1. We now consider the following formulae:

- $\theta_1 = (\forall x)\neg\neg U(x) \mathbin{\&} \neg(\forall x)U(x)$.
- $\theta_2 = (\forall x)(\forall y)((U(x) \to (U(y) \mathbin{\&} U(x))) \to U(y))$.
- The conjunction of all formulae $\sigma^{\neg\neg}$ such that σ is an axiom of Q^+. We denote such conjunction by θ_3.[2]
- The formula $\theta_4 = (\forall x)(U(S(x)) \leftrightarrow ((\forall y)((y \leq x)^{\neg\neg} \to U(y)))^3$.

LEMMA 3.0.24. *If* $\boldsymbol{A}$ *is a standard* BL*-algebra and* $\mathbf{M}$ *is an* $\boldsymbol{A}$*-structure such that* $\|\theta_1 \mathbin{\&} \theta_2\|^{\boldsymbol{A}}_{\mathbf{M}} > 0$*, then* $\boldsymbol{A}$ *begins with* Π *or is isomorphic to* $[0,1]_\Pi$.

Proof. Suppose first that $\boldsymbol{A}$ begins with a Łukasiewicz component (or is isomorphic to $[0,1]_Ł$). If for some b, $\|U(b)\|^{\boldsymbol{A}}_{\mathbf{M}}$ is in the first component, then $\|(\forall x)U(x)\|^{\boldsymbol{A}}_{\mathbf{M}} = \|(\forall x)\neg\neg U(x)\|^{\boldsymbol{A}}_{\mathbf{M}}$, and $\|\theta_1\|^{\boldsymbol{A}}_{\mathbf{M}} = 0$. If either $\boldsymbol{A}$ begins with a Gödel component or has no first t-norm component, then we must have $\inf\{\|U(d)\| \mid d \in M\} = 0$ and $\|U(d)\| > 0$ for all $d \in M$, otherwise $\|\theta_1\| = 0$. Hence, if either $\boldsymbol{A}$ begins with a Gödel component or has no first t-norm component, then for all $d \in M$ there must be $d' \in M$ such that $\|U(d')\|^{\boldsymbol{A}}_{\mathbf{M}} < \|U(d)\|^{\boldsymbol{A}}_{\mathbf{M}}$ and $\|U(d)\|^{\boldsymbol{A}}_{\mathbf{M}}$ and $\|U(d')\|^{\boldsymbol{A}}_{\mathbf{M}}$ are not in the same Wajsberg component. This implies $\|(U(d') \to ((U(d') \mathbin{\&} U(d))) \to U(d)\|^{\boldsymbol{A}}_{\mathbf{M}} = \|U(d)\|^{\boldsymbol{A}}_{\mathbf{M}}$, and $\|\theta_2\|^{\boldsymbol{A}}_{\mathbf{M}} = 0$. □

[2]We assume that θ_3 includes $\gamma^{\neg\neg}$ when γ is an axiom of equality or the crispness axiom $x = y \vee \neg(x = y)$ for $=$.

LEMMA 3.0.25.
If $\boldsymbol{A}$ is a standard BL*-algebra,* $\mathbf{M}$ *is an* $\boldsymbol{A}$*-model, and* $\|\theta_1 \,\&\, \theta_2 \,\&\, \theta_3 \,\&\, \theta_4\|^{\boldsymbol{A}}_{\mathbf{M}} > 0$*, then:*

(1) For every formula $\gamma(x_1, \ldots, x_n)$ *in the language of PA whose free variables are among* $x_1, \ldots, x_n$ *and for all* $d_1, \ldots, d_n \in M$, $\|\gamma^{\neg\neg}(d_1, \ldots, d_n)\|^{\boldsymbol{A}}_{\mathbf{M}} \in \{0, 1\}$.

(2) Let $\mathbf{M}^{\neg\neg}$ *be the classical structure whose domain*[3] *is* M*, in which function and constant symbols are interpreted as in* $\mathbf{M}$ *and such that for every* n*-ary predicate symbol* P *and for all* $d_1, \ldots, d_n \in M$*, one has* $\mathbf{M}^{\neg\neg} \models P(d_1, \ldots, d_n)$ *iff* $\|P^{\neg\neg}(d_1, \ldots, d_n)\|^{\boldsymbol{A}}_{\mathbf{M}} = 1$*. Then* $\mathbf{M}^{\neg\neg}$ *is a (classical) model of* Q^+.

(3) $\mathbf{M}^{\neg\neg}$ *is isomorphic to the standard model* $\mathbf{N}$ *of natural numbers.*

Proof. (1) Let us write $\| \ldots \|$ instead of $\| \ldots \|^{\boldsymbol{A}}_{\mathbf{M}}$. By Lemma 3.0.24, $\boldsymbol{A}$ begins with Π or is isomorphic to $[0,1]_\Pi$, and hence it is an SBL-chain. Therefore, for every sentence ψ, $\|\psi^{\neg\neg}\| \in \{0, 1\}$.

(2) By an easy induction, we see that $\mathbf{M}^{\neg\neg} \models \psi$ iff $\|\psi^{\neg\neg}\| = 1$ for every sentence ψ. Moreover by (1), $\|\theta_3\| \in \{0, 1\}$, and by our assumptions, $\|\theta_3\| > 0$. Hence, $\|\theta_3\| = 1$. It follows that every axiom of Q^+ is true in $\mathbf{M}^{\neg\neg}$.

(3) Suppose, by the way of contradiction, that $\mathbf{M}^{\neg\neg}$ is a non-standard model of Q^+. First of all, note that

(+) $\quad \inf\{\|U(d)\| \mid d \in M\} = 0$ and for all $d \in M$, $\|U(d)\| > 0$,

(otherwise $\|\theta_1\| = 0$). We claim that there is a $c \in M$ such that for all $b \in M$,

(++) $\quad$ if $\mathbf{M}^{\neg\neg} \models c \leq b$, then $\|U(S(b))\| \leq (\|(\forall x)((x \leq^{\neg\neg} b) \to U(x))\|)^2$.

Indeed, if

(*) $\quad \|U(S(b))\| > (\|(\forall x)((x \leq^{\neg\neg} b) \to U(x))\|)^2$,

then $\|U(S(b)) \to ((\forall x)(x \leq^{\neg\neg} b \to U(x)))^3\| \leq \|(\forall x)(x \leq^{\neg\neg} b \to U(x))\|$, and if (*) holds for unboundedly many b, (that is, if for all $c \in M$ there is a $b \in M$ such that $\mathbf{M}^{\neg\neg} \models c \leq b$ and (*) holds), then $\|\theta_4\| = 0$, against our assumption.

Since $\inf\{\|U(d) \mid d \in M\} = 0$, there is a $d \in M$ such that $\|U(d)\| < 1$, and since if condition (++) holds for $c \in M$, then it holds for all $c' \in M$ such that $\mathbf{M}^{\neg\neg} \models c \leq c'$, we can suppose, without loss of generality, $\mathbf{M}^{\neg\neg} \models d < c$, and by (++), $\|U(c)\| \leq \|U(d)\|^2 < 1$. Moreover, by the previous observation (that is, condition (*) is upward preserved), we may assume without loss of generality that c is non-standard. Hence, by an iterated use of (++) we see that for every natural number n, $\|U(c+n)\| \leq \|U(c)\|^{2^n}$, and $\inf\{\|U(c+n)\| \mid n \in \mathsf{N}\} = 0$. But since c is non-standard, then $\mathbf{M}^{\neg\neg} \models c + c > c + n$ for every $n \in \mathsf{N}$, and hence, by (*), $\|U(c+c)\| \leq (\|U(c+n)\|)^2$ for every natural number n. It follows $\|U(c+c)\| = 0$, contradicting condition (+).

We have derived a contradiction from our assumption that $\mathbf{M}^{\neg\neg}$ was not standard, and hence $\mathbf{M}^{\neg\neg}$ is isomorphic to $\mathbf{N}$. □

[3] If the predicate $=$ for equality is not interpreted as crisp equality, then instead of M we have to take as domain the set of all equivalence classes of elements of M modulo the equivalence $\equiv$ defined by $a \equiv b$ iff $\|a =^{\neg\neg} b\|^{\boldsymbol{A}}_{\mathbf{M}} = 1$.

We conclude the proof of Theorem 3.0.23. Since the set of sentences of PA which are true in $\mathbf{N}$ is not arithmetical, it suffices to prove:

LEMMA 3.0.26. *Let ψ be any PA-sentence. Then,* $\mathbf{N} \models \psi$ *iff* $\theta_1 \& \theta_2 \& \theta_3 \& \theta_4 \& \psi^{\neg\neg} \in \mathrm{SAT}(\mathbb{K})$ *iff* $\theta_1 \ \& \ \theta_2 \ \& \ \theta_3 \ \& \ \theta_4 \ \& \ \psi^{\neg\neg} \in \mathrm{SAT}_{\mathrm{pos}}(\mathbb{K})$.

Proof. If $\mathbf{N} \not\models \psi$, then we have seen that if $\boldsymbol{A} \in \mathbb{K}$ and $\mathbf{M}$ is an $\boldsymbol{A}$-structure such that $\|\theta_1 \ \& \ \theta_2 \ \& \ \theta_3 \ \& \ \theta_4\|^{\boldsymbol{A}}_{\mathbf{M}} > 0$, then $\mathbf{M}^{\neg\neg}$ is isomorphic to $\mathbf{N}$. Moreover, an easy induction shows that for every formula $\gamma(x_1, \ldots, x_n)$ of T and for all $d_1, \ldots, d_n \in M$, one has $\mathbf{M}^{\neg\neg} \models \gamma(d_1, \ldots, d_n)$ iff $\|\gamma^{\neg\neg}(d_1, \ldots, d_n)\|^{\boldsymbol{A}}_{\mathbf{M}} = 1$. Hence, $\|\psi^{\neg\neg}\|^{\boldsymbol{A}}_{\mathbf{M}} = 0$, and $\theta_1 \& \theta_2 \& \theta_3 \& \theta_4 \& \psi^{\neg\neg} \notin \mathrm{SAT}_{\mathrm{pos}}(\mathbb{K})$. *A fortiori*, $\theta_1 \& \theta_2 \& \theta_3 \& \theta_4 \& \psi^{\neg\neg} \notin \mathrm{SAT}(\mathbb{K})$.

Conversely, assume $\mathbf{N} \models \psi$. Take $\boldsymbol{A} \in \mathbb{K}$ which either begins with Π or is isomorphic to $[0,1]_\Pi$, and define an $\boldsymbol{A}$-structure $\mathbf{M}$ as follows: the domain M of $\mathbf{M}$ is N and the constants and the function symbols of PA are interpreted as in $\mathbf{N}$. Moreover, for $d_1, \ldots, d_n \in M$ and for every n-ary predicate of the language of PA, we set $P_{\mathbf{M}}(d_1, \ldots, d_n) = 1$ if $\mathbf{N} \models P(d_1, \ldots, d_n)$ and $P_{\mathbf{M}}(d_1, \ldots, d_n) = 0$ otherwise. Finally, $U_{\mathbf{M}}(n) = (\frac{1}{2})^{3^n}$, where each real number is thought of as an element of the first component $[0,1]_\Pi$ of $\boldsymbol{A}$.

It is easily seen that $\|\theta_1 \& \theta_2 \& \theta_3 \& \theta_4 \& \psi^{\neg\neg}\|^{\boldsymbol{A}}_{\mathbf{M}} = 1$. Hence, $\theta_1 \& \theta_2 \& \theta_3 \& \theta_4 \& \psi^{\neg\neg} \in \mathrm{SAT}(\mathbb{K})$, and *a fortiori* $\theta_1 \ \& \ \theta_2 \ \& \ \theta_3 \ \& \ \theta_4 \ \& \ \psi^{\neg\neg} \in \mathrm{SAT}_{\mathrm{pos}}(\mathbb{K})$. This ends the proof of Lemma 3.0.26. □

At this point the proof of Theorem 3.0.23 is immediate. □

COROLLARY 3.0.27. *For every* $\mathrm{L} \in \{\Pi, \mathrm{BL}, \mathrm{SBL}, \Pi\oplus\}$*, the sets* $\mathrm{stSAT}(\mathrm{L}\forall)$ *and* $\mathrm{stSAT}_{\mathrm{pos}}(\mathrm{L}\forall)$ *are not arithmetical.*

We now consider the complexity of $\mathrm{TAUT}(\mathbb{K})$ and $\mathrm{TAUT}_{\mathrm{pos}}(\mathbb{K})$ where $\mathbb{K}$ is a set of standard BL-chains such that at least one of them begins with Π or is isomorphic to $[0,1]_\Pi$. Let T, θ_1, θ_2, θ_3 and θ_4 be as in the proof of Theorem 3.0.23.

THEOREM 3.0.28. *Let ψ be a sentence of PA and $\psi^\star$ the sentence* $(\theta_1 \& \theta_2 \& \theta_3 \& \theta_4) \to \psi^{\neg\neg}$. *Then,* $\mathbf{N} \models \psi$ *iff* $\psi^\star \in \mathrm{TAUT}(\mathbb{K})$ *iff* $\psi^\star \in \mathrm{TAUT}_{\mathrm{pos}}(\mathbb{K})$. *Hence,* $\mathrm{TAUT}(\mathbb{K})$ *and* $\mathrm{TAUT}_{\mathrm{pos}}(\mathbb{K})$ *are not arithmetical.*

Proof. Suppose first $\mathbf{N} \not\models \psi$. Let $\boldsymbol{A} \in \mathbb{K}$ be such that $\boldsymbol{A}$ begins with Π or is isomorphic to $[0,1]_\Pi$, and consider an $\boldsymbol{A}$-structure $\mathbf{M}$ as follows: the domain M of $\mathbf{M}$ is N and the constants and the function symbols of T are interpreted as in $\mathbf{N}$. Moreover, for every n-ary predicate of PA and for all $k_1, \ldots, k_n \in \mathsf{N}$, we stipulate that $P_{\mathbf{M}}(k_1, \ldots, k_n) = 1$ if $\mathbf{N} \models P(k_1, \ldots, k_n)$ and $P_{\mathbf{M}}(k_1, \ldots, k_n) = 0$ otherwise. Finally, $U_{\mathbf{M}}(n) = (\frac{1}{2})^{3^n}$, where each real number is thought of as an element of the first component $[0,1]_\Pi$ of $\boldsymbol{A}$. It is easily seen that $\|\theta_1 \ \& \ \theta_2 \ \& \ \theta_3 \ \& \ \theta_4\|^{\boldsymbol{A}}_{\mathbf{M}} = 1$, and $\|\psi^{\neg\neg}\|^{\boldsymbol{A}}_{\mathbf{M}} = 0$. Hence, $\psi^\star \notin \mathrm{TAUT}_{\mathrm{pos}}(\mathbb{K})$, and *a fortiori* $\psi^\star \notin \mathrm{TAUT}(\mathbb{K})$.

Now suppose that $\mathbf{N} \models \psi$, and let us prove that $\psi^\star \in \mathrm{TAUT}(\mathbb{K})$ (hence, *a fortiori*, $\psi^\star \in \mathrm{TAUT}_{\mathrm{pos}}(\mathbb{K})$). Thus, let $\boldsymbol{A} \in \mathbb{K}$ and $\mathbf{M}$ be any $\boldsymbol{A}$-structure, and let us prove that $\|\psi^\star\|^{\boldsymbol{A}}_{\mathbf{M}} = 1$. Once again,we will write $\|\ldots\|$ instead of $\|\ldots\|^{\boldsymbol{A}}_{\mathbf{M}}$. Clearly, $\|\psi^\star\| = 1$ if $\|\theta_1 \ \& \ \theta_2 \ \& \ \theta_3 \ \& \ \theta_4\| = 0$. Hence, we may assume, without loss of generality, that $\|\theta_1 \ \& \ \theta_2 \ \& \ \theta_3 \ \& \ \theta_4\| > 0$. By Lemma 3.0.24 we have that either $\boldsymbol{A}$ begins with Π or is isomorphic to $[0,1]_\Pi$, and by Lemma 3.0.25 we can construct a classical model $\mathbf{M}^{\neg\neg}$ which is isomorphic to $\mathbf{N}$ and such that for every sentence γ in the language of PA, one has $\mathbf{M}^{\neg\neg} \models \gamma$ iff $\|\gamma^{\neg\neg}\| = 1$. Hence, if $\mathbf{N} \models \psi$, then $\|\psi^{\neg\neg}\| = 1$, and finally $\|\psi^\star\| = 1$. Summing up, if $\mathbf{N} \models \psi$, then $\psi^\star \in \mathrm{TAUT}(\mathbb{K})$, and the claim is proved. □

COROLLARY 3.0.29. *For every* $\mathrm{L} \in \{\Pi, \mathrm{BL}, \mathrm{SBL}, \Pi\oplus\}$*, the sets* $\mathrm{stTAUT}(\mathrm{L}\forall)$ *and* $\mathrm{stTAUT}_{\mathrm{pos}}(\mathrm{L}\forall)$ *are not arithmetical.*

We now prove that with a finite number of exceptions, for all classes $\mathbb{K}$ of standard BL-algebras, $\mathrm{TAUT}(\mathbb{K})$ is not arithmetical. We first prove that $\mathrm{TAUT}(\mathbb{K})$ is not arithmetical when $\mathbb{K}$ contains a standard BL-algebra with a product component (not necessarily the first component).

From the proof of Theorem 3.0.28 it follows that the set of standard tautologies of $\Pi\forall$ of the form $\psi = \theta_3 \to \gamma$ (where θ_3 is as in the proof of Theorem 3.0.28) is not arithmetical. Hence, it suffices to find an algorithm that associates to every sentence ψ of the form shown above a sentence $\psi^\star$ such that $\psi \in \mathrm{stTAUT}(\Pi\forall)$ if, only if, $\psi^\star \in \mathrm{TAUT}(\mathbb{K})$. Once again, U denotes a unary predicate symbol not in the language of $\Pi\forall$, and we define $\phi = (\forall x)U(x)$. Moreover, let us define:

$$\theta = (\forall x)(U(x) \uparrow \phi),$$

$$\sigma = (\forall x)(\forall y)((U(x) \uparrow U(y)) \leftrightarrow (U(y) \uparrow U(x))).$$

For every sentence ψ in the language of $\Pi\forall$ of the form $\theta_3 \to \gamma$, we define $\psi^\star = ((\theta \,\&\, \sigma) \to \psi^\phi) \vee (\exists x)(\psi^\phi \uparrow U(x))$.

THEOREM 3.0.30. *Let* $\mathbb{K}$ *a set of standard* BL*-algebras containing one with a product component (not necessarily the first component). Then for every sentence* ψ *of the form* $\theta_3 \to \gamma$ *and not containing the symbol* U*, one has:* $\psi \in \mathrm{stTAUT}(\Pi\forall)$ *iff* $\psi^\star \in \mathrm{TAUT}(\mathbb{K})$*. Thus (by Theorem 3.0.28)* $\mathrm{TAUT}(\mathbb{K})$ *is not arithmetical.*

Proof. We start with one lemma.

LEMMA 3.0.31. *Let* $\boldsymbol{A}$ *be a standard* BL*-algebra, let* $\star$ *and* $\to$ *be its monoid operation and its residual respectively, and let* $\mathbf{M}$ *be an* $\boldsymbol{A}$*-structure such that* $\|\psi^\star\|^{\boldsymbol{A}}_{\mathbf{M}} \neq 1$*. Let us write* $\|\dots\|$ *for* $\|\dots\|^{\boldsymbol{A}}_{\mathbf{M}}$*. Then:*

(i) For all $a \in M$*,* $\|U(a)\|$ *and* $\|\phi\|$ *do not belong to the same Wajsberg component.*

(ii) The set $\{\|U(a)\| \mid a \in M\}$ *has no minimum (hence it is infinite).*

(iii) There is an $a \in M$ *such that the set* $\{\|U(b))\| \mid \|U(b))\| \leq \|U(a)\|\}$ *is included in a single Wajsberg component* $\boldsymbol{W}$ *of* $\boldsymbol{A}$*, which is necessarily a cancellative component. Moreover,* $\|\psi^\phi\| \in (W \cup \{\|\phi\|\}) \setminus \{1\}$*.*

Proof. (i) Suppose that $\|U(a)\|$ and $\|\phi\|$ are in the same Wajsberg component. Then since $\|U(b)\| \geq \|\phi\|$ for all $b \in M$, by Lemma 3.0.19 $\|U(a) \uparrow \phi\| = \|U(a)\|$, and $\|(\forall x)(U(x) \uparrow \phi)\| = \inf\{\|U(a)\| \mid a \in M\} = \|\phi\|$. Since $\|\phi\| \leq \|\psi^\phi\|$, one would have $\|\psi^\star\| = 1$, a contradiction.

(ii) If $\{\|U(a)\| \mid a \in M\}$ has a minimum, then this minimum is equal to $\|\phi\|$, which is in contradiction with (i).

(iii) Suppose, by way of contradiction, that for every $a \in M$ there is a $b \in M$ such that $\|U(b)\| < \|U(a)\|$ and $\|U(a)\|, \|U(b)\|$ do not belong to the same Wajsberg component. Then, by Lemma 3.0.19, we have $\|U(b) \uparrow U(a)\| = \|U(a)\|$, and $\|U(a) \uparrow U(b)\| = 1$. Hence $\|\sigma\| = \inf\{\|U(a)\| \mid a \in M\} = \|\phi\|$. It follows that $\|\sigma\| \leq \|\psi^\phi\|$, and $\|\psi^\star\| = 1$, a contradiction. Thus there are $a \in M$ and a Wajsberg component $\boldsymbol{W}$ of $\boldsymbol{A}$ such that for all $b \in M$, if $\|U(b)\| \leq \|U(a)\|$, then $\|U(b)\| \in W$. Now $\boldsymbol{W}$ is either a cancellative hoop or (the reduct of) a Wajsberg algebra. In the latter case $\|\phi\|$, being the infimum of a subset of W (namely, of $\{\|U(b)\| \mid \|U(b)\| \in W\}$) would be in $\boldsymbol{W}$. This contradicts (i). Hence $\boldsymbol{W}$ is a cancellative component. Finally, suppose $\|\psi^\phi\| \notin (W \cup \{\|\phi\|\}) \setminus \{1\}$. Then clearly $\|\psi^\phi\| \neq 1$, otherwise $\|\psi^\star\| = 1$. Thus $\|\psi^\phi\| \notin W \cup \{\|\phi\|\}$. Now $\|\psi^\phi\| \geq \|\phi\|$. Thus if $b \in M$ is such that $\|U(b)\| \in W \setminus \{1\}$, then $\|\psi^\phi\| > \|U(b)\|$, and $\|\psi^\phi\|$ and $\|U(b)\|$ do not belong to the same component. Therefore, $\|\psi^\phi \uparrow U(b)\| = 1$, and $\|(\exists x)(\psi^\phi \uparrow U(x))\| = 1$. It follows that $\|\psi^\star\| = 1$, a contradiction. □

The rest of the proof of Theorem 3.0.30 If $\psi^\star \notin \mathrm{TAUT}(\mathbb{K})$, then by Lemma 3.0.31, there is an $a \in M$ such that the set $\{\|U(b))\| \mid \|U(b))\| \leq \|U(a)\|\}$ is included in a single cancellative component $\boldsymbol{W}$ of $\boldsymbol{A}$. Moreover, $\|\psi^\phi\| \in (W \cup \{\|\phi\|\}) \setminus \{1\}$. Let $m = \sup(W)$, and let $C = (W \setminus \{1\}) \cup \{\|\phi\|, m\}$. Then C is the domain of a product component $\boldsymbol{C}$ of $\boldsymbol{A}$. Let $\boldsymbol{C}^\phi$ be constructed from $\boldsymbol{C}$ as in the proof of Lemma 3.0.20. Then, since $\|\phi\| = \min(\boldsymbol{C})$, $\boldsymbol{C}^\phi = \boldsymbol{C}$. Now let $S = \{a \in A \mid \forall b \in C(b \leq a)\}$, and let $\|\ldots\|_\phi$ be defined as in Lemma 3.0.20, taking into account that $\boldsymbol{C}^\phi = \boldsymbol{C}$. Then by Lemma 3.0.31 (iii), $\|\psi^\phi\| \notin S$, and by Lemma 3.0.20 we obtain $\|\psi\|_\phi \neq 1$. Hence $\psi \notin \mathrm{stTAUT}(\Pi\forall)$.

Conversely, suppose that $\psi \notin \mathrm{stTAUT}(\Pi\forall)$. Then there is a $[0,1]_\Pi$-structure $\mathbf{M}$ such that $\|\psi\|_{\mathbf{M}}^{[0,1]_\Pi} \neq 1$. Then we must have $\|\theta_3\|_{\mathbf{M}}^{[0,1]_\Pi} = 1$, and hence the domain M of $\mathbf{M}$ must be infinite (its domain is a model of Q^+).

Now take an element $\boldsymbol{A} \in \mathbb{K}$ with a product t-norm component $\boldsymbol{C}$. Up to isomorphism, we may assume that $\mathbf{M}$ is a $\boldsymbol{C}$-structure. We define an $\boldsymbol{A}$-structure $\mathbf{M}'$ such that $\|\psi^\star\|_{\mathbf{M}'}^{\boldsymbol{A}} \neq 1$ as follows. The domain M' of $\mathbf{M}'$ is the domain M of $\mathbf{M}$. Moreover, for all $d \in M$, let $U_{\mathbf{M}'}(d)$ be such that $\overline{0}^{\boldsymbol{C}} < U_{\mathbf{M}'}(d)$, $\sup\{U_{\mathbf{M}'}(d)) \mid d \in M\} < \overline{1}^{\boldsymbol{C}}$, and $\inf\{U_{\mathbf{M}'}(d) \mid d \in M\} = \overline{0}^{\boldsymbol{C}}$. Then $\|\phi\|_{\mathbf{M}'}^{\boldsymbol{A}} = \overline{0}^{\boldsymbol{C}}$, and $\boldsymbol{C}^\phi = \boldsymbol{C}$ (cf. the proof of Theorem 3.0.28 for the construction of $\boldsymbol{C}^\phi$ from $\boldsymbol{C}$). Moreover, for every n-ary predicate P different from U, and for every $d_1, \ldots, d_n \in M$, we define $P_{\mathbf{M}'}(d_1, \ldots, d_n) = h(P_{\mathbf{M}}(d_1, \ldots, d_n))$, where h is an isomorphism between $[0,1]_\Pi$ and $\boldsymbol{C}$. Thus if $P_{\mathbf{M}}(d_1, \ldots, d_n) = 1$, then $P_{\mathbf{M}'}(d_1, \ldots, d_n) = \overline{1}^{\boldsymbol{C}} = \sup(C)$. Note that the interpretation $\|\ldots\|_\phi$ constructed from $\|\ldots\| = \|\ldots\|_{\mathbf{M}'}^{\boldsymbol{A}}$ according to Lemma 3.0.20 is just $\|\ldots\|_{\mathbf{M}}^{\boldsymbol{C}}$. Hence, by Lemma 3.0.20, we obtain that $\|\psi^\phi\|_{\mathbf{M}'}^{\boldsymbol{A}} = \|\psi\|_{\mathbf{M}}^{\boldsymbol{C}} \in \boldsymbol{C} \setminus \{\overline{1}^{\boldsymbol{C}}\}$. Moreover, for all $d \in M$, $\|\phi\|_{\mathbf{M}}^{\boldsymbol{A}} < \|U(d)\|_{\mathbf{M}}^{\boldsymbol{A}}$, and $\|\phi\|_{\mathbf{M}}^{\boldsymbol{A}}$ is not in the same Wajsberg component as $\|U(d)\|_{\mathbf{M}}^{\boldsymbol{A}}$. Thus, by Lemma 3.0.19, $\|\sigma\|_{\mathbf{M}}^{\boldsymbol{A}} = 1$. Also, all elements of the form $\|U(d)\|^{\boldsymbol{A}}_{\mathbf{M}}$ with $d \in M$ are in the same Wajsberg component. Hence, by Lemma 3.0.19 again, $\|\theta\|^{\boldsymbol{A}}_{\mathbf{M}} = 1$, and $\|(\sigma \,\&\, \theta) \to \psi^{\phi}\|^{\boldsymbol{A}}_{\mathbf{M}} < 1$. Finally,

$$\begin{aligned}\|(\exists x)(\psi^{\phi} \uparrow U(x))\|^{\boldsymbol{A}}_{\mathbf{M}} &= \sup\{\|\psi^{\phi} \uparrow U(d)\|^{\boldsymbol{A}}_{\mathbf{M}} \mid d \in M\} = \\ &= \sup\{\max\{\|\psi^{\phi}\|^{\boldsymbol{A}}_{\mathbf{M}}, \|U(d)\|^{\boldsymbol{A}}_{\mathbf{M}}\} \mid d \in M\} = \\ &= \max\{\|\psi^{\phi}\|^{\boldsymbol{A}}_{\mathbf{M}}, \sup\{\|U(d)\|^{\boldsymbol{A}}_{\mathbf{M}} \mid d \in M\}\} < 1.\end{aligned}$$

It follows that $\|\psi^{\star}\|^{\boldsymbol{A}}_{\mathbf{M}} < 1$, and $\psi^{\star} \notin \mathrm{TAUT}(\mathbb{K})$. This concludes the proof of Theorem 3.0.30. □

Now we extend this non-arithmeticity result to the case where $\mathbb{K}$ contains a standard algebra with at least one Łukasiewicz t-norm component which is neither the first nor the last component. Clearly, we may assume that there is no algebra in $\mathbb{K}$ with a product component, otherwise we already know that $\mathrm{TAUT}(\mathbb{K})$ is not arithmetical. In order to prove our claim, we will directly reduce the set of sentences which are valid in the standard model $\mathbf{N}$ of natural numbers to $\mathrm{TAUT}(\mathbb{K})$.

Let θ_3, U and $^{\neg\neg}$ be as in the proof of Theorem 3.0.23, let $\phi = (\forall x)U(x)$ and let $^{\phi}$ be as in Theorem 3.0.21. Consider the following formulae:

- $\gamma_1 = (\forall x)\neg\neg U(x)$.
- γ_2 is defined to be the conjunction of all formulae of the form

  $$\forall x_1 \ldots \forall x_n(\neg P(x_1, \ldots, x_n) \vee \neg\neg P(x_1, \ldots, x_n)),$$

  where P is a predicate symbol of PA, including equality if it is not assumed to be crisp.
- $\delta_1 = (\exists x)((U(x) \uparrow \phi) \vee (U(x) \to \phi^2))$.
- $\delta_2 = (\exists x)(U(S(x))^2 \to (\exists y)(y \leq^{\neg\neg} x \,\&\, U(y)))$.

Now let for every formula ψ in the language of PA,

$$\psi^{\star} = (\gamma_1 \,\&\, \gamma_2 \,\&\, \theta_3) \to (\psi^{\neg\neg} \vee \phi \vee \delta_1 \vee \delta_2).$$

THEOREM 3.0.32. *Let $\mathbb{K}$ be a class of standard* BL*-algebras containing an element with a Łukasiewicz component which is neither the first one nor the last one. Then for every formula ψ of the language of arithmetic, one has: $\mathbf{N} \models \psi$ iff $\psi^{\star} \in \mathrm{TAUT}(\mathbb{K})$.*

Proof. ($\Rightarrow$). We argue contrapositively. Suppose that $\psi^{\star} \notin \mathrm{TAUT}(\mathbb{K})$. Let $\boldsymbol{A} \in \mathbb{K}$ and let $\mathbf{M}$ be an $\boldsymbol{A}$-structure such that $\|\psi^{\star}\|^{\boldsymbol{A}}_{\mathbf{M}} \neq 1$. Once again, let us write $\|\ldots\|$ instead of $\|\ldots\|^{\boldsymbol{A}}_{\mathbf{M}}$.

LEMMA 3.0.33.

(i) There is a Wajsberg component $\boldsymbol{W}$ of $\boldsymbol{A}$, isomorphic to $[0,1]_{Ł}$, which is not the first component of $\boldsymbol{A}$, and for all $d \in M$, $\|U(d)\| \in W\backslash\{1\}$. Moreover, $\|\phi\| \in W$.

(ii) For every PA-sentence with parameters in M, $\|\lambda^{\neg\neg}\| \in \{0,1\}$.

(iii) Define a model $\mathbf{M}^{\neg\neg}$ from $\mathbf{M}$ as in the proof of Theorem 3.0.23. Then $\mathbf{M}^{\neg\neg}$ is isomorphic to the standard model of natural numbers.

Proof. (i) For all $d \in M$, $\|U(d)\| \neq 1$, otherwise $\|\psi^\star\| \geq \|\delta_1\| = 1$. Moreover all the elements of the form $\|U(d)\|$ with $d \in M$ belong to the same Wajsberg component $\boldsymbol{W}$, because if $\|U(a)\| < \|U(b)\|$ and $\|U(a)\|$, $\|U(b)\|$ do not belong to the same component, then since for all $a \in M$, $\|\phi\| \leq \|U(a)\|$, by Lemma 3.0.19 we obtain that $\|U(b) \uparrow \phi\| = 1$, and $\|\psi^\star\| \geq \|\delta_1\| = 1$. For the same reason, $\|\phi\| \in W$. Now $\boldsymbol{W}$ cannot be a cancellative component, because by our assumptions $\boldsymbol{A}$ contains no product component. Moreover it cannot be isomorphic to $\boldsymbol{B}_2$, otherwise for all $d \in M$ we would have $\|U(d) \to \phi^2\| = 1$, and $\|\psi^\star\| \geq \|\delta_1\| = 1$. Hence $\boldsymbol{W}$ must be an isomorphic copy of $[0,1]_{\text{Ł}}$. Now if $\boldsymbol{W}$ were the first component, then for all $d \in M$, $\|\neg\neg U(d)\| = \|U(d)\|$, $\|\gamma_1\| = \|\phi\|$, and $\|\psi^\star\| \geq \|\gamma_1 \to (\psi^{\neg\neg} \vee \phi)\| = 1$, which is impossible.

(ii) The proof is by induction on the formula λ. Actually, the induction steps are immediate, so it suffices to prove the claim for atomic formulae $P(d_1, \ldots, d_n)$. If $P^{\neg\neg}(d_1, \ldots, d_n) \notin \{0,1\}$, then $P^{\neg\neg}(d_1, \ldots, d_n)$ is in the first Wajsberg component, otherwise by Lemma 3.0.19 $\|\neg\neg P(d_1, \ldots, d_n)\| = 1$. It would follow that $\|\gamma_2\|$ is in the first Wajsberg component and is different from 1. Since $\|\phi\|$ is the infimum of a Wajsberg component which is not the first component, $\|\gamma_2\| \leq \|\phi\| \leq \|\psi^{\neg\neg} \vee \phi\|$, and $\|\psi^\star\| = 1$, which is impossible.

(iii) First of all, we must have $\|\theta_3\| > 0$, otherwise $\|\psi^\star\| = 1$. Since $\|\theta_3\| \in \{0,1\}$, $\|\theta_3\| = 1$, and hence $\mathbf{M}^{\neg\neg}$ is a model of Q^+ (recall that if λ is a sentence of *PA*, then $\mathbf{M}^{\neg\neg} \models \lambda$ iff $\|\lambda^{\neg\neg}\| = 1$). Now $\|\delta_2\| < 1$, otherwise $\|\psi^\star\| = 1$. Hence, for all $d \in M$ we must have $\|U(S(d))\|^2 > \sup\{\|U(b)\| \mid \mathbf{M}^{\neg\neg} \models b \leq d\}$. Now let for $n \in \mathsf{N}$, d_n be the realization of n in $\mathbf{M}^{\neg\neg}$. Then $\|U(d_1)\|^2 > \|U(d_0)\|$, $\|U(d_2)\|^2 > \|U(d_1)\|$, etc. Continuing, since by (i), for every n, $\|U(d_n)\| \in W \setminus \{1\}$ and since $\boldsymbol{W}$ is isomorphic to $[0,1]_{\text{Ł}}$, we easily obtain that $\sup\{\|U(d_n)\| \mid n \in \mathsf{N}\} = \sup(W \setminus \{1\})$. Now suppose that d is a non-standard element in $\mathbf{M}^{\neg\neg}$. Then we should have that $\|U(S(d))\|^2 > \sup(W \setminus \{1\})$, which is impossible, because by (i) we have $\|U(S(d))\| \in \boldsymbol{W} \setminus \{1\}$. □

The rest of the proof of Theorem 3.0.32 At this point, the proof of ($\Rightarrow$) is immediate: if $\|\psi^\star\| \neq 1$, then $\|\psi^{\neg\neg}\| = 0$, and $\mathbf{M}^{\neg\neg} \not\models \psi$. Since $\mathbf{M}^{\neg\neg}$ is isomorphic to $\mathbf{N}$, we conclude that $\mathbf{N} \not\models \psi$.

($\Leftarrow$) Suppose that $\mathbf{N} \not\models \psi$. Take $\boldsymbol{A} \in \mathbb{K}$ with a Łukasiewicz t-norm component $\boldsymbol{C}$ which is neither the first nor the last component. Clearly $\boldsymbol{C}$ is isomorphic to the Łukasiewicz t-norm. Let for every $q \in \mathsf{Q} \cap [0,1]$, $q^{\boldsymbol{C}}$ denote the isomorphic copy of q in $\boldsymbol{C}$. Define an $\boldsymbol{A}$-structure $\mathbf{M}$ as follows: the domain, M, of $\mathbf{M}$ is N, the function symbols and the constants of *PA* are interpreted as in $\mathbf{N}$, $=$ is interpreted as crisp equality (that is, $n =_{\mathbf{M}} m = 1$ if $n = m$ and $n =_{\mathbf{M}} m = 0$ otherwise, and $\leq$ is interpreted similarly as the usual order in $\mathbf{N}$. Then clearly $\|\psi^{\neg\neg}\|^{\boldsymbol{A}}_{\mathbf{M}} = 0$.

Now define $U_{\mathbf{M}}(n)$ recursively by $U_{\mathbf{M}}(0) = (\frac{1}{2})^{\boldsymbol{C}}$, and $U_{\mathbf{M}}(n+1)) = \left(\frac{U_{\mathbf{M}}(n)+3}{4}\right)^{\boldsymbol{C}}$. For all $n \in \mathsf{N}$ we have $(\frac{1}{2})^{\boldsymbol{C}} \leq \|U(n)\| < \|U(n+1)\| < \overline{1}^{\boldsymbol{C}}$. Thus $\|U(n)\| \in W \setminus \{1\}$. Moreover $\|U(n+1)^2\| = \frac{\|U(n)\|+1}{2} > \|U(n)\|$ and therefore $\|U(S(n))^2 \to (\exists x)(x \leq^{\neg\neg} n \,\&\, U(x))\| < \overline{1}^{\boldsymbol{C}}$. It follows that $\|\delta_2\| = \overline{1}^{\boldsymbol{C}} < 1$, because $\boldsymbol{C}$ is not the last component. Also, it is easily seen that $\|\gamma_1 \,\&\, \gamma_2\| = 1$ and that $\|(\psi^{\neg\neg} \vee \phi)\| = (\frac{1}{2})^{\boldsymbol{C}}$. Finally, for every $n \in \mathsf{N}$, $\|U(n) \to \phi^2\| \leq (\frac{1}{2})^{\boldsymbol{C}}$, and $\|U(n) \uparrow \phi\| = \|U(n)\|$. Thus $\|\delta_1\| = \sup\{\|U(n)\| \mid n \in \mathsf{N}\} = \overline{1}^{\boldsymbol{C}} < 1$. Hence, $\|\psi^\star\| = \overline{1}^{\boldsymbol{C}} < 1$. □

It follows:

THEOREM 3.0.34. *Let* $\mathbb{K}$ *be a class of standard* BL*-algebras containing an element* not *isomorphic to any of* $[0,1]_{\mathrm{G}}$, $[0,1]_{\text{Ł}}$, $[0,1]_{\text{Ł}} \oplus [0,1]_{\mathrm{G}}$, $[0,1]_{\mathrm{G}} \oplus [0,1]_{\text{Ł}}$, $[0,1]_{\text{Ł}} \oplus [0,1]_{\text{Ł}}$ *and* $[0,1]_{\text{Ł}} \oplus [0,1]_{\mathrm{G}} \oplus [0,1]_{\text{Ł}}$. *Then* $\mathrm{TAUT}(\mathbb{K})$ *is not arithmetical.*

Proof. If a standard BL-algebra is not among the ones shown above, then either it contains a product t-norm component or it contains a Łukasiewicz t-norm component which is neither the first component nor the last component. The claim follows from Theorems 3.0.30 and 3.0.32. □

In the final part of this section we consider the arithmetical complexity of the standard semantics of logics of left-continuous t-norms in proper sense, i.e. logics extending $\mathrm{MTL}\forall$ but not $\mathrm{BL}\forall$, such as $\mathrm{MTL}\forall$ itself, $\mathrm{IMTL}\forall$ or $\mathrm{SMTL}\forall$. Since these logics are typically introduced as the logics of certain semantics of t-norms all their real chains are actually intended models and hence standard chains.

THEOREM 3.0.35. *Let* $\mathrm{L}\forall$ *be a logic enjoying the* $\mathrm{FS}\mathbb{K}\mathrm{C}$ *for* $\mathbb{K}$ *being the class of all real* L*-chains. Then* $\mathrm{realTAUT}(\mathrm{L}\forall)$ *and* $\mathrm{realTAUT}_{\mathrm{pos}}(\mathrm{L}\forall)$ *are* $\mathbf{\Sigma_1}$*-complete, while* $\mathrm{realSAT}(\mathrm{L}\forall)$ *and* $\mathrm{realSAT}_{\mathrm{pos}}(\mathrm{L}\forall)$ *are* $\mathbf{\Pi_1}$*-complete.*

Proof. It is a consequence of Propositions 2.0.8 and 2.0.21, and Theorems 2.0.13 and 2.0.20. □

The prominent logics of left-continuous t-norms collected in Table 2 fall under the scope of this theorem and, thus, the complexities thereof are justified.

4 Complexity of finite and rational-chain semantics

Let $\boldsymbol{A}$ be any finite chain and let $\overline{0} = a_1 < \cdots < a_n = \overline{1}$ be the elements of A in increasing order. $\mathrm{L}_{\boldsymbol{A}}$, the first-order many-valued logic based on $\boldsymbol{A}$, is defined semantically as follows: $\mathrm{L}_{\boldsymbol{A}}$ has a language $\mathcal{P}_{\boldsymbol{A}}$ containing, besides parentheses, variables, predicate symbols, function symbols, a k-ary connective F for each k-ary operation $F^{\boldsymbol{A}}$ on $\boldsymbol{A}$ (different symbols for different operations), plus the quantifiers $\exists$ and $\forall$. For each connective F introduced in this way, we refer to $F^{\boldsymbol{A}}$ as *the realization of* F *in* $\boldsymbol{A}$. Since $\boldsymbol{A}$ is finite, each $\boldsymbol{A}$-structure $\mathbf{M} = \langle M, \langle P_{\mathbf{M}} \rangle_{P \in \mathcal{P}_{\boldsymbol{A}}}, \langle f_{\mathbf{M}} \rangle_{f \in \mathcal{P}_{\boldsymbol{A}}} \rangle$ is safe, because universal quantifiers are interpreted by taking the minimum value of instances, and existential quantifiers by taking the maximum value of instances.

For every set $T \cup \{\phi\}$ of sentences of $\mathrm{L}_{\boldsymbol{A}}$, the consequence relation $\models_{\mathrm{L}_{\boldsymbol{A}}}$ in $\mathrm{L}_{\boldsymbol{A}}$ is defined as follows: $T \models_{\mathrm{L}_{\boldsymbol{A}}} \phi$ iff for every $\boldsymbol{A}$-structure $\mathbf{M}$, one has that if $\langle \boldsymbol{A}, \mathbf{M} \rangle \models \psi$ for all $\psi \in T$, then $\langle \boldsymbol{A}, \mathbf{M} \rangle \models \phi$. If $\emptyset \models_{\mathrm{L}_{\boldsymbol{A}}} \phi$, then we say that ϕ is an $\boldsymbol{A}$*-tautology* and we write $\models_{\mathrm{L}_{\boldsymbol{A}}} \phi$.

THEOREM 4.0.1. *For every finite chain* $\boldsymbol{A}$*, the set of* $\boldsymbol{A}$*-tautologies is* $\mathbf{\Sigma_1}$.

Proof. We will associate to $\boldsymbol{A}$ a recursively axiomatized classical first-order theory $T_{\boldsymbol{A}}$ and to every sentence ϕ of $\mathrm{L}_{\boldsymbol{A}}$ a formula ϕ^T in the language of $T_{\boldsymbol{A}}$ such that the map $\phi \mapsto \phi^T$ is computable and ϕ is an $\boldsymbol{A}$-tautology iff ϕ^T is a theorem of $T_{\boldsymbol{A}}$. This will clearly suffice to prove the theorem. First of all, the theory $T_{\boldsymbol{A}}$ has all function symbols in $\mathrm{L}_{\boldsymbol{A}}$. Moreover $T_{\boldsymbol{A}}$ has a constant symbol c^T for each element c of A, plus an additional constant u (for undefined) and an additional k-ary functional symbol f_ϕ for each formula ϕ with k free variables (the intended meaning is that $f_\phi(d_1,\dots,d_k) = \|\phi(d_1,\dots,d_k)\|^{\boldsymbol{A}}_{\mathbf{M}}$; in particular, if ϕ is a sentence, then f_ϕ is a constant). $T_{\boldsymbol{A}}$ has two binary predicate symbols $=$ and $<$. The intended meaning of $x = y$ is that x is equal to y, and the intended meaning of $x < y$ is that $x, y \in A$ and x is less than y in the order of $\boldsymbol{A}$. Finally $T_{\boldsymbol{A}}$ has two unary predicate symbols M and A. The intended meanings of $M(v)$ and of $A(v)$ are: v is in the domain M of individuals of the first-order structure we are referring to, and v is an element of the algebra $\boldsymbol{A}$, respectively.

It is a little bit boring to write all the axioms of $T_{\boldsymbol{A}}$, therefore we only describe them informally and we leave the obvious formal translation to the reader.

(0) Identity axioms for $=$.

(1) A group of axioms which say that the domain M of individuals is disjoint from A and u is neither in M nor in A.

(2) An axiom saying that every element is either in A or in M or u.

(3) Axioms describing the structure of $\boldsymbol{A}$, that is:
- (3a) $c_i^T < c_j^T$ for each $1 \le i < j \le n$, and $\neg(c_i^T < c_j^T)$ for each $j \le i$;
- (3b) axioms of the form $\neg(c_i^T = c_j^T)$ for each $i \ne j$;
- (3c) axioms of the form $F(e_1^T,\dots,e_k^T) = e^T$ for each k-ary connective F and for all $e_1,\dots,e_k,e \in A$ such that $F^{\boldsymbol{A}}(e_1,\dots,e_k) = e$;
- (3d) $(\forall v)(A(v) \leftrightarrow (v = c_1^T \vee \dots \vee v = c_n^T)$ saying that $A = \{c_1,\dots,c_n\}$;
- (3e) axioms saying that for every connective F corresponding to an operation $F^{\boldsymbol{A}}$, $F(x_1,\dots,x_k)$ is undefined (i.e. it is equal to u) if some of the x_i is not in A;
- (3f) an axiom saying that if $x < y$ then $x, y \in A$.

(4) Axioms describing the structure $\mathbf{M}$, that is for every:
- (4a) constant symbol d of $\mathrm{L}_{\boldsymbol{A}}$, an axiom saying that $d \in M$;
- (4b) k-ary function symbol g of $\mathrm{L}_{\boldsymbol{A}}$, an axiom saying that for all $x_1,\dots,x_k$, $g(x_1,\dots,x_k) \in M$ if $x_1,\dots,x_k \in M$ and $g(x_1,\dots,x_k) = u$ otherwise.

(5) Axioms describing the behavior of $\|\phi(v_1,\dots,v_k)\|^{\boldsymbol{A}}_{\mathbf{M}}$, that is:
- (5a) if $v_1,\dots,v_k$ are all in M, then $f_\phi(v_1,\dots,v_k)$ is in M, otherwise we define $f_\phi(v_1,\dots,v_k) = u$;
- (5b) for every k-ary connective F of $\mathrm{L}_{\boldsymbol{A}}$ we define $f_{F(\phi_1,\dots,\phi_k)}(v_1,\dots,v_l) = F(f_{\phi_1}(v_1,\dots,v_l),\dots,f_{\phi_k}(v_1,\dots,v_l))$ (thus $f_{F(\phi_1,\dots,\phi_k)}(v_1,\dots,v_l) = u$ if for some i, $f_{\phi_i}(v_1,\dots,v_l) = u$, otherwise $f_{F(\phi_1,\dots,\phi_k)}(v_1,\dots,v_l) \in A$);
- (5c) an axiom saying that for $j = 1,\dots,n$, $f_{(\forall v)\phi}(v_1,\dots,v_k) = c_j$ iff (i) $v_1,\dots,v_k \in M$, (ii) for all $v \in M$, $f_\phi(v,v_1,\dots,v_k) \ge c_j$ and (iii) for some $v \in M$, $f_\phi(v,v_1,\dots,v_k) = c_j$;
- (5d) an axiom saying that for $j = 1,\dots,n$, $f_{(\exists v)\phi}(v_1,\dots,v_k) = c_j$ iff (i) $v_1,\dots,v_k \in M$, (ii) for all $v \in M$, $f_\phi(v,v_1,\dots,v_k) \le c_j$ and (iii) for some $v \in M$, $f_\phi(v,v_1,\dots,v_k) = c_j$.

LEMMA 4.0.2.

(a) Let $\mathbf{M}$ *be an* $\boldsymbol{A}$*-structure for* $\mathrm{L}_{\boldsymbol{A}}$*. Then there is a model* $\mathbf{M}^*$ *of* $T_{\boldsymbol{A}}$ *(in the sense of classical logic) such that for every sentence* ϕ *of* $\mathrm{L}_{\boldsymbol{A}}$ *and for every* $c_i \in A$ *one has:* $\mathbf{M}^* \models f_\phi = c_i^T$ *iff* $\|\phi\|_{\mathbf{M}}^{\boldsymbol{A}} = c_i$*.*

(b) Let $\mathbf{H}$ *be a model of* $T_{\boldsymbol{A}}$ *(again, in the sense of classical logic). Then there is an* $\boldsymbol{A}$*-structure* $\mathbf{H}^+$ *for* $\mathrm{L}_{\boldsymbol{A}}$ *such that for every sentence* ϕ *of* $\mathrm{L}_{\boldsymbol{A}}$ *and for every* $c_i \in A$ *one has:* $\mathbf{H} \models f_\phi = c_i^T$ *iff* $\|\phi\|_{\mathbf{H}^+}^{\boldsymbol{A}} = c_i$*.*

Proof. (a) Given $\mathbf{M}$, we can assume without loss of generality that $M \cap A = \emptyset$. Let $u^* \notin M \cup A$, and consider the model $\mathbf{M}^*$ whose universe is $M^* = M \cup A \cup \{u^*\}$ and whose constants, operations and predicates are as follows:

(i) If c_i^T is a constant for an element of A, then $(c_i^T)^{\mathbf{M}^*} = c_i$; if c is a constant of $\mathrm{L}_{\boldsymbol{A}}$, then $c^{\mathbf{M}^*} = c^{\mathbf{M}}$; if c is a constant of the form f_ϕ, ϕ a sentence of $\mathrm{L}_{\boldsymbol{A}}$, then $c^{\mathbf{M}^*} = \|\phi\|_{\mathbf{M}}^{\boldsymbol{A}}$. Finally, u is interpreted as u^*.

(ii) If f is a k-ary function symbol in $\mathrm{L}_{\boldsymbol{A}}$, then $f^{\mathbf{M}^*}$ is defined by $f^{\mathbf{M}^*}(d_1,\ldots,d_k) = f^{\mathbf{M}}(d_1,\ldots,d_k)$ if $d_{1,\ldots,}d_k \in M$, and $f^{\mathbf{M}^*}(d_1,\ldots,d_k) = u^*$ otherwise; if F is a k-ary connective of $\mathrm{L}_{\boldsymbol{A}}$, then $F^{\mathbf{M}^*}(d_1,\ldots,d_k) = F^{\boldsymbol{A}}(d_1,\ldots,d_k)$ if $d_1,\ldots,d_k \in A$, and $F^{\mathbf{M}^*}(d_1,\ldots,d_k) = u^*$ otherwise; if $\phi(v_1,\ldots,v_k)$ is a formula of $\mathrm{L}_{\boldsymbol{A}}$ with free variables $v_1,\ldots,v_k$, then $f_\phi^{\mathbf{M}^*}(d_1,\ldots,d_k) = \|\phi(d_1,\ldots,d_k)\|_{\mathbf{M}}^{\boldsymbol{A}}$ if $d_1,\ldots,d_k \in M$, and $f_\phi^{\mathbf{M}^*}(d_1,\ldots,d_k) = u^*$ otherwise.

(iii) $\mathbf{M}^* \models d = e$ iff d is equal to e; $\mathbf{M}^* \models d < e$ iff $d, e \in A$ and $d < e$ in the order of $\boldsymbol{A}$; $\mathbf{M}^* \models M(d)$ iff $d \in M$ and $\mathbf{M}^* \models A(d)$ iff $d \in A$.

It is clear that for every formula $\phi(v_1,\ldots,v_k)$, for every $c_i \in A$ and for every $d_1,\ldots,d_k \in M$: $\mathbf{M}^* \models f_\phi(d_1,\ldots,d_k) = c_i^T$ iff $\|\phi(d_1,\ldots,d_k)\|_{\mathbf{M}}^{\boldsymbol{A}} = c_i$, and (a) follows.

(b) Let $\mathbf{H}$ be a model of $T_{\boldsymbol{A}}$; we define an algebra $\boldsymbol{A}^+$ and an $\boldsymbol{A}^+$-structure $\mathbf{H}^+$ based on $\boldsymbol{A}^+$ as follows:

(i) The domain A^+ of $\boldsymbol{A}^+$ is the set $\{d \in H \mid \mathbf{H} \models A(d)\}$ and the operations of $\boldsymbol{A}^+$ are the restrictions to A^+ of the operations $F^{\mathbf{H}}$ of $\mathbf{H}$ such that F is a connective of $\mathrm{L}_{\boldsymbol{A}}$. Trivially, $\boldsymbol{A}^+$ is isomorphic to $\boldsymbol{A}$ (here we use in a crucial way the fact that $\boldsymbol{A}$ is finite).

(ii) $H^+ = \{d \in H \mid \mathbf{H} \models M(d)\}$; for every constant c of $\mathrm{L}_{\boldsymbol{A}}$, $c^{\mathbf{H}^+} = c^{\mathbf{H}}$; for every k-ary function symbol g of $\mathrm{L}_{\boldsymbol{A}}$, $g^{\mathbf{H}^+}$ is the function from $(H^+)^k$ into H^+ defined for all $d_1,\ldots,d_k \in H^+$, by $g^{\mathbf{H}^+}(d_1,\ldots,d_k) = g^{\mathbf{H}}(d_1,\ldots,d_k)$ (i.e. $g^{\mathbf{H}^+}$ is the restriction of $g^{\mathbf{H}}$ to $(H^+)^k$).

(iii) For every k-ary predicate P and every $d_1,\ldots,d_k \in H^+$, $\|P(d_1,\ldots,d_k)\|_{\mathbf{H}^+}^{\boldsymbol{A}} = f_P^{\mathbf{H}}(d_1,\ldots,d_k)$.

Then $\|.\|_{\mathbf{H}^+}^{\boldsymbol{A}}$ uniquely extends to all formulae in such a way that for every formula $\phi(v_1,\ldots,v_k)$, for every $c_i \in A$ and for every $d_1,\ldots,d_k \in M$: $\mathbf{H} \models f_\phi(d_1,\ldots,d_k) = c_i^T$ iff $\|\phi(d_1,\ldots,d_k)\|_{\mathbf{H}^+}^{\boldsymbol{A}} = c_i$, and (b) follows. □

The rest of the proof of Theorem 4.0.1 It suffices to associate to every sentence ϕ of $\mathrm{L}_{\boldsymbol{A}}$ the formula $f_\phi = \overline{1}^T$ (remind that $\overline{1}$ is the top element of $\boldsymbol{A}$). Then by Lemma 4.0.2 we have that the following are equivalent:

(i) There is an $\boldsymbol{A}$-structure $\mathbf{M}$ such that $\|\phi\|^{\boldsymbol{A}}_{\mathbf{M}} \neq \overline{1}$.

(ii) There is a model $\mathbf{H}$ of $T_{\boldsymbol{A}}$ such that $f_\phi = \overline{1}^T$ is not valid in $\mathbf{H}$.

Thus we conclude that ϕ is an $\boldsymbol{A}$-tautology iff $T_{\boldsymbol{A}} \vdash f_\phi = \overline{1}^T$, and the set of $\boldsymbol{A}$-tautologies is Σ_1. □

We have seen that $\mathrm{TAUT}(\boldsymbol{A})$ is $\boldsymbol{\Sigma_1}$. Similar arguments show that for every sentence ϕ of $\mathrm{L}_{\boldsymbol{A}}$ we have:

- $\phi \in \mathrm{TAUT}_{\mathrm{pos}}(\boldsymbol{A})$ iff $T_{\boldsymbol{A}} \vdash \overline{0}^T < f_\phi$,
- $\phi \in \mathrm{SAT}(\boldsymbol{A})$ iff $T_{\boldsymbol{A}}$ plus $f_\phi = \overline{1}^T$ is consistent,
- $\phi \in \mathrm{SAT}_{\mathrm{pos}}(\boldsymbol{A})$ iff $T_{\boldsymbol{A}}$ plus $f_\phi > \overline{0}^T$ is consistent.

THEOREM 4.0.3. *Let $\boldsymbol{A}$ be a finite chain.* $\mathrm{TAUT}(\boldsymbol{A})$ *and* $\mathrm{TAUT}_{\mathrm{pos}}(\boldsymbol{A})$ *are in* $\boldsymbol{\Sigma_1}$. *Moreover,* $\mathrm{SAT}(\boldsymbol{A})$ *and* $\mathrm{SAT}_{\mathrm{pos}}(\boldsymbol{A})$ *are in* $\boldsymbol{\Pi_1}$.

Observe that the proof of this theorem would be completely analogous if instead of a linearly ordered algebra $\boldsymbol{A}$ would be an arbitrary finite algebra (in a finite language), as this was the essential requirement to build the classical first-order theory $T_{\boldsymbol{A}}$.

By the general hardness results from Section 2 we obtain:

COROLLARY 4.0.4. *For every finite chain $\boldsymbol{A}$,*

1. $\mathrm{TAUT}(\boldsymbol{A})$ *and* $\mathrm{TAUT}_{\mathrm{pos}}(\boldsymbol{A})$ *are* $\boldsymbol{\Sigma_1}$*-complete,*
2. $\mathrm{SAT}(\boldsymbol{A})$ *and* $\mathrm{SAT}_{\mathrm{pos}}(\boldsymbol{A})$ *are* $\boldsymbol{\Pi_1}$*-complete.*

From these results, we can obtain some upper bounds for the arithmetical complexities with respect to the finite-chain semantics, when the class of finite chains is recursively enumerable:

THEOREM 4.0.5. *Suppose that* L *is a* ($\triangle$-)*core fuzzy logic such that there is a computable enumeration of all (up to isomorphism) finite* L*-chains. Then:*

(a) $\mathrm{finTAUT}(\mathrm{L}\forall)$ *and* $\mathrm{finTAUT}_{\mathrm{pos}}(\mathrm{L}\forall)$ *are in* $\boldsymbol{\Pi_2}$.

(b) $\mathrm{finSAT}(\mathrm{L}\forall)$ *and* $\mathrm{finSAT}_{\mathrm{pos}}(\mathrm{L}\forall)$ *are in* $\boldsymbol{\Sigma_2}$.

Proof. Let $\boldsymbol{A}_1, \boldsymbol{A}_2, \ldots, \boldsymbol{A}_n, \ldots$ be a computable enumeration of all finite L-chains. Then $\phi \in \mathrm{finTAUT}(\mathrm{L}\forall)$ iff $(\forall n)(\phi \in \mathrm{TAUT}(\boldsymbol{A}_n))$ and $\phi \in \mathrm{finTAUT}_{\mathrm{pos}}(\mathrm{L}\forall)$ iff $(\forall n)(\phi \in \mathrm{TAUT}_{\mathrm{pos}}(\boldsymbol{A}_n))$. Since the sequence $\langle \boldsymbol{A}_n \mid n \in \mathsf{N}\rangle$ is computable, by Theorem 4.0.3, $\{\langle \phi, n\rangle \mid \phi \in \mathrm{TAUT}(\boldsymbol{A}_n)\}$ and $\{\langle \phi, n\rangle \mid \phi \in \mathrm{TAUT}_{\mathrm{pos}}(\boldsymbol{A}_n)\}$ are in Σ_1, and claim (a) follows.

Regarding claim (b), we have that $\phi \in \mathrm{finSAT}_{\mathrm{pos}}(\mathrm{L}\forall)$ iff $(\exists n)(\phi \in \mathrm{SAT}_{\mathrm{pos}}(\boldsymbol{A}_n))$, and $\phi \in \mathrm{finSAT}(\mathrm{L}\forall)$ iff $(\exists n)(\phi \in \mathrm{SAT}(\boldsymbol{A}_n))$, and the claim follows from the computability of the sequence $\langle \boldsymbol{A}_n \mid n \in \mathsf{N}\rangle$ and from Theorem 4.0.3 (note that if $R(n, x)$ is Π_1, then $(\exists n)R(n, x)$ is in turn Σ_2). □

Problem	Complexity
$\mathrm{finTAUT}(\mathrm{L}\forall)$	$\mathbf{\Sigma_1}$-hard, $\mathbf{\Pi_2}$
$\mathrm{finSAT}(\mathrm{L}\forall)$	$\mathbf{\Pi_1}$-hard, $\mathbf{\Sigma_2}$
$\mathrm{finTAUT}_{\mathrm{pos}}(\mathrm{L}\forall)$	$\mathbf{\Sigma_1}$-hard, $\mathbf{\Pi_2}$
$\mathrm{finSAT}_{\mathrm{pos}}(\mathrm{L}\forall)$	$\mathbf{\Pi_1}$-hard, $\mathbf{\Sigma_2}$

Table 3. Arithmetical complexity bounds for the finite-chain semantics when L is recursively axiomatizable.

THEOREM 4.0.6. *If* L *is a finitely axiomatizable (*$\triangle$*-)core fuzzy logic, then there is a computable enumeration of all (up to isomorphism) finite* L*-chains.*

Proof. We can obtain a computable enumeration of all finite L-chains as follows: clearly there is a computable enumeration of all the finite algebras of the signature of L (first put the trivial algebra in the list, then enumerate all the (finitely many) structures with two elements, 0 and 1, then all the (finitely many) structures with three elements, $0, 1$ and 2, etc.). Let $\boldsymbol{C}_1, \boldsymbol{C}_2, \ldots, \boldsymbol{C}_n, \ldots$ be the computable list of structures obtained in this way, and assume without loss of generality that $\boldsymbol{C}_1$ is the trivial algebra. Now let $\boldsymbol{A}_1 = \boldsymbol{C}_1$ (note that the trivial algebra is a totally ordered algebraic model of any logic with that signature); then for every n, check whether $\boldsymbol{C}_n$ is a chain and whether it satisfies the finite axiomatization of L. This can be done with a finite computation. If so, let $\boldsymbol{A}_n = \boldsymbol{C}_n$; otherwise, let $\boldsymbol{A}_n = \boldsymbol{C}_1$. □

From the last two theorems, together with the general results in Section 2, we can obtain uniform bounds for the complexity of finite-chain semantics in recursively axiomatizable ($\triangle$-)core fuzzy logics; see the results in Table 3. It applies, in particular, to all the prominent fuzzy logics, for instance if L is Ł, G, Π, BL, SBL, MTL, SMTL, IMTL, ΠMTL, WCMTL, C$_n$MTL, C$_n$IMTL, WNM, or NM, $\mathrm{finTAUT}(\mathrm{L}\forall)$ is $\mathbf{\Pi_2}$, etc. Note that the sets $\mathrm{finTAUT}(\mathrm{L}\forall)$ for these logics may have repetitions, e.g. if L has only finitely many totally ordered algebraic models (this is the case for $\mathrm{L} = \Pi$ or for $\mathrm{L} = \Pi\mathrm{MTL}$). In this case, $\mathrm{finTAUT}(\mathrm{L}\forall)$ is not only $\mathbf{\Pi_2}$, but even $\mathbf{\Sigma_1}$: for instance $\mathrm{finTAUT}(\Pi\forall)$ and $\mathrm{finTAUT}(\Pi\mathrm{MTL}\forall)$ coincide with the set of classical first-order tautologies, which is $\mathbf{\Sigma_1}$-complete. Next will show that in some cases the upper bounds are reached as well.

THEOREM 4.0.7. *Let* L *be a recursively axiomatizable (*$\triangle$*-)core fuzzy logic such that the following conditions hold:*

(1) For every finite cardinal m*, there is a finite* L*-chain with at least* m *elements.*

(2) There is an L*-formula* $\phi(p)$ *such that for every* L*-chain* $\boldsymbol{A}$ *and for every* $\boldsymbol{A}$*-evaluation* v*,* $v(\phi(p)) \in \{\overline{0}^{\boldsymbol{A}}, \overline{1}^{\boldsymbol{A}}\}$*, and there are evaluations* v_0 *and* v_1 *such that* $v_0(\phi(p)) = \overline{0}^{\boldsymbol{A}}$ *and* $v_1(\phi(p)) = \overline{1}^{\boldsymbol{A}}$*.*

Then the set $\mathrm{finTAUT}(\mathrm{L}\forall)$ *is* $\mathbf{\Pi_2}$*-complete.*

Proof. Let Φ denote the conjunction of all axioms of Q^+ and let for every formula γ, γ^+ be the result of replacing in γ every atomic formula δ by $\phi(\delta)$. Notice that for every model $\langle \boldsymbol{A}, \mathbf{M}\rangle$ the value $\|\gamma^+\|^{\boldsymbol{A}}_{\mathbf{M}}$ is crisp.

Let $\mathbf{M}$ be a first-order safe structure over an L-chain $\boldsymbol{A}$ such that $\|\Phi^+\|^{\boldsymbol{A}}_{\mathbf{M}} = \overline{1}^{\boldsymbol{A}}$. We define a classical model $\mathbf{M}^{Q^+}$ for the language of Q^+ as follows. The domain of $\mathbf{M}^{Q^+}$ is the domain M of $\mathbf{M}$ modulo the equivalence $\sim$ defined as: $d \sim d'$ if, and only if, $\|\phi(d = d')\|^{\boldsymbol{A}}_{\mathbf{M}} = 1$. For every n-ary function symbol f of Q^+ and for every $d_1, \ldots, d_n \in M$, $f^{\mathbf{M}^{Q^+}}([d_1], \ldots, [d_n]) = [f^{\mathbf{M}}(d_1, \ldots, d_n)]$, where for each $d \in M$, $[d]$ denotes its equivalence class modulo $\sim$. Finally, for every n-ary predicate symbol P of Q^+ and for every $d_1, \ldots, d_n \in M$, we stipulate that $\langle [d_1], \ldots, [d_n]\rangle \in P^{\mathbf{M}^{Q^+}}$ if, and only if, $\|\phi(P(d_1, \ldots, d_n))\|^{\boldsymbol{A}}_{\mathbf{M}} = \overline{1}^{\boldsymbol{A}}$. By induction on δ we can easily prove:

Claim 1: For each formula $\delta(x_1, \ldots, x_n)$ and any elements $d_1, \ldots, d_n \in M$ we have: $\|\delta(d_1, \ldots, d_n)^+\|^{\boldsymbol{A}}_{\mathbf{M}} = \overline{1}^{\boldsymbol{A}}$ iff $\mathbf{M}^{Q^+} \models \delta([d_1], \ldots, [d_n])$.

Conversely, given a model $\mathbf{H}$ of Q^+ and an L-chain $\boldsymbol{A}$ we define a first-order structure $\mathbf{H}^{\boldsymbol{A}}$ on $\boldsymbol{A}$ (restricted to the language of Q^+) as follows: the domain of $\mathbf{H}^{\boldsymbol{A}}$ coincides with the domain H of $\mathbf{H}$ and the function symbols and the constants are interpreted as in $\mathbf{H}$; moreover, let z, o be elements of A such that for every evaluation v, we have $v(\phi(p)) = \overline{0}^{\boldsymbol{A}}$ if $v(p) = z$ and $v(\phi(p)) = \overline{1}^{\boldsymbol{A}}$ if $v(p) = o$. Then for every n-ary predicate symbol P and for every $d_1, \ldots, d_n \in H$, we define $\|\delta(d_1, \ldots, d_n)\|^{\boldsymbol{A}}_{\mathbf{H}^{\boldsymbol{A}}} = o$ if $\mathbf{H} \models \delta(d_1, \ldots, d_n)$ and $\|\delta(d_1, \ldots, d_n)\|^{\boldsymbol{A}}_{\mathbf{H}^{\boldsymbol{A}}} = z$ otherwise. Then, again by induction on δ, we can easily prove:

Claim 2: For every formula $\delta(x_1, \ldots, x_n)$ of Q^+ and for every $d_1, \ldots, d_n \in H$ one has: $\|\delta(d_1, \ldots, d_n)^+\|^{\boldsymbol{A}}_{\mathbf{H}^{\boldsymbol{A}}} = \overline{1}^{\boldsymbol{A}}$ iff $\mathbf{H} \models \delta(d_1, \ldots, d_n)$.

Now let $X = \{n \mid (\forall m)(\exists k)R(n, m, k)\}$, with R recursive, be a $\mathbf{\Pi_2}$-complete set. Let $R'(x, y, z)$ be a formula of Q^+ representing R in Q^+, that is, for all n, m, k, if $R(n, m, k)$ is true, then $R'(\overline{n}, \overline{m}, \overline{k})$ is provable in Q^+ and if $R(n, m, k)$ is false, then $\neg R'(\overline{n}, \overline{m}, \overline{k})$ is provable in Q^+. Let R^+ be the formula obtained from R' by replacing every atomic subformula ψ by $\phi(\psi)$. Then R^+ behaves as a crisp formula. Finally, let P be a new unary predicate, and let $\Psi(x)$ be the formula

$$\Phi^+ \to (\forall y)((\exists u)((u \leq y)^+ \wedge (P(S(u)) \to P(u))) \vee (\exists z)R^+(x, y, z)).$$

We claim that for every n, $n \in X$ iff $\Psi(\overline{n})$ is true in every first-order model over a finite L-chain. Indeed, suppose $n \in X$. Let $\boldsymbol{A}$ be an L-chain with m elements, and let $\mathbf{M}$ be a first-order model over $\boldsymbol{A}$. If $\|\Phi^+\|^{\boldsymbol{A}}_{\mathbf{M}} = \overline{0}^{\boldsymbol{A}}$, then $\|\Psi(\overline{n})\|^{\boldsymbol{A}}_{\mathbf{M}} = \overline{1}^{\boldsymbol{A}}$. Otherwise, $\|\Phi^+\|^{\boldsymbol{A}}_{\mathbf{M}} = \overline{1}^{\boldsymbol{A}}$, that is, the translation of every axiom of Q^+ is true in $\langle \boldsymbol{A}, \mathbf{M}\rangle$.

Claim 3: If $\|\Phi^+\|^{\boldsymbol{A}}_{\mathbf{M}} = \overline{1}^{\boldsymbol{A}}$, then for every theorem ψ of Q^+, $\|\psi^+\|^{\boldsymbol{A}}_{\mathbf{M}} = \overline{1}^{\boldsymbol{A}}$.

Proof of Claim 3: Suppose not. Then, by Claim 1, $\mathbf{M}^{Q^+}$ would be a model of Q^+ which does not satisfy ψ, a contradiction.

Now let y be an element of the universe of $\mathbf{M}$. If $\|(y \leq \overline{m})^+\|^{\boldsymbol{A}}_{\mathbf{M}} = \overline{1}^{\boldsymbol{A}}$, then by Claim 3, $\|(y = \overline{0} \vee y = \overline{1} \vee \cdots \vee y = \overline{m})^+\|^{\boldsymbol{A}}_{\mathbf{M}} = \overline{1}^{\boldsymbol{A}}$, because $Q^+ \vdash (\forall x)(x \leq \overline{m} \leftrightarrow (x = \overline{0} \vee x = \overline{1} \vee \cdots \vee x = \overline{m}))$.

Moreover since $n \in X$, for $y = 0, 1, \ldots, m$, there is a k_y such that $R(n, y, k_y)$ is true. Then for such k_y, $R'(\overline{n}, \overline{y}, \overline{k_y})$ is provable in Q^+ and $\|R^+(\overline{n}, \overline{y}, \overline{k_y})\|^{\boldsymbol{A}}_{\mathbf{M}} = \overline{1}^{\boldsymbol{A}}$, again by Claim 3. If $\|(y > \overline{m})^+\|^{\boldsymbol{A}}_{\mathbf{M}} = \overline{1}^{\boldsymbol{A}}$, then for some i such that $\|(\bar{i} \leq y)^+\|^{\boldsymbol{A}}_{\mathbf{M}} = \overline{1}^{\boldsymbol{A}}$, we must have $\|P(S(\bar{i})) \to P(\bar{i})\|^{\boldsymbol{A}}_{\mathbf{M}} = \overline{1}^{\boldsymbol{A}}$, otherwise $\|P(\overline{0})\|^{\boldsymbol{A}}_{\mathbf{M}} < \|P(\overline{1})\|^{\boldsymbol{A}}_{\mathbf{M}} < \cdots < \|P(\overline{m+1})\|^{\boldsymbol{A}}_{\mathbf{M}}$ and A would have more than m elements. Thus in this case $(\exists u)((u \leq y)^+ \wedge (P(S(u)) \to P(u)))$. In any case, if $\|\Phi^+\|^{\boldsymbol{A}}_{\mathbf{M}} = \overline{1}^{\boldsymbol{A}}$ and $n \in X$ then $\|(\forall y)((\exists u)((u \leq y)^+ \wedge (P(S(u)) \to P(u))) \vee (\exists z) R^+(x, y, z))\|^{\boldsymbol{A}}_{\mathbf{M}} = \overline{1}^{\boldsymbol{A}}$. Thus $\Psi(\overline{n})$ has truth value $\overline{1}^{\boldsymbol{A}}$.

Now suppose that $n \notin X$. Then for some m there is no k such that $R(n, m, k)$. Let $\boldsymbol{A}$ be an L-chain with more than m elements. Let $\overline{0}^{\boldsymbol{A}} = a_0 < a_1 < \cdots < a_h = \overline{1}^{\boldsymbol{A}}$ with $h \geq m$, be the elements of A. Consider the first-order structure $\mathbf{N}^{\boldsymbol{A}}$ over $\boldsymbol{A}$ obtained from the standard model $\mathbf{N}$ of natural numbers according to Claim 2. Moreover, let us set, for $i = 0, \ldots, h$, $P^{\mathbf{N}^{\boldsymbol{A}}}(i) = a_i$ and for $i > h$, $P^{\mathbf{N}^{\boldsymbol{A}}}(i) = \overline{1}^{\boldsymbol{A}}$. Then by Claim 2, $\|\Phi^+\|^{\boldsymbol{A}}_{\mathbf{N}^{\boldsymbol{A}}} = \overline{1}^{\boldsymbol{A}}$, $\|(\exists u)((u \leq \overline{m})^+ \wedge (P(S(u)) \to P(u)))\|^{\boldsymbol{A}}_{\mathbf{N}^{\boldsymbol{A}}} = a_{h-1} < \overline{1}^{\boldsymbol{A}}$ and $\|(\exists z) R^+(\overline{n}, \overline{m}, z))\|^{\boldsymbol{A}}_{\mathbf{N}^{\boldsymbol{A}}} = \overline{0}^{\boldsymbol{A}}$. It follows that $\|\Psi(\overline{n})\|^{\boldsymbol{A}}_{\mathbf{N}^{\boldsymbol{A}}} = a_{h-1} < \overline{1}^{\boldsymbol{A}}$. □

COROLLARY 4.0.8. *Let* L *be a recursively axiomatizable (△-)core fuzzy logic such that for every finite cardinal* m*, there is a finite* L*-chain with at least* m *elements. Then* $\mathrm{finTAUT}(\mathrm{L}\forall)$ *is* $\mathbf{\Pi_2}$*-complete if one of the following sufficient conditions is satisfied:*

1. L *has a strict negation* $\sim$.
2. L *expands* WNM.
3. L *is a* △*-core fuzzy logic.*

Proof. By the hypothesis, all logics above satisfy condition (1) of Theorem 4.0.7. As regards to condition (2), for logics with a strict negation $\sim$, take $\phi(p) = {\sim}{\sim}p$, and note that for every evaluation v in an L-chain, if $v(p) = \overline{0}$, then $v(\phi(p)) = \overline{0}$, otherwise $v(\phi(p)) = \overline{1}$. For logics expanding WNM, take $\phi(p) = \neg((\neg((\neg\neg p)^2))^2)$ and note that for any evaluation v in an L-chain, if $v(\neg\neg p) \leq v(\neg p)$, then $v(\phi(p)) = \overline{0}$, otherwise $v(\phi(p)) = \overline{1}$. As regards to △-core fuzzy logics, it is clear that $\phi(p) = \triangle(p)$ satisfies condition (2) of Theorem 4.0.7. □

COROLLARY 4.0.9. *For every* $\mathrm{L} \in \{\mathrm{SMTL}, \mathrm{NM}, \mathrm{WNM}, \mathrm{SBL}, \mathrm{G}\}$*, we have that* $\mathrm{finTAUT}(\mathrm{L}\forall)$ *is* $\mathbf{\Pi_2}$*-complete.*

COROLLARY 4.0.10. *Let* L *be a* △*-core fuzzy logic such that for every finite cardinal* m*, there is a finite* L*-chain with at least* m *elements. Then* $\mathrm{finTAUT_{pos}}(\mathrm{L}\forall)$ *is* $\mathbf{\Pi_2}$*-complete, and* $\mathrm{finSAT}(\mathrm{L}\forall)$ *and* $\mathrm{finSAT_{pos}}(\mathrm{L}\forall)$ *are* $\mathbf{\Sigma_2}$*-complete.*

Proof. It follows from Corollary 4.0.8 by using some relations in Lemma 2.0.7:

- $\varphi \in \mathrm{finTAUT_{pos}}(\mathrm{L}\forall)$ iff $\neg\triangle(\neg\varphi) \in \mathrm{TAUT}(\mathrm{L}\forall)$,
- $\varphi \in \mathrm{finTAUT}(\mathrm{L}\forall)$ iff $\neg\triangle\varphi \notin \mathrm{finSAT}(\mathrm{L}\forall)$ iff $\neg\triangle\varphi \notin \mathrm{finSAT_{pos}}(\mathrm{L}\forall)$. □

From the general results in Section 2 and the real and rational completeness properties we obtain many arithmetical complexity results with respect to the rational semantics for prominent logics as collected in Table 4. In the case of $\mathrm{BL}\forall$ and $\mathrm{SBL}\forall$ we need an additional result:

PROPOSITION 4.0.11. *The sets* $\mathrm{ratTAUT}(\mathrm{BL}\forall)$ *and* $\mathrm{ratTAUT}(\mathrm{SBL}\forall)$ *are in* $\mathbf{\Sigma_1}$*, and hence* $\mathrm{ratSAT_{pos}}(\mathrm{BL}\forall)$ *and* $\mathrm{ratSAT_{pos}}(\mathrm{SBL}\forall)$ *are in* $\mathbf{\Pi_1}$.

Proof. Consider the extensions of $\mathrm{BL}\forall$ and $\mathrm{SBL}\forall$ by the schema $\Phi = (\forall x)(\chi \mathbin{\&} \varphi) \to (\chi \mathbin{\&} (\forall x)\varphi)$, where x is not free in χ. Call them $\mathrm{BL}\forall^+$ and $\mathrm{SBL}\forall^+$, respectively. Φ is valid in every model on a densely ordered BL-chain, but it is not a tautology for all BL-chains (see [21]). It is easy to see that $\mathrm{BL}\forall^+$ (resp. $\mathrm{SBL}\forall^+$) enjoys strong completeness with respect to models over rational BL-chains (resp. SBL-chains) (see Chapter V) and it is not necessary to require that those models satisfy the additional schema because their chains are densely ordered. Therefore, $\mathrm{ratTAUT}(\mathrm{BL}\forall)$ turns out to be the set of theorems of the logic $\mathrm{BL}\forall^+$, and analogously for $\mathrm{ratTAUT}(\mathrm{SBL}\forall)$; this proves the result. □

On the other hand, as we prove later in this section, $\mathrm{finTAUT}(\text{Ł}\forall)$ is $\mathbf{\Pi_2}$-complete. This allows to prove the following result:

PROPOSITION 4.0.12. $\mathrm{finTAUT}(\mathrm{BL}\forall)$ *is* $\mathbf{\Pi_2}$*-complete.*

Proof. For every sentence φ we consider the formula $\varphi^{\neg\neg}$ resulting from φ by adding double negation $\neg\neg$ to all atoms. Then for every $\varphi \in \mathrm{Sent}_{\mathcal{P}}$: $\varphi^{\neg\neg} \in \mathrm{finTAUT}(\mathrm{BL}\forall)$ iff $\varphi \in \mathrm{finTAUT}(\text{Ł}\forall)$. Indeed, the left-to-right implication is obvious because the negation is involutive in Łukasiewicz logic; as for the converse one let us assume that $\varphi \in \mathrm{finTAUT}(\text{Ł}\forall)$ and consider any model $\mathbf{M}$ over a finite BL-chain $\boldsymbol{A}$. Taking into account the structure of BL-chains described in previous chapters, it is enough to distinguish two cases:

(1) Assume that $\boldsymbol{A}$ is an ordinal sum $\boldsymbol{C}_1 \oplus \boldsymbol{C}_2$ where $\boldsymbol{C}_1$ is a finite MV-chain. Then we define a model $\mathbf{M}'$ over $\boldsymbol{C}_1$ from $\mathbf{M}$ in the following way: take the same domain, the same interpretation of constants and functionals, and for every n-ary predicate symbol P and elements $a_1, \ldots, a_n$ in the domain set $P_{\mathbf{M}'}(a_1, \ldots, a_n) = P_{\mathbf{M}}(a_1, \ldots, a_n)$ if $P_{\mathbf{M}}(a_1, \ldots, a_n) \in C_1$ and $P_{\mathbf{M}'}(a_1, \ldots, a_n) = \overline{1}^{\boldsymbol{A}}$ otherwise. Now it is easy to prove by induction that for every formula α and every evaluation v: $\|\alpha^{\neg\neg}\|^{\boldsymbol{A}}_{\mathbf{M},\mathrm{v}} = \|\alpha\|^{\boldsymbol{C}_1}_{\mathbf{M}',\mathrm{v}}$. Hence $\|\varphi^{\neg\neg}\|^{\boldsymbol{A}}_{\mathbf{M}} = \|\varphi\|^{\boldsymbol{C}_1}_{\mathbf{M}'} = \overline{1}^{\boldsymbol{A}}$.

(2) Assume that $\boldsymbol{A}$ is an SBL-chain (i.e. its negation is strict). Then we define a model $\mathbf{M}'$ over $\boldsymbol{B}_2$ from $\mathbf{M}$ in the following way: take the same domain, the same interpretation of constants and functionals, and for every n-ary predicate symbol P and elements $a_1, \ldots, a_n$ in the domain set $P_{\mathbf{M}'}(a_1, \ldots, a_n) = 0$ if $P_{\mathbf{M}}(a_1, \ldots, a_n) = \overline{0}^{\boldsymbol{A}}$ and $P_{\mathbf{M}'}(a_1, \ldots, a_n) = 1$ otherwise. Now we have: $\|\varphi^{\neg\neg}\|^{\boldsymbol{A}}_{\mathbf{M}} = \|\varphi^{\neg\neg}\|^{\boldsymbol{B}_2}_{\mathbf{M}'} = \|\varphi\|^{\boldsymbol{B}_2}_{\mathbf{M}'} = 1$. Therefore, we have proved that $\mathrm{finTAUT}(\mathrm{BL}\forall)$ is $\mathbf{\Pi_2}$-hard. The $\mathbf{\Pi_2}$ containment follows from Theorems 4.0.5 and 4.0.6. □

Some more results on complexity of finite-chain semantics will be obtained soon when comparing such semantics with the real and rational ones.

Logic	ratTAUT	ratSAT	$\mathrm{ratTAUT_{pos}}$	$\mathrm{ratSAT_{pos}}$
MTL∀	$\mathbf{\Sigma_1}$-complete	$\mathbf{\Pi_1}$-complete	$\mathbf{\Sigma_1}$-complete	$\mathbf{\Pi_1}$-complete
IMTL∀	$\mathbf{\Sigma_1}$-complete	$\mathbf{\Pi_1}$-complete	$\mathbf{\Sigma_1}$-complete	$\mathbf{\Pi_1}$-complete
SMTL∀	$\mathbf{\Sigma_1}$-complete	$\mathbf{\Pi_1}$-complete	$\mathbf{\Sigma_1}$-complete	$\mathbf{\Pi_1}$-complete
WCMTL∀	$\mathbf{\Sigma_1}$-hard	$\mathbf{\Pi_1}$-hard	$\mathbf{\Sigma_1}$-hard	$\mathbf{\Pi_1}$-hard
ΠMTL∀	$\mathbf{\Sigma_1}$-hard	$\mathbf{\Pi_1}$-hard	$\mathbf{\Sigma_1}$-hard	$\mathbf{\Pi_1}$-hard
BL∀	$\mathbf{\Sigma_1}$-complete	$\mathbf{\Pi_1}$-hard	$\mathbf{\Sigma_1}$-hard	$\mathbf{\Pi_1}$-complete
SBL∀	$\mathbf{\Sigma_1}$-complete	$\mathbf{\Pi_1}$-hard	$\mathbf{\Sigma_1}$-hard	$\mathbf{\Pi_1}$-complete
Ł∀	$\mathbf{\Sigma_1}$-complete	$\mathbf{\Pi_1}$-complete	$\mathbf{\Sigma_1}$-complete	$\mathbf{\Pi_1}$-complete
Π∀	$\mathbf{\Sigma_1}$-complete	$\mathbf{\Pi_1}$-complete	$\mathbf{\Sigma_1}$-complete	$\mathbf{\Pi_1}$-complete
G∀	$\mathbf{\Sigma_1}$-complete	$\mathbf{\Pi_1}$-complete	$\mathbf{\Sigma_1}$-complete	$\mathbf{\Pi_1}$-complete
C_nMTL∀	$\mathbf{\Sigma_1}$-complete	$\mathbf{\Pi_1}$-complete	$\mathbf{\Sigma_1}$-complete	$\mathbf{\Pi_1}$-complete
C_nIMTL∀	$\mathbf{\Sigma_1}$-complete	$\mathbf{\Pi_1}$-complete	$\mathbf{\Sigma_1}$-complete	$\mathbf{\Pi_1}$-complete
WNM∀	$\mathbf{\Sigma_1}$-complete	$\mathbf{\Pi_1}$-complete	$\mathbf{\Sigma_1}$-complete	$\mathbf{\Pi_1}$-complete
NM∀	$\mathbf{\Sigma_1}$-complete	$\mathbf{\Pi_1}$-complete	$\mathbf{\Sigma_1}$-complete	$\mathbf{\Pi_1}$-complete

Table 4. Complexity results for the rational semantics.

Observe that the completeness properties imply that for some prominent logics we have $\mathrm{genTAUT(L\forall)} = \mathrm{realTAUT(L\forall)} = \mathrm{ratTAUT(L\forall)}$. Now, in addition, we will consider the semantics given by intended rational chains. Of course, this can only be done for those logics where it makes sense to have an intended semantics over the rational unit interval, i.e. logics L_* given by a left-continuous t-norm $*$ such that its restriction to $[0,1]^{\mathsf{Q}}$ is well-defined. We denote the corresponding algebra as $[0,1]^{\mathsf{Q}}_*$. This can be done, for instance, for the logic NM corresponding to the nilpotent minimum t-norm. By inspecting the usual the proof of the fact that every countable NM-chain can be σ-embedded into $[0,1]_{\mathrm{NM}}$ one realizes that the embedding can be in fact defined into the rationals and thus we have $\mathrm{genTAUT(NM\forall)} = \mathrm{stTAUT(NM\forall)} = \mathrm{intratTAUT(NM\forall)}$ and they are $\mathbf{\Sigma_1}$-complete. The three main continuous t-norms satisfy the required property as well, i.e. we have well-defined algebras over the rationals $[0,1]^{\mathsf{Q}}_{\text{Ł}}$, $[0,1]^{\mathsf{Q}}_{\Pi}$ and $[0,1]^{\mathsf{Q}}_{\mathrm{G}}$; the same goes for their ordinal sums. Let $\mathcal{Q}$ be the set of ordinal sums of these three rational BL-chains. Given $\mathbb{K} \subseteq \mathcal{Q}$, $\overline{\mathbb{K}}$ will denote the subset of $\mathcal{R}$ given by the substitution in the elements of $\mathbb{K}$ of each component for its corresponding basic real chain. We start with the case of $[0,1]^{\mathsf{Q}}_{\text{Ł}}$.

LEMMA 4.0.13. *Let $\mathbf{M}$ and $\mathbf{M}'$ be two first-order structures with the same domain M over $[0,1]_{\text{Ł}}$, and let ϕ, ψ be first-order sentences with parameters from M and $\delta(x)$ be a first-order formula with parameters from M and with x as its only free variable. Let for any two real numbers α, β, $d(\alpha,\beta)$ denote the distance between α and β, that is, $d(\alpha,\beta) = \max\{\alpha-\beta, \beta-\alpha\}$. Then for every positive real number γ we have:*

(i) If $d(\|\phi\|_{\mathbf{M}}, \|\phi\|_{\mathbf{M}'}) \leq \gamma$, then $d(\|\neg\phi\|_{\mathbf{M}}, \|\neg\phi\|_{\mathbf{M}'}) \leq \gamma$.

(ii) If $d(\|\phi\|_{\mathbf{M}}, \|\phi\|_{\mathbf{M}'}) \leq \gamma$ and $d(\|\psi\|_{\mathbf{M}}, \|\psi\|_{\mathbf{M}'}) \leq \gamma$, then $d(\|\phi \,\&\, \psi\|_{\mathbf{M}}, \|\phi \,\&\, \psi\|_{\mathbf{M}'}) \leq 2\gamma$.

(iii) If for all $a \in M$, $d(\|\delta(a)\|_{\mathbf{M}}, \|\delta(a)\|_{\mathbf{M}'}) \leq \gamma$,
then $d(\|(\forall x)\delta(x)\|_{\mathbf{M}}, \|(\forall x)\delta(x)\|_{\mathbf{M}'}) \leq \gamma$.

Proof. Almost trivial. □

COROLLARY 4.0.14. *If ϕ is a sentence of complexity k and $\mathbf{M}$ and $\mathbf{M}'$ are first-order structures with the same domain M over $[0,1]_{Ł}$ such that for every closed atomic subformula ψ of ϕ, $d(\|\psi\|_{\mathbf{M}}, \|\psi\|_{\mathbf{M}'}) \leq \gamma$, then $d(\|\phi\|_{\mathbf{M}}, \|\phi\|_{\mathbf{M}'}) \leq 2^k\gamma$.*

LEMMA 4.0.15. *Let ϕ be a first-order sentence of complexity k and let for every n, $\mathbf{Ł}_n$ denote the finite MV-chain with $n+1$ elements. Let $\mathbf{M}$ be a first-order structure over $[0,1]_{Ł}$ with domain M such that $\|\phi\|_{\mathbf{M}} < 1$ and let n be such that $2^{-n} < 1 - \|\phi\|_{\mathbf{M}}$. Then there is a first-order structure $\mathbf{M}'$ over $\mathbf{Ł}_{2^{k+n}}$ such $\|\phi\|_{\mathbf{M}'} < 1$.*

Proof. Let for every atomic formula ψ with parameters in M, $m(\psi)$ denote the maximum natural number such that $\frac{m(\psi)}{2^{n+k}} \leq \|\psi\|_{\mathbf{M}}$. Define a new first-order structure $\mathbf{M}'$ with domain M letting for every atomic formula ψ with parameters in M, $\|\psi\|_{\mathbf{M}'} = \frac{m(\psi)}{2^{n+k}}$. Then $d(\|\psi\|_{\mathbf{M}}, \|\psi\|_{\mathbf{M}'}) < \frac{1}{2^{n+k}}$ and by Corollary 4.0.14, $d(\|\phi\|_{\mathbf{M}}, \|\phi\|_{\mathbf{M}'}) \leq 2^k \frac{1}{2^{n+k}} = \frac{1}{2^n} < 1 - \|\phi\|_{\mathbf{M}}$. It follows that $\|\phi\|_{\mathbf{M}'} < 1$. Now $\mathbf{M}'$ has been defined as a first-order structure over $[0,1]_{Ł}$, but since for every sentence δ, $\|\delta\|_{\mathbf{M}'} \in \mathbf{Ł}_{2^{k+n}}$, $\mathbf{M}'$ can be also regarded as a first-order structure over $\mathbf{Ł}_{2^{k+n}}$. □

THEOREM 4.0.16. $\mathrm{stTAUT}(Ł\forall) = \mathrm{intratTAUT}(Ł\forall) = \mathrm{finTAUT}(Ł\forall)$ *and they are $\mathbf{\Pi_2}$-complete.*

Proof. The Inclusions from left to right follow from Lemma 2.0.3, therefore it suffices to prove that $\mathrm{finTAUT}(Ł\forall) \subseteq \mathrm{stTAUT}(Ł\forall)$. But this is immediate from Lemma 4.0.15. □

Recall that if $\mathbb{K}$ is a class of MV-chains, then $\mathrm{SAT}_{\mathrm{pos}}(\mathbb{K}) = \{\phi \mid \neg\phi \notin \mathrm{TAUT}(\mathbb{K})\}$. Thus we obtain:

THEOREM 4.0.17. $\mathrm{stSAT}_{\mathrm{pos}}(Ł\forall) = \mathrm{intratSAT}_{\mathrm{pos}}(Ł\forall) = \mathrm{finSAT}_{\mathrm{pos}}(Ł\forall)$ *and they are $\mathbf{\Sigma_2}$-complete.*

In the SAT case the situation is different:

THEOREM 4.0.18. $\mathrm{finSAT}(Ł\forall) \subsetneq \mathrm{intratSAT}(Ł\forall)$.

Proof. The inclusion is obvious; let us show the difference with an example. Let S be a unary function symbol, P be a unary predicate symbol and $\mathbf{0}$ be a constant symbol. Consider the following sentence:

$$\Phi = (P(\mathbf{0}) \leftrightarrow \neg P(\mathbf{0}))\ \&\ (\forall x)(P(x) \leftrightarrow ((P(S(x)) \oplus P(S(x))))$$

On the one hand, Φ is satisfiable in $[0,1]_{Ł}^{\mathbf{Q}}$. Indeed, take the structure $\mathbf{M}$ on $[0,1]_{Ł}$ whose domain is the set of natural numbers, $\mathbf{0}$ and S are respectively interpreted as 0 and the successor function, and $\|P(n)\|_{\mathbf{M}} = \frac{1}{2^{n+1}}$. On the other hand Φ is not satisfiable in models over finite MV-chains. Indeed, let $\mathbf{M}$ be a first-order structure over a finite chain $\mathbf{Ł}_n$. The first conjunct of Φ is satisfiable if the chain has a negation fixpoint (i.e. it has an odd number of truth-values); assume it. A contradiction will be reached when combining it with the satisfiability of the second conjunct. Thus, assume, in addition, that $\|(\forall x)(P(x) \leftrightarrow ((P(S(x)) \oplus P(S(x)))\|_{\mathbf{M}} = 1$. This means that for every $a \in M$, $P_{\mathbf{M}}(a) = P_{\mathbf{M}}(S_{\mathbf{M}}(a)) \oplus P_{\mathbf{M}}(S_{\mathbf{M}}(a))$; therefore, if $P_{\mathbf{M}}(a) \neq 1$, then $P_{\mathbf{M}}(a) > P_{\mathbf{M}}(S_{\mathbf{M}}(a))$. So we obtain $P_{\mathbf{M}}(0) > P_{\mathbf{M}}(S_{\mathbf{M}}(0)) > P_{\mathbf{M}}(S_{\mathbf{M}}(S_{\mathbf{M}}(0))) > \cdots$, a contradiction with the finiteness of $\mathbf{Ł}_n$. □

Similarly, Theorems 4.0.16 and 4.0.17 do not extend to finite consequence relation:

COROLLARY 4.0.19. *There are formulae ϕ and ψ of Łukasiewicz logic such that $\phi \in \mathrm{finCons}(Ł, \psi)$ and $\phi \notin \mathrm{intratCons}(Ł, \psi)$.*

Proof. It is immediate from the last theorem. Indeed, take any $\varphi \in \mathrm{intratSAT}(Ł\forall) \setminus \mathrm{finSAT}(Ł\forall)$, and then it is clear than $\overline{0} \in \mathrm{finCons}(Ł, \varphi)$ and $\overline{0} \notin \mathrm{intratCons}(Ł, \varphi)$. □

We consider now other intended rational chains.

THEOREM 4.0.20. *Let $\mathbb{K} \subseteq \mathcal{Q}$ such that there exists $\boldsymbol{A} \in \mathbb{K}$ whose first component is product. Then $\mathrm{TAUT}(\mathbb{K})$, $\mathrm{SAT}(\mathbb{K})$, $\mathrm{TAUT}_{\mathrm{pos}}(\mathbb{K})$ and $\mathrm{SAT}_{\mathrm{pos}}(\mathbb{K})$ are non-arithmetical.*

Proof. Let T, θ_1, θ_2, θ_3 and θ_4 be as in the proof of Theorem 3.0.23. Then one can prove the following claims for every Φ in the language of PA:

$$\begin{aligned}\mathbf{N} \models \Phi \ &\text{iff } \theta_1 \mathbin{\&} \theta_2 \mathbin{\&} \theta_3 \mathbin{\&} \theta_4 \to \Phi^{\neg\neg} \in \mathrm{TAUT}(\mathbb{K})\\ &\text{iff } \theta_1 \mathbin{\&} \theta_2 \mathbin{\&} \theta_3 \mathbin{\&} \theta_4 \to \Phi^{\neg\neg} \in \mathrm{TAUT}_{\mathrm{pos}}(\mathbb{K}).\end{aligned}$$

$$\begin{aligned}\mathbf{N} \models \Phi \ &\text{iff } \theta_1 \mathbin{\&} \theta_2 \mathbin{\&} \theta_3 \mathbin{\&} \theta_4 \mathbin{\&} \Phi^{\neg\neg} \in \mathrm{SAT}(\mathbb{K})\\ &\text{iff } \theta_1 \mathbin{\&} \theta_2 \mathbin{\&} \theta_3 \mathbin{\&} \theta_4 \mathbin{\&} \Phi^{\neg\neg} \in \mathrm{SAT}_{\mathrm{pos}}(\mathbb{K}).\end{aligned}$$

We justify the first claim (the second one is proved analogously). Assume that $\mathbf{N} \models \Phi$. By Lemma 3.0.26, this implies that $\theta_1 \& \theta_2 \& \theta_3 \& \theta_4 \to \Phi^{\neg\neg} \in \mathrm{TAUT}(\overline{\mathbb{K}})$, and hence $\theta_1 \& \theta_2 \& \theta_3 \& \theta_4 \to \Phi^{\neg\neg} \in \mathrm{TAUT}(\mathbb{K})$ and $\theta_1 \& \theta_2 \& \theta_3 \& \theta_4 \to \Phi^{\neg\neg} \in \mathrm{TAUT}_{\mathrm{pos}}(\mathbb{K})$. Assume now that $\mathbf{N} \not\models \Phi$. Then we construct a countermodel $\mathbf{M}$ over $\boldsymbol{A}$ as follows:

(a) The domain of $\mathbf{M}$ is the set of natural numbers, and the constant 0 and the function symbols of Q^+ are interpreted as in the standard model $\mathbf{N}$ of natural numbers.

(b) If P is an n-ary predicate symbol of Q^+ and $k_1, \ldots, k_n$ are natural numbers, then $P_{\mathbf{M}}(k_1, \ldots, k_n) = 1$ if $P(k_1, \ldots, k_n)$ is true in $\mathbf{N}$ and $P_{\mathbf{M}}(k_1, \ldots, k_n) = 0$ otherwise.

(c) Let f be the affine bijective transformation from $[0,1]^{\mathbf{Q}}$ to the interval where the first component of $\boldsymbol{A}$ is defined. For every natural number n, $U_{\mathbf{M}}(n) = f(2^{-3^n})$.

It is readily seen that $\|\theta_1 \mathbin{\&} \theta_2 \mathbin{\&} \theta_3 \mathbin{\&} \theta_4\|^{\boldsymbol{A}}_{\mathbf{M}} = 1$ and $\|\Phi^{\neg\neg}\|^{\boldsymbol{A}}_{\mathbf{M}} = 0$. Therefore, $\|\theta_1 \mathbin{\&} \theta_2 \mathbin{\&} \theta_3 \mathbin{\&} \theta_4 \to \Phi^{\neg\neg}\|^{\boldsymbol{A}}_{\mathbf{M}} = 0$ and hence $\theta_1 \mathbin{\&} \theta_2 \mathbin{\&} \theta_3 \mathbin{\&} \theta_4 \to \Phi^{\neg\neg} \notin \mathrm{TAUT}(\mathbb{K})$ and $\theta_1 \mathbin{\&} \theta_2 \mathbin{\&} \theta_3 \mathbin{\&} \theta_4 \to \Phi^{\neg\neg} \notin \mathrm{TAUT}_{\mathrm{pos}}(\mathbb{K})$. □

THEOREM 4.0.21.

1. *If* $\boldsymbol{C}$ *is the first component of a chain* $\boldsymbol{A} \in \mathcal{Q}$*, then* $\mathrm{TAUT_{pos}}(\boldsymbol{A}) = \mathrm{TAUT_{pos}}(\boldsymbol{C})$ *and* $\mathrm{SAT_{pos}}(\boldsymbol{A}) = \mathrm{SAT_{pos}}(\boldsymbol{C})$.
2. $\mathrm{SAT_{pos}}([0,1]_{\mathrm{G}}^{\mathbf{Q}}) = \mathrm{SAT}([0,1]_{\mathrm{G}}^{\mathbf{Q}})$ *and it is* $\mathbf{\Pi_1}$*-complete.*
3. *If* $\boldsymbol{A} \in \mathcal{Q}$ *begins with a Gödel component, then* $\mathrm{SAT}(\boldsymbol{A}) = \mathrm{SAT}([0,1]_{\mathrm{G}}^{\mathbf{Q}})$.
4. $\mathrm{TAUT_{pos}}([0,1]_{\mathrm{G}}^{\mathbf{Q}})$ *is* $\mathbf{\Sigma_1}$*-complete.*
5. $\mathrm{TAUT}([0,1]_{\Pi}) = \mathrm{TAUT}([0,1]_{\Pi}^{\mathbf{Q}})$.
6. *If* $\mathbb{K} \subseteq \mathcal{Q}$ *contains some component non-isomorphic to* $[0,1]_{\mathrm{G}}^{\mathbf{Q}}$*, then* $\mathrm{TAUT}(\mathbb{K})$ *is* $\mathbf{\Pi_2}$*-hard.*
7. *If* $\mathbb{K} \subseteq \mathcal{Q}$ *contains at least one algebra non-isomorphic to any of* $[0,1]_{\mathrm{G}}^{\mathbf{Q}}$, $[0,1]_{\mathrm{Ł}}^{\mathbf{Q}}$, $[0,1]_{\mathrm{Ł}}^{\mathbf{Q}} \oplus [0,1]_{\mathrm{G}}^{\mathbf{Q}}$, $[0,1]_{\mathrm{G}}^{\mathbf{Q}} \oplus [0,1]_{\mathrm{Ł}}^{\mathbf{Q}}$, $[0,1]_{\mathrm{Ł}}^{\mathbf{Q}} \oplus [0,1]_{\mathrm{Ł}}^{\mathbf{Q}}$ *and* $[0,1]_{\mathrm{Ł}}^{\mathbf{Q}} \oplus [0,1]_{\mathrm{G}}^{\mathbf{Q}} \oplus [0,1]_{\mathrm{Ł}}^{\mathbf{Q}}$*, then* $\mathrm{TAUT}(\mathbb{K})$ *is non-arithmetical.*

Proof. The first four claims are proved by checking that the proofs of the corresponding results for the standard semantics actually work as well for intended rational semantics. 5 is proved in the appendix of [4]. Point 6 is shown by reducing the problem to $\mathrm{TAUT}([0,1]_{\mathrm{Ł}}^{\mathbf{Q}})$, which we know is $\mathbf{\Pi_2}$-hard, as in the proof of Theorem 3.0.21. Similarly, for the last point we use the fact, proved in the previous theorem, that the set $\mathrm{TAUT}([0,1]_{\Pi}^{\mathbf{Q}})$ is non-arithmetical and perform the analogous reduction to that problem as in Theorem 3.0.30. □

As for the relation between tautologies over real and finite chains, we have the following result:

THEOREM 4.0.22. *Let* L *be a consistent (*$\triangle$*-)core fuzzy logic. If there exist* L*-chains over* $[0,1]$ *whose t-norm is not isomorphic to Łukasiewicz, then* $\mathrm{realTAUT}(\mathrm{L}\forall) \neq \mathrm{finTAUT}(\mathrm{L}\forall)$ *and* $\mathrm{genTAUT}(\mathrm{L}\forall) \neq \mathrm{finTAUT}(\mathrm{L}\forall)$.

Proof. Let $[0,1]_*$ be a real L-chain defined by a left-continuous t-norm $*$ non-isomorphic to the Łukasiewicz t-norm. Then:

(i) If $*$ is continuous, then the formula

$$(C\forall) \quad (\exists x)(P(x) \to (\forall y)P(y))$$

is an $\boldsymbol{A}$-tautology for any finite L-chain $\boldsymbol{A}$, but it is not a $[0,1]_*$-tautology.

(ii) If $*$ is not continuous, then the formula Φ

$$(\forall x)(\chi \mathbin{\&} \psi) \to (\chi \mathbin{\&} (\forall x)\psi), \text{ where } x \text{ is not free in } \chi$$

is an $\boldsymbol{A}$-tautology for any finite L-chain $\boldsymbol{A}$, but it is not a $[0,1]_*$-tautology.

Indeed:

(i) It is well known that, for any continuous t-norm $*$ which is not isomorphic to Łukasiewicz t-norm, the corresponding negation $n_*(x) = x \Rightarrow_* 0$ is not (right) continuous at $x = 0$. Let $b = lim_{x \to 0^+} n_*(x)$. We know that $b < 1$. Take an infinite decreasing sequence $1 > a_1 > a_2 > \cdots > a_n \cdots > 0$ with limit 0. Consider the $[0,1]_*$-model $\mathbf{M} = \langle \mathsf{N}, P_\mathbf{M} \rangle$ where $P_\mathbf{M}(n) = a_n$. Then we have $\|(\exists x)(P(x) \to (\forall y)P(y))\|^{[0,1]_*}_{\mathbf{M},e} = \sup_n\{a_n \Rightarrow_* (\inf_n a_n)\} = \sup_n\{a_n \Rightarrow_* 0\} = \sup_n n_*(a_n) = b < 1$. On the other hand, it is clear that the formula has value $\overline{1}^{\boldsymbol{A}}$ in any structure over a finite L-chain $\boldsymbol{A}$.

(ii) For simplicity, let us take the following instance of Φ:

$$(\forall x)(P(c) \mathbin{\&} Q(x)) \to P(c) \mathbin{\&} (\forall x)Q(x)$$

where c is a 0-ary functional symbol. If the t-norm is not right-continuous there is a sequence $\langle a_n \mid n \geq 1 \rangle$ and an element b such that $b * \inf\{a_n \mid n \geq 1\} < \inf\{b * a_n \mid n \geq 1\}$. Consider the $[0,1]_*$-model $\mathbf{M} = \langle \mathsf{N}, P_\mathbf{M}, Q_\mathbf{M} \rangle$ and an evaluation of variables e such that $P_\mathbf{M}(c_\mathbf{M}) = b$ and $Q_\mathbf{M}(n) = a_n$ for every n. Then $\|(\forall x)(P(c) \mathbin{\&} Q(x)) \to P(c) \mathbin{\&} (\forall x)Q(x)\|^{[0,1]_*}_{\mathbf{M},e} = \inf\{b * a_n, n \geq 1\} \Rightarrow_* (b * \inf\{a_n, n \geq 1\}) < 1$. But an easy computation shows that for any finite chain the formula is a tautology (take into account that the $\inf$ becomes a $\min$). □

5 Further topics on arithmetical hierarchy in first-order logics

5.1 Semantics of witnessed models

The semantics of witnessed models has been introduced in Chapter II; we quickly recall a few necessary notions for the reader's convenience. For each continuous t-norm $*$, the propositional logic given by it (Definition 1.1.19 of Chapter I) is denoted L_* and the corresponding predicate logic is denoted $\mathrm{L}_*\forall$. Let $\boldsymbol{A}$ be a BL-chain. A model $\langle \boldsymbol{A}, \mathbf{M} \rangle$ is *witnessed* if for each formula $\varphi(x, y, \dots)$ and for each $b, \dots \in M$,

$$\|(\forall x)\varphi(x, b, \dots)\|^{\boldsymbol{A}}_{\mathbf{M}} = \min_a \|\varphi(a, b, \dots)\|^{\boldsymbol{A}}_{\mathbf{M}},$$
$$\|(\exists x)\varphi(x, b, \dots)\|^{\boldsymbol{A}}_{\mathbf{M}} = \max_a \|\varphi(a, b, \dots)\|^{\boldsymbol{A}}_{\mathbf{M}},$$

(i.e. there is an a with minimal (maximal) value of $\|\varphi(a, b, \dots)\|$). Alternatively we say that $\mathbf{M}$ is $\boldsymbol{A}$-witnessed.

FACT 5.1.1 ([20]). *Over the Łukasiewicz logic* $Ł\forall$, *each countable standard model* $\mathbf{M}$ *is an elementary submodel of a witnessed standard model* $\mathbf{M}'$.

Given a logic L, we denote by $\mathrm{L}\forall^w$ the logic $\mathrm{L}\forall$ extended by the following axioms:

$(C\forall)$	$(\exists x)(\varphi(x) \to (\forall y)\varphi(y))$,
$(C\exists)$	$(\exists x)((\exists y)\varphi(y) \to \varphi(x))$.

FACT 5.1.2 (Chapter II and [20]).

(1) For our logics, the logic $\mathrm{L}\forall^w$ *is strongly complete w.r.t. witnessed models.*

(2) A model $\langle \boldsymbol{A}, \mathbf{M}\rangle$ *is elementarily embeddable into a witnessed model iff* $(C\forall)$ *and* $(C\exists)$ *are true in* $\langle \boldsymbol{A}, \mathbf{M}\rangle$.

(3) The following are equivalent:

- $(C\forall), (C\exists) \in \mathrm{genTAUT}(\mathrm{L}_*\forall)$,
- $(C\forall), (C\exists) \in \mathrm{stTAUT}(\mathrm{L}_*\forall)$,
- $*$ *is the Łukasiewicz t-norm.*

For each L_*-chain $\boldsymbol{A}$ let $\boldsymbol{A}^{\neg\neg}$ be the homomorphic image of $\boldsymbol{A}$ defined by the mapping $f(x) = \neg\neg x$. For each L_*-model $\langle \boldsymbol{A}, \mathbf{M}\rangle$, $\mathbf{M} = \langle M, \langle P_{\mathbf{M}}\rangle_P, \langle f_{\mathbf{M}}\rangle_f\rangle$, let $\mathbf{M}^{\neg\neg}$ be the $\boldsymbol{A}^{\neg\neg}$-model $\mathbf{M}^{\neg\neg} = \langle M, \langle P^{\neg\neg}_{\mathbf{M}}\rangle_P, \langle f_{\mathbf{M}}\rangle_f\rangle$, where, for each P and $\mathbf{a}$, $P^{\neg\neg}_{\mathbf{M}}(\mathbf{a}) = f(P_{\mathbf{M}}(\mathbf{a})) = \neg\neg P_{\mathbf{M}}(\mathbf{a})$. For a t-norm with Gödel negation $\boldsymbol{A}^{\neg\neg}$ is the two-element Boolean algebra; for a t-norm beginning by a Łukasiewicz component $\boldsymbol{A}^{\neg\neg}$ is Łukasiewicz.

FACT 5.1.3. *Let* $\boldsymbol{A}$ *be an* L_**-chain and* $\langle \boldsymbol{A}, \mathbf{M}\rangle$ *a witnessed model. Then* $\langle \boldsymbol{A}^{\neg\neg}, \mathbf{M}^{\neg\neg}\rangle$ *is a witnessed model such that for each formula* φ *and each tuple* $\mathbf{a}$ *of elements of* M, $\|\varphi(\mathbf{a})\|^{\boldsymbol{A}^{\neg\neg}}_{\mathbf{M}^{\neg\neg}} = \|\neg\neg\varphi(\mathbf{a})\|^{\boldsymbol{A}}_{\mathbf{M}}$.

Now we start to discuss the arithmetical complexity of general and standard semantics of the predicate logics given by continuous t-norms with semantics restricted to (standard and general) witnessed models. We write $\mathrm{wgenTAUT}(\mathrm{L}_*\forall)$ for the set of formulae true in all witnessed general models, $\mathrm{wstTAUT}$, $\mathrm{wgenSAT}$ and wstSAT in the obvious analogous sense.

THEOREM 5.1.4. *For each* $*$, $\mathrm{wgenTAUT}(\mathrm{L}_*\forall)$ *is* $\mathbf{\Sigma_1}$*-complete and* $\mathrm{wgenSAT}(\mathrm{L}_*\forall)$ *is* $\mathbf{\Pi_1}$*-complete.*

Proof. The fact that $\mathrm{wgenTAUT}(\mathrm{L}_*\forall)$ is in $\mathbf{\Sigma_1}$ follows from the completeness theorems for the corresponding logic. Similarly, $\mathrm{wgenSAT}(\mathrm{L}_*\forall)$ is in $\mathbf{\Pi_1}$ since satisfiability is equivalent to consistency.

Concerning the $\mathbf{\Sigma_1}$-completeness of $\mathrm{wgenTAUT}(\mathrm{L}_*\forall)$: For Łukasiewicz it follows from the fact that tautologies are the same as witnessed tautologies. For $*$ with strict negation observe that φ is a Boolean tautology iff $\varphi^{\neg\neg}$ is a witnessed general tautology of $\mathrm{L}_*\forall$. Finally for a t-norm $*$ beginning by Łukasiewicz, φ is a (witnessed) general tautology of Łukasiewicz logic iff $\varphi^{\neg\neg}$ is a witnessed general tautology of $\mathrm{L}_*\forall$. This reduces a $\mathbf{\Sigma_1}$-complete set to our set $\mathrm{wgenTAUT}(\mathrm{L}_*\forall)$. □

THEOREM 5.1.5. *For every* $*$ *with strict negation:*

(1) The following five sets of formulae are equal: $\mathrm{wgenSAT}$, $\mathrm{wgenSAT_{pos}}$, wstSAT, $\mathrm{wstSAT_{pos}}$, $\mathrm{SAT}(\boldsymbol{B}_2)$.

(2) The following sets are equal: $\mathrm{wgenTAUT_{pos}}$, $\mathrm{wstTAUT_{pos}}$, $\mathrm{TAUT}(\boldsymbol{B}_2)$ *(but different from* $\mathrm{wstTAUT}$, $\mathrm{wgenTAUT}$*).*

Proof. (1) All sets include $\mathrm{SAT}(\boldsymbol{B}_2)$ and are included in $\mathrm{wgenSAT}_{\mathrm{pos}}$. Thus if a formula is Boolean satisfiable it is in all sets in question. Conversely, if $\varphi \in \mathrm{wgenSAT}_{\mathrm{pos}}$, thus for some witnessed $\langle \mathbf{M}, \boldsymbol{A} \rangle$ it holds $\|\varphi\|^{\boldsymbol{A}}_{\mathbf{M}} > 0$, then $\|\neg\neg\varphi\|^{\boldsymbol{A}}_{\mathbf{M}} = 1$, hence $\|\varphi\|^{\boldsymbol{B}_2}_{\mathbf{M}^{\neg\neg}} = 1$, i.e. φ is Boolean satisfiable.

(2) Clearly, $\mathrm{wgenTAUT}_{\mathrm{pos}} \subseteq \mathrm{wstTAUT}_{\mathrm{pos}} \subseteq \mathrm{TAUT}(\boldsymbol{B}_2)$. Conversely, if φ is not in $\mathrm{wgenTAUT}_{\mathrm{pos}}$, thus for some witnessed $\langle \boldsymbol{A}, \mathbf{M} \rangle$ it holds $\|\varphi\|^{\boldsymbol{A}}_{\mathbf{M}} = 0$, then $\|\neg\neg\varphi\|^{\boldsymbol{A}}_{\mathbf{M}} = 0$ hence $\|\varphi\|^{\boldsymbol{B}_2}_{\mathbf{M}^{\neg\neg}} = 0$, i.e. φ is not a Boolean tautology. Finally, the *tertium non datur* formula $(\forall x)(P(x) \vee \neg P(x))$ is a formula in $\mathrm{TAUT}(\boldsymbol{B}_2)$ but not in $\mathrm{wstTAUT}$, hence not in $\mathrm{wgenTAUT}$. □

Now recall Definition 3.0.10 of the mapping h (for all $x \in [0,1]$, let $h(x) = 2x$ for $x \leq 1/2$ and $h(x) = 1$ for $x \geq 1/2$), the model $h(\mathbf{M})$ for each standard model $\mathbf{M}$ and the notation $\boldsymbol{C}\oplus$ for a continuous t-norm whose first component is $\boldsymbol{C}$ on $[0, \frac{1}{2}]$. Also recall the mapping f from (the proof of) 3.0.14 ($f(x) = \frac{x}{2}$ for $x < 1$, $f(1) = 1$).

LEMMA 5.1.6. *Let $\mathbf{M}$ be a standard model, $\varphi(a_1, \ldots, a_n)$ a formula with $a_1, \ldots, a_n \in M$ substituted for free variables. Write φ for $\varphi(a_1, \ldots, a_n)$ for brevity.*

(1) $h(\|\varphi\|^{\boldsymbol{C}\oplus}_{\mathbf{M}}) = \|\varphi\|^{\boldsymbol{C}}_{h(\mathbf{M})}$.

(2) If $\mathbf{M}$ is $\boldsymbol{C}$-witnessed then $\|\varphi\|^{\boldsymbol{C}\oplus}_{f(\mathbf{M})} = f(\|\varphi\|^{\boldsymbol{C}}_{\mathbf{M}})$.

(3) Consequently, if $\mathbf{M}$ is $\boldsymbol{C}\oplus$-witnessed then $h(\mathbf{M})$ is $\boldsymbol{C}$-witnessed; if $\mathbf{M}$ is $\boldsymbol{C}$-witnessed then $f(\mathbf{M})$ is $\boldsymbol{C}\oplus$-witnessed.

Proof. Easy; compare it with the proofs of 3.0.10 and 3.0.14. □

THEOREM 5.1.7. *For a continuous t-norm $\boldsymbol{C}\oplus$:*

(1) $\mathrm{wstSAT}_{\mathrm{pos}}(\boldsymbol{C}\oplus) = \mathrm{wstSAT}_{\mathrm{pos}}(\boldsymbol{C})$,

(2) $\mathrm{wstTAUT}_{\mathrm{pos}}(\boldsymbol{C}\oplus) = \mathrm{wstTAUT}_{\mathrm{pos}}(\boldsymbol{C})$,

(3) $\mathrm{wstSAT}(\boldsymbol{C}\oplus) = \mathrm{wstSAT}(\boldsymbol{C})$.

Proof. For (1) observe that, for $\mathbf{M}$ $[\boldsymbol{C}]$-witnessed, $\|\varphi\|^{\boldsymbol{C}}_{\mathbf{M}} > 0$ implies $\|\varphi\|^{\boldsymbol{C}\oplus}_{j(\mathbf{M})} > 0$ and for $\mathbf{M}$ $[\boldsymbol{C}\oplus]$-witnessed, $\|\varphi\|^{\boldsymbol{C}\oplus}_{\mathbf{M}} > 0$ implies $\|\varphi\|^{\boldsymbol{C}}_{h(\mathbf{M})} > 0$.

For (2) replace $>$ by $=$. For (3) observe that, similarly to the above and under the respective witnessedness, $\|\varphi\|^{\boldsymbol{C}}_{\mathbf{M}} = 1$ implies $\|\varphi\|^{\boldsymbol{C}\oplus}_{j(\mathbf{M})} = 1$ and $\|\varphi\|^{\boldsymbol{C}\oplus}_{\mathbf{M}} = 1$ implies $\|\varphi\|^{\boldsymbol{C}}_{h(\mathbf{M})} = 1$, □

THEOREM 5.1.8. *For $\bullet$ being* 'wst' *or* 'wgen'*, and each continuous t-norm $*$,*

(1) $\varphi \in \bullet\mathrm{TAUT}_{\mathrm{pos}}(\mathrm{L}_*\forall)$ *iff* $\neg\varphi \notin \bullet\mathrm{SAT}(\mathrm{L}_*\forall)$,

(2) $\varphi \in \bullet\mathrm{SAT}_{\mathrm{pos}}(\mathrm{L}_*\forall)$ *iff* $\neg\varphi \notin \bullet\mathrm{TAUT}(\mathrm{L}_*\forall)$.

Proof. This is true since $\|\neg\varphi\|^{\boldsymbol{A}}_{\mathbf{M}} = 1$ iff $\|\varphi\|^{\boldsymbol{A}}_{\mathbf{M}} = 0$, for any BL-chain $\boldsymbol{A}$. □

	wstTAUT	wstSAT	$\mathrm{wstTAUT_{pos}}$	$\mathrm{wstSAT_{pos}}$
Ł∀	$\mathbf{\Pi_2}$-complete	$\mathbf{\Pi_1}$-complete	$\mathbf{\Sigma_1}$-complete	$\mathbf{\Sigma_2}$-complete
G∀	$\mathbf{\Sigma_1}$-complete	$\mathbf{\Pi_1}$-complete	$\mathbf{\Sigma_1}$-complete	$\mathbf{\Pi_1}$-complete
Π∀	$\mathbf{\Pi_2}$-hard	$\mathbf{\Pi_1}$-complete	$\mathbf{\Sigma_1}$-complete	$\mathbf{\Pi_1}$-complete
(Ł⊕)∀	$\mathbf{\Pi_2}$-hard	$\mathbf{\Pi_1}$-complete	$\mathbf{\Sigma_1}$-complete	$\mathbf{\Sigma_2}$-complete
(G⊕)∀	$\mathbf{\Sigma_1}$-hard	$\mathbf{\Pi_1}$-complete	$\mathbf{\Sigma_1}$-complete	$\mathbf{\Pi_1}$-complete
(Π⊕)∀	$\mathbf{\Sigma_2}$-hard	$\mathbf{\Pi_1}$-complete	$\mathbf{\Sigma_1}$-complete	$\mathbf{\Pi_1}$-complete

Table 5. Complexity of standard witnessed semantics.

COROLLARY 5.1.9. *For each continuous t-norm* $*$*,*

(1) $\mathrm{wgenTAUT_{pos}}(\mathrm{L}_*\forall)$ *is* $\mathbf{\Sigma_1}$*-complete.*

(2) $\mathrm{wgenSAT_{pos}}(\mathrm{L}_*\forall)$ *is* $\mathbf{\Pi_1}$*-complete.*

Proof. By the preceding theorem and Theorem 5.1.4. □

COROLLARY 5.1.10.

(1) If $\mathrm{wstSAT}(\mathrm{L}_*\forall) = \mathrm{wgenSAT}(\mathrm{L}_*\forall)$*, then*
$$\mathrm{wstTAUT_{pos}}(\mathrm{L}_*\forall) = \mathrm{wgenTAUT_{pos}}(\mathrm{L}_*\forall).$$

(2) If $\mathrm{wstTAUT}(\mathrm{L}_*\forall) = \mathrm{wgenTAUT}(\mathrm{L}_*\forall)$*, then*
$$\mathrm{wstSAT_{pos}}(\mathrm{L}_*\forall) = \mathrm{wgenSAT_{pos}}(\mathrm{L}_*\forall).$$

THEOREM 5.1.11. *The set* $\mathrm{wstTAUT}(\text{Ł}\forall)$ *recursively reduces to* $\mathrm{wstTAUT}((\text{Ł}\oplus)\forall)$*.*

Proof. Let φ be an arbitrary sentence. If $\mathbf{M}$ is a witnessed Ł-model with $\|\varphi\|^{\text{Ł}}_{\mathbf{M}} < 1$ then for the mapping f as above, $f(\mathbf{M})$ is a witnessed (Ł⊕)-model with $\|\neg\neg\varphi\|^{\text{Ł}\oplus}_{f(\mathbf{M})} < 1$. On the other hand, $\mathbf{M}$ is a witnessed (Ł⊕)-model with $\|\neg\neg\varphi\|^{\text{Ł}\oplus}_{\mathbf{M}} < 1$ then $h(\mathbf{M}^{\neg\neg})$ is a witnessed Ł-model in which the value of φ is < 1. Thus the mapping assigning to each φ its double negation is the claimed reduction. □

The complexity results for the witnessed semantics we can obtain are collected in Table 5. Let us justify them. First, the values for witnessed Ł∀ (first row) are the same as for Ł∀ (without assuming witnessed) due to Fact 1 above. Now for the first column: For G∀ see Theorem 5.1.4; for Π∀ see [16]. For (Ł⊕)∀ see Theorem 5.1.11. For (G⊕)∀ and (Π⊕)∀ the only thing we know is by 2.0.13; it is a problem what more can be shown. Theorem 5.1.11 gives also the rest for Ł⊕; for wstSAT and $\mathrm{wstSAT_{pos}}$ the results follows by Theorem 5.1.5 (1) and for $\mathrm{wstTAUT_{pos}}$ by Theorem 5.1.5 (2).

5.2 Semantics of models over complete chains

In this subsection we present an overview of results about complexity of the semantics over complete signs (for more details see [21]). We start from a very simple observation. Let L be a (recursively axiomatizable) axiomatic extension of MTL such that the MacNeille completion of every L-chain is an L-chain. Then every L-chain can be σ-embedded into a complete L-chain. Hence, $\mathrm{L}\forall$ is strongly complete with respect to the class of complete L-chains. Let $\mathrm{complTAUT}(\mathrm{L}\forall)$ denote the set of all sentences which are valid in every complete L-chain, and let $\mathrm{complTAUT}_{\mathrm{pos}}(\mathrm{L}\forall)$, $\mathrm{complSAT}(\mathrm{L}\forall)$, $\mathrm{complSAT}_{\mathrm{pos}}(\mathrm{L}\forall)$ be defined analogously.

THEOREM 5.2.1. *Let* L *be a (recursively axiomatizable) axiomatic extension of* MTL *such that the MacNeille completion of every* L*-chain is an* L*-chain. Then:*

(1) $\mathrm{complTAUT}(\mathrm{L}\forall) = \mathrm{genTAUT}(\mathrm{L}\forall)$*, and hence it is* $\mathbf{\Sigma_1}$*-complete.*

(2) $\mathrm{complTAUT}_{\mathrm{pos}}(\mathrm{L}\forall) = \mathrm{genTAUT}_{\mathrm{pos}}(\mathrm{L}\forall)$*, and hence it is* $\mathbf{\Sigma_1}$*-complete.*

(3) $\mathrm{complSAT}(\mathrm{L}\forall) = \mathrm{genSAT}(\mathrm{L}\forall)$*, and hence it is* $\mathbf{\Pi_1}$*-complete.*

(4) $\mathrm{complSAT}_{\mathrm{pos}}(\mathrm{L}\forall) = \mathrm{genSAT}_{\mathrm{pos}}(\mathrm{L}\forall)$*, and hence it is in* $\mathbf{\Pi_1}$*.*

The theorem applies to several prominent fuzzy logics, including the following: MTL, IMTL, SMTL, NM, or $\mathrm{C}_n\mathrm{BL}$ for $n > 0$ (i.e. BL plus the n-contraction schema $\phi^n \to \phi^{n+1}$).

We now investigate the first-order logics of complete chains of extensions of BL. First, we prove the non-arithmeticity of sets of the form $\mathrm{TAUT}(\mathbb{K})$ or $\mathrm{TAUT}_{\mathrm{pos}}(\mathbb{K})$ or $\mathrm{SAT}(\mathbb{K})$ or $\mathrm{SAT}_{\mathrm{pos}}(\mathbb{K})$, where $\mathbb{K}$ is a set of complete BL-chains which either contains an infinite product chain or contains a BL-chain of the form $\boldsymbol{B}\oplus\boldsymbol{C}$ where $\boldsymbol{B}$ is an infinite product chain and $\boldsymbol{C}$ is a BL-chain. In this case, we will say that $\mathbb{K}$ *does not exclude* Π. Moreover, we will give a characterization of recursively axiomatizable extensions L of BL such that set of first-order sentences valid in all complete L-chains is recursively axiomatizable. Namely, we will prove that if L is a recursively axiomatizable schematic extensions of BL and $\mathbb{K}$ is the class of complete L-chains, then $\mathrm{TAUT}(\mathbb{K})$ is recursively axiomatizable iff L proves the n-contraction schema $\phi^n \to \phi^{n+1}$ for some n. We start from the non-arithmeticity results.

THEOREM 5.2.2. *Suppose that* $\mathbb{K}$ *is a class of complete* BL*-chains which does not exclude* Π*. Then the followings sets are non-arithmetical:* $\mathrm{TAUT}(\mathbb{K})$*,* $\mathrm{TAUT}_{\mathrm{pos}}(\mathbb{K})$*,* $\mathrm{SAT}(\mathbb{K})$*, and* $\mathrm{SAT}_{\mathrm{pos}}(\mathbb{K})$*.*

Proof. The proof is similar to the proofs of Theorem 3.0.23 and of Theorem 3.0.28, and hence we will only point out the parts where the proofs diverge. Let U, $^{\neg\neg}$, θ_1, θ_2, θ_3 and θ_4 be as in the proof of Theorem 3.0.23. As in the proofs of Theorems 3.0.23 and 3.0.28, it suffices to prove:

Claim: Let ψ be any *PA*-sentence, and let $\theta = (\theta_1)^2 \mathbin{\&} \theta_2 \mathbin{\&} \theta_3 \mathbin{\&} \theta_4$. (Note that we have replaced θ_1 by θ_1^2 for reasons that will become clear later). Then:

(1) If $\mathbf{N} \models \psi$, then $\theta \to \psi^{\neg\neg} \in \mathrm{TAUT}(\mathbb{K})$, and if $\mathbf{N} \not\models \psi$, then $\theta \mathbin{\&} \psi^{\neg\neg} \notin \mathrm{SAT}_{\mathrm{pos}}(\mathbb{K})$.

(2) If $\mathbf{N} \models \psi$, then $\theta \mathbin{\&} \psi^{\neg\neg} \in \mathrm{SAT}(\mathbb{K})$ and if $\mathbf{N} \not\models \psi$, then $\theta \to \psi^{\neg\neg} \notin \mathrm{TAUT}_{\mathrm{pos}}(\mathbb{K})$.

Proof of the claim: (1) Let $\boldsymbol{A} \in \mathbb{K}$ and $\mathbf{M}$ be any $\boldsymbol{A}$-structure. Let us write $\|\dots\|$ instead of $\|\dots\|^{\boldsymbol{A}}_{\mathbf{M}}$. The claim is clear if $\|\theta\| = 0$, and hence assume $\|\theta\| > 0$. We first prove that $\boldsymbol{A}$ is either an infinite product chain or is the ordinal sum of an infinite product chain and a BL-chain. Consider $a = \|(\forall x)U(x)\|$. Then a must belong to the first Wajsberg component, $\boldsymbol{W}$, of $\boldsymbol{A}$, otherwise $\|\theta_1\| = 0$.

We now prove that W has only two elements. Suppose not. If for some $d \in M$, $\|U(d)\| \in W \setminus \{0,1\}$, then $\|(\forall x)\neg\neg U(x)\| = \|(\forall x)U(x)\|$. Hence, $\|\theta_1\| = 0$, and $\|\theta\| = 0$. The same is true if for some $d \in M$, $\|U(d)\| = 0$. It remains to consider the case where for all $d \in M$, either $\|U(d)\| \notin W$ or $\|U(d)\| = 1$, but $\|(\forall x)U(x)\| \in W \setminus \{1\}$. But in this case, $\|(\forall x)U(x)\|$ is the unique coatom, a, of $\boldsymbol{W}$, $\|\neg(\forall x)U(x)\|$ is the unique atom of $\boldsymbol{W}$, and $(\neg a)^2 = 0$. Hence $\|\theta_1^2\| = 0$ (here is the place where we need θ_1^2). Hence, $\boldsymbol{A}$ is an SBL-algebra.

It follows that $\inf\{\|U(d)\| \mid d \in M\} = 0$ and for all $d \in M$, $\|U(d)\| > 0$ (otherwise, $\|\theta_1\| = 0$). Using $\|\theta_2\| > 0$, we can derive that $\boldsymbol{A}$ is either an infinite product chain or the ordinal sum of an infinite product chain and a BL-chain, by an argument which is very similar to the one used in the proof of Lemma 3.0.24.

We now can prove an analogue of Lemma 3.0.25. That is:

(a) For every sentence γ of PA, $\|\gamma^{\neg\neg}\| \in \{0,1\}$.

(b) Since $\|\theta_3\| > 0$, by (1) we have $\|\theta_3\| = 1$. Moreover we can obtain as in Lemma 3.0.25 a model $\mathbf{M}^{\neg\neg}$ from $\mathbf{M}$ such that for every sentence γ of PA, one has $\mathbf{M}^{\neg\neg} \models \gamma$ iff $\|\gamma^{\neg\neg}\| = 1$. Hence, $\mathbf{M}^{\neg\neg} \models Q^+$.

(c) We can prove as in Lemma 3.0.25 that $\mathbf{M}^{\neg\neg}$ is isomorphic to $\mathbf{N}$. Analogously to Lemma 3.0.25, we can see that there is a $c \in M$ such that $\|U(c)\| < 1$ and for all $d \in M$, if $\mathbf{M}^{\neg\neg} \models c \le d$, then $\|U(S(d))\| \le \|(\forall x)(x \le^{\neg\neg} d \to U(x))\|^2$. It follows $\|U(c+n)\| \le \|U(c)\|^{2^n}$. Now the first component of $\boldsymbol{A}$ is a complete product chain, and in a complete product chain, if $a < 1$, then $\inf\{a^{2^n} \mid n \in \mathsf{N}\} = 0$. Hence, the proof proceeds as in Lemma 3.0.25.

At this point, since $\mathbf{N} \models \psi$ and $\|\theta\| > 0$, we must have $\|\psi^{\neg\neg}\| = 1$, and also $\|\theta \to \psi^{\neg\neg}\| = 1$. Therefore, if $\mathbf{N} \models \psi$, then $\theta \to \psi^{\neg\neg} \in \mathrm{TAUT}(\mathbb{K})$, and hence $\theta \to \psi^{\neg\neg} \in \mathrm{TAUT}_{\mathrm{pos}}(\mathbb{K})$. Moreover, if $\mathbf{N} \not\models \psi$, then $\|\theta\| > 0$ implies $\|\psi^{\neg\neg}\| = 0$, and $\theta \,\&\, \psi^{\neg\neg} \notin \mathrm{SAT}_{\mathrm{pos}}(\mathbb{K})$. *A fortiori*, $\theta \,\&\, \psi^{\neg\neg} \notin \mathrm{SAT}_{\mathrm{pos}}(\mathbb{K})$.

(2) Let $\boldsymbol{A} \in \mathbb{K}$ be either an infinite product chain or the ordinal sum of an infinite product chain and a BL-chain. Let $a \in A$ be such that $0 < a < 1$ and a is in the first product component of $\boldsymbol{A}$. Take an $\boldsymbol{A}$-structure $\mathbf{M}$ whose universe is N and whose constants and function symbols are interpreted as in $\mathbf{N}$ (if we do not want function symbols, we may replace them by predicates as usual). Moreover, for every n-ary predicate P and for $d_1, \dots, d_n \in \mathsf{N}$ we define $P_{\mathbf{M}}(d_1, \dots, d_n) = 1$ if $\mathbf{N} \models P(d_1, \dots, d_n)$ and $P_{\mathbf{M}}(d_1, \dots, d_n) = 0$ otherwise. Moreover let for all $n \in \mathsf{N}$, $U_{\mathbf{M}}(n) = a^{3^n}$. It is readily seen that $\|\theta\|^{\boldsymbol{A}}_{\mathbf{M}} = 1$ and $\|\psi^{\neg\neg}\|^{\boldsymbol{A}}_{\mathbf{M}} = 1$ if $\mathbf{N} \models \psi$ and $\|\psi^{\neg\neg}\|^{\boldsymbol{A}}_{\mathbf{M}} = 0$ otherwise. Hence, if $\mathbf{N} \models \psi$, then $\theta \,\&\, \psi^{\neg\neg} \in \mathrm{SAT}(\mathbb{K})$, and *a fortiori* $\theta \,\&\, \psi^{\neg\neg} \in \mathrm{SAT}_{\mathrm{pos}}(\mathbb{K})$. Moreover if $\mathbf{N} \not\models \psi$, then $\theta \to \psi \notin \mathrm{TAUT}_{\mathrm{pos}}(\mathbb{K})$, and *a fortiori* $\theta \to \psi \notin \mathrm{TAUT}(\mathbb{K})$. □

COROLLARY 5.2.3. *For each logic* $\mathrm{L} \in \{\Pi, \mathrm{BL}, \mathrm{SBL}\}$*, the following sets of sentences are not arithmetical:* $\mathrm{complSAT}(\mathrm{L}\forall)$*,* $\mathrm{complSAT}_{\mathrm{pos}}(\mathrm{L}\forall)$*,* $\mathrm{complTAUT}(\mathrm{L}\forall)$*, and* $\mathrm{complTAUT}_{\mathrm{pos}}(\mathrm{L}\forall)$.

The next theorem, which describes the complexity of satisfiability and of tautologicity problems for Łukasiewicz logic with respect to the class of complete MV-chains, is easy to prove:

THEOREM 5.2.4.

(i) $\mathrm{complTAUT}(\text{Ł}\forall) = \mathrm{stTAUT}(\text{Ł}\forall)$*, and so it is* $\mathbf{\Pi_2}$*-complete.*

(ii) $\mathrm{complSAT}(\text{Ł}\forall) = \mathrm{stSAT}(\text{Ł}\forall)$*, and so it is* $\mathbf{\Pi_1}$*-complete.*

(iii) $\mathrm{complTAUT}_{\mathrm{pos}}(\text{Ł}\forall) = \mathrm{stTAUT}_{\mathrm{pos}}(\text{Ł}\forall)$*, and so it is* $\mathbf{\Sigma_1}$*-complete.*

(iv) $\mathrm{complSAT}_{\mathrm{pos}}(\text{Ł}\forall) = \mathrm{stSAT}_{\mathrm{pos}}(\text{Ł}\forall)$*, and so it is* $\mathbf{\Sigma_2}$*-complete.*

Proof. Every complete MV-chain is either isomorphic to $[0,1]_{\text{Ł}}$ or a finite MV-chain, and hence it is a complete subalgebra of $[0,1]_{\text{Ł}}$. Hence, the semantics based on the class of complete MV-chains is equivalent to the standard semantics for predicate Łukasiewicz logic. □

We now characterize the schematic extensions L of BL such that $\mathrm{complTAUT}(\mathrm{L}\forall)$ is recursively axiomatizable.

THEOREM 5.2.5. *Let* L *be a recursively axiomatizable schematic extension of* BL*, and assume that* L*, as a propositional logic, is complete with respect to the class of complete* L*-chains. Then the following are equivalent:*

(1) For some natural number $n \geq 1$*,* L *proves the* n*-contraction schema* $\phi^n \to \phi^{n+1}$.

(2) $\mathrm{complTAUT}(\mathrm{L}\forall)$ *is recursively axiomatizable.*

Proof. $(1) \Rightarrow (2)$ Follows from Theorem 5.2.1, since in Chapter V it is proved that for any variety $\mathbb{V}$ of $\mathrm{C}_n\mathrm{BL}$-chains, the class of all chains in $\mathbb{V}$ is closed under MacNeille completions.

$(2) \Rightarrow (1)$ Let $\mathbb{V}$ be the variety equivalent to L, and assume that for no natural number n, L proves the n-contraction schema. Then (see Chapter V) either $\mathbb{V}$ contains all finite MV-chains, and hence it contains $[0,1]_{\text{Ł}}$, or it contains an infinite product chain. In the latter case, it contains $[0,1]_{\Pi}$, and by Theorem 5.2.2, $\mathrm{complTAUT}(\mathrm{L}\forall)$ is not arithmetical. In the former case, (2) follows from the following claim.

Claim: Let $\mathbb{K}$ be a class of complete BL-chains. If $[0,1]_{\text{Ł}} \in \mathbb{K}$, then $\mathrm{TAUT}(\mathbb{K})$ is $\mathbf{\Pi_2}$-hard.

Proof of the claim: Let, for every formula ψ, $\psi^{\neg\neg}$ be the formula obtained by replacing in ψ every atomic subformula γ by $\neg\neg\gamma$. We claim that for every sentence ψ we have $\psi \in \mathrm{stTAUT}(\text{Ł}\forall)$ iff $\neg\neg\psi^{\neg\neg} \in \mathrm{TAUT}(\mathbb{K})$. This clearly implies the claim of the lemma and hence of Theorem 5.2.5.

If $\neg\neg\psi^{\neg\neg} \in \mathrm{TAUT}(\mathbb{K})$, then since $[0,1]_{\text{Ł}} \in \mathbb{K}$, we have $\neg\neg\psi^{\neg\neg} \in \mathrm{stTAUT}(\text{Ł}\forall)$, and $\psi \in \mathrm{stTAUT}(\text{Ł}\forall)$, as $\vdash_{\text{Ł}} \psi \leftrightarrow \neg\neg\psi^{\neg\neg}$.

Conversely, consider any complete BL-chain $\boldsymbol{A}$. Then $\boldsymbol{A}$ has a complete first Wajsberg component $\boldsymbol{W}$, which is necessarily a complete Wajsberg algebra, and hence it can be σ-embedded into $[0,1]_{Ł}$. Moreover for every $\boldsymbol{A}$-structure $\mathbf{M}$, let us define $\mathbf{M}'$ as follows: $\mathbf{M}'$ has the same domain as $\mathbf{M}$ and the constants and the function symbols are interpreted as in $\mathbf{M}$; moreover for every n-ary predicate P and for every $a_1, \ldots, a_n \in M$, $P_{\mathbf{M}'}(a_1, \ldots, a_n) = \|\neg\neg P(a_1, \ldots, a_n)\|$. Then we can prove by induction on ψ, that for every sentence ψ, the following conditions hold:

(1) If $\sup(W \setminus \{1\}) \in W$, then $\|\psi^{\neg\neg}\|^{\boldsymbol{A}}_{\mathbf{M}} = \|\psi\|^{\boldsymbol{W}}_{\mathbf{M}'}$.
(2) If $\sup(W \setminus \{1\}) \notin W$ and $\|\psi\|^{\boldsymbol{W}}_{\mathbf{M}'} < 1$, then $\|\psi\|^{\boldsymbol{W}}_{\mathbf{M}'} = \|\psi^{\neg\neg}\|^{\boldsymbol{A}}_{\mathbf{M}}$.
(3) If $\sup(W \setminus \{1\}) \notin W$ and $\|\psi\|^{\boldsymbol{W}}_{\mathbf{M}'} = 1$, then either $\|\psi^{\neg\neg}\|^{\boldsymbol{A}}_{\mathbf{M}} = 1$ or $\|\psi^{\neg\neg}\|^{\boldsymbol{A}}_{\mathbf{M}} = \sup(W \setminus \{1\})$.

In any case, we have $\|\neg\neg\psi^{\neg\neg}\|^{\boldsymbol{A}}_{\mathbf{M}} = \|\psi\|^{\boldsymbol{W}}_{\mathbf{M}'}$. Hence, if $\psi \in \mathrm{stTAUT}(Ł\forall)$, then $\neg\neg\psi^{\neg\neg} \in \mathrm{TAUT}(\mathbb{K})$. □

5.3 Fragments with implication and negation

Here we shall understand that a logic $\mathrm{L}_1\forall$ is a fragment of the logic $\mathrm{L}_2\forall$ given by the subset C of connectives of $\mathrm{L}_2\forall$ if $\mathrm{L}_1\forall$-formulae are $\mathrm{L}_2\forall$-formulae not containing any connective which is not in C, and each $\mathrm{L}_1\forall$-formula φ is $\mathrm{L}_1\forall$-provable iff it is $\mathrm{L}_2\forall$-provable. If this is the case then for each $\mathrm{L}_1\forall$-theory T and formula φ it follows that T proves φ over $\mathrm{L}_1\forall$ iff T proves φ over $\mathrm{L}_2\forall$; moreover each $\mathrm{L}_1\forall$-algebra (chain) is an $\mathrm{L}_2\forall$-algebra (chain). We are going to discuss fragments of some extensions of $\mathrm{BL}\forall$ given by the connectives $\to, \neg$ (equivalently, $\to, \overline{0}$) and by all connectives except $\overline{0}$ (and of course except $\neg$)—the hoop logics.

We shall discuss logics extending $\mathrm{BL}\forall$, particularly Łukasiewicz, Gödel, product logic, $\mathrm{SBL}\forall$ and $\mathrm{BL}\forall$ itself. For such a logic $\mathrm{L}\forall$ we denote its fragment given by the connectives $\to, \neg$ as $\mathrm{L}\forall\restriction(\to, \neg)$. These logics are recursively axiomatized, i.e. the set $\mathrm{genTAUT}(\mathrm{L}\forall\restriction(\to, \neg))$ of general tautologies is $\boldsymbol{\Sigma_1}$ and the set $\mathrm{genSAT}(\mathrm{L}\forall\restriction(\to, \neg))$ of general satisfiable formulae (= consistent formulae) is $\boldsymbol{\Pi_1}$.

To prove hardness, recall that each $Ł\forall$-formula is equivalent to some $Ł\forall\restriction(\to, \neg)$-formula; thus the set $\mathrm{genTAUT}(Ł\forall)$ recursively reduces to $\mathrm{genTAUT}(Ł\forall\restriction(\to, \neg))$ and similarly for genSAT (as well as for stTAUT, stSAT, as we shall need later).

For Gödel logic the set $\mathrm{SAT}(\boldsymbol{B}_2\restriction(\to, \neg))$ of classically satisfiable formulae with connectives only $\to, \neg$ recursively reduces to $\mathrm{genSAT}(\mathrm{G}\forall\restriction(\to, \neg))$ by the mapping associating to each formula φ the formula $\varphi^{\neg\neg}$ resulting from φ by adding double negation to each atomic subformula; analogously for $\mathrm{genTAUT}$.

This works also for any logic with strict negation. On the other hand, for a logic $\mathrm{L}_*\forall$ given by any continuous t-norm $*$ whose first component is Łukasiewicz observe that the mapping sending any φ to $\varphi^{\neg\neg}$ recursively reduces $\mathrm{genSAT}(Ł\forall\restriction(\to, \neg))$ to $\mathrm{genSAT}(\mathrm{L}_*\forall\restriction(\to, \neg))$ and analogously for $\mathrm{genTAUT}$.

LEMMA 5.3.1.
The set $\mathrm{genTAUT}(\mathrm{BL}\forall\restriction(\to, \neg))$ *is* $\boldsymbol{\Sigma_1}$*-hard and* $\mathrm{genSAT}(\mathrm{BL}\forall\restriction(\to, \neg))$ *is* $\boldsymbol{\Pi_1}$*-hard.*

Proof. Let $\varphi^{\neg}$ result from φ by replacing each atom by its negation. Recall that for each sentence φ not containing the existential quantifier, φ is a general tautology of

Ł∀ iff $\varphi^{\neg}$ is a general tautology of BL∀ [12]. One can prove in the same way that φ is generally satisfiable in Ł∀ iff $\varphi^{\neg}$ is generally satisfiable in BL∀. Clearly if φ is a $(\rightarrow, \overline{0})$-formula then so is $\varphi^{\neg}$. The set of all $(\rightarrow, \overline{0})$-sentences not containing $\exists$ that are general Ł∀ tautologies is $\mathbf{\Sigma_1}$-complete and the set of all such formulae that are generally Ł∀-satisfiable is $\mathbf{\Pi_1}$-complete (since in Ł∀ each sentence is equivalent to a $(\rightarrow, \overline{0})$-sentence not containing $\exists$). This gives the result. □

We focus now on the standard semantics. In the case of first-order Łukasiewicz logic we have that $\mathrm{stTAUT}(Ł\forall)$ recursively reduces to $\mathrm{stTAUT}(Ł\forall\restriction(\rightarrow, \neg))$ and conversely; thus the set $\mathrm{stTAUT}(Ł\forall\restriction(\rightarrow, \neg))$ is $\mathbf{\Pi_2}$-complete. Similarly to genSAT, we see that the set $\mathrm{stSAT}(Ł\forall\restriction(\rightarrow, \neg))$ is $\mathbf{\Pi_1}$-complete.

Finally we survey some results on non-arithmeticity. We shall only sketch their proofs since they are rather laborious; for details see [18] and references thereof.

DEFINITION 5.3.2. *For a set $\mathbb{K}$ of continuous t-norms, we denote the set of all finite sets Σ of predicate $(\rightarrow, \overline{0})$-formulae that are standardly $\mathbb{K}$-satisfiable (i.e. for some t-norm $*$ from $\mathbb{K}$ there is a standard interpretation $\mathbf{M}$ with $\|\alpha\|^*_{\mathbf{M}} = 1$ for each $\alpha \in \Sigma$) by $\mathrm{stSAT}_{(f)}(\mathbb{K}\forall\restriction(\rightarrow, \overline{0}))$. By $\|\Sigma\|^*_{\mathbf{M}}$ we denote the minimum of the values $\|\alpha\|^*_{\mathbf{M}}$ for $\alpha \in \Sigma$.*

THEOREM 5.3.3. *For each set $\mathbb{K}$ of continuous t-norms containing the product t-norm, the set $\mathrm{stSAT}_{(f)}(\mathbb{K}\forall\restriction(\rightarrow, \overline{0}))$ is not arithmetical.*

Let us sketch the proof. Recall the notation $\varphi^{\neg\neg}$ for the formula resulting from φ by putting double negation to all atomic subformulae of φ; for a set Ψ of formulae, $\Psi^{\neg\neg}$ is the set $\{\varphi^{\neg\neg} | \varphi \in \Psi\}$. For a standard interpretation $\mathbf{M}$, let $\mathbf{M}^{\neg\neg}$ result from $\mathbf{M}$ by replacing each positive value by 1. Clearly, if $*$ is a continuous t-norm with Gödel negation then for each standard interpretation $\mathbf{M}$ and each formula φ we have $\|\varphi^{\neg\neg}\|^*_{\mathbf{M}} = \|\varphi\|^{\boldsymbol{B}_2}_{\mathbf{M}^{\neg\neg}}$.

Now recall the finite axiomatic system Q^+ (see Introduction). Since it is a system in classical logic we may assume that all axioms are $(\rightarrow, \overline{0})$-formulae (other connectives being classically definable). Evidently, if $*$ has Gödel negation then for each (standard) interpretation $\mathbf{M}$, $\|(Q^+)^{\neg\neg}\|^*_{\mathbf{M}} > 0$ iff $\mathbf{M}^{\neg\neg}$ is a classical model of Q^+ (possibly with non-absolute equality—you may factorize).

DEFINITION 5.3.4. *Let U be a unary predicate distinct from the symbols of Q^+. We introduce the following axioms:*

Ψ_0 $\quad \neg(\forall x)U(x)$

Ψ_1 $\quad (\forall x)(\forall y)(\forall z)[((U(x) \rightarrow U(y)) \rightarrow U(y)) \rightarrow$
$[(U(y) \rightarrow U(z)) \rightarrow ((U(y) \rightarrow U(x)) \rightarrow U(x)) \rightarrow ((U(y) \rightarrow U(x)) \rightarrow U(x))]]$

Ψ_2 $\quad (\forall x)\neg\neg U(x)$

Ψ_3 $\quad (\forall x)((U(x) \rightarrow U(S(x))) \rightarrow U(x))$ $\qquad$ (*S is the successor from Q^+*)

Ψ_4 $\quad (\forall x)(\forall y)(\forall z)(\neg\neg(x \leq y) \rightarrow (U(y) \rightarrow U(x))$. $\qquad$ (*$<$ from Q^+*))

Finally, Ψ will stand for the set $\{\Psi_i \; i = 0, \ldots, 4\} \cup (Q^+)^{\neg\neg}$

Note that all formulae in Ψ are $(\rightarrow, \overline{0})$-formulae.

LEMMA 5.3.5. *Let Φ be a $(\rightarrow, \overline{0})$-formula of arithmetic. Φ is true in the standard model $\mathbf{N}$ of arithmetic iff the finite set $\Psi \cup \{\Phi^{\neg\neg}\}$ is standardly $\mathrm{L}_{\mathbb{K}}\forall\restriction(\rightarrow, \neg)$-satisfiable.*

	Ł	G	Π	SBL	BL
genTAUT	$\mathbf{\Sigma_1}$-compl.	$\mathbf{\Sigma_1}$-compl.	$\mathbf{\Sigma_1}$-compl.	$\mathbf{\Sigma_1}$-compl.	$\mathbf{\Sigma_1}$-compl.
genSAT	$\mathbf{\Pi_1}$-compl.	$\mathbf{\Pi_1}$-compl.	$\mathbf{\Pi_1}$-compl.	$\mathbf{\Pi_1}$-compl.	$\mathbf{\Pi_1}$-compl.
stTAUT	$\mathbf{\Pi_2}$-compl.	$\mathbf{\Sigma_1}$-compl.	NA	NA	NA
stSAT	$\mathbf{\Pi_1}$-compl.	$\mathbf{\Pi_1}$-compl.	NA	NA	NA

Table 6. Arithmetical complexity of $(\to, \neg)$-fragments.

The theorem follows: the preceding lemma gives a recursive reduction of the set of $(\to, \neg)$-formulae true in the standard model to our set $\mathrm{stSAT}_{(f)}(\mathrm{L}_{\mathbb{K}}\forall\!\upharpoonright\!(\to, \overline{0}))$.

THEOREM 5.3.6. *The set of standard tautologies of* $\Pi\forall\!\upharpoonright\!(\to, \overline{0})$ *is not arithmetical.*

We again sketch a proof. Here we shall use Ψ_0 and Ψ_2 as well as Q^+ defined above. Let Ψ_3' be the axiom

$$(\forall x))(U(S(x)) \lhd (\forall z)(\neg\neg(z < x) \to U(z)))$$

where $\alpha \lhd \beta$ is $(\beta \to (\beta \to \alpha)) \to \beta$. Ψ will be the finite set $\{\Psi_0, \Psi_2, \Psi_3'\} \cup (Q^+)^{\neg\neg}$. And let us agree that if T is a finite set $\{\alpha_1, \ldots, \alpha_n\}$ of formulae and β is a formula then $T \to \beta$ means $(\alpha_1 \to (\alpha_2 \to \ldots (\alpha_n \to \beta) \ldots))$.

LEMMA 5.3.7. *Let* Φ *be a sentence of arithmetic.* $\mathbf{N} \models \Phi$ *iff the formula* $\Psi \to \Phi^{\neg\neg}$ *is a standard tautology of* $\Pi\forall\!\upharpoonright\!(\to, \neg)$.

This shows that the function assigning to each $(\to, \overline{0})$-sentence Φ of arithmetic the sentence $\Psi \to \Phi^{\neg\neg}$ recursively reduces the non-arithmetical set of $(\to, \overline{0})$-sentences true in $\mathbf{N}$ to the set of standard tautologies of $\Pi\forall\!\upharpoonright\!(\to, \neg)$. This completes the (sketch of a) proof of Theorem 5.3.6.

THEOREM 5.3.8. *For each set* $\mathbb{K}$ *of continuous t-norms containing the product t-norm, the set of standard tautologies of the logic* $\mathrm{L}_{\mathbb{K}}\forall\!\upharpoonright\!(\to, \neg)$ *is non-arithmetical.*

For the proof sketch, we use the notation: $\varphi \uparrow \psi$ stands for $(\varphi \to \psi) \to \psi$. Θ is the set $\{\Psi_0, \Psi_2, \Sigma\}$ of formulae where Ψ_0, Ψ_2 is as above (i.e. $(\forall x)\neg\neg U(x)$, $\neg(\forall x)U(x)$) and Σ is the formula $(\forall x, y)((U(x) \uparrow U(y)) \to (U(y) \uparrow U(x)))$. Now, for each $(\to, \overline{0})$-formula Φ not containing the predicate U take the pair $\{(\Theta \to \Phi), (\exists x)(\Phi \uparrow U(x))\}$. Observe that both formulae are $(\to, \overline{0})$-formulae. Their disjunction will be denoted by $\Phi^{\#}$. This is not a $(\to, \overline{0})$-formula, but to say for some α, β that $\alpha \vee \beta$ is a tautology is the same as to say that $(\alpha \uparrow \beta) \wedge (\beta \uparrow \alpha)$ is a tautology and this is to same as say that both $\alpha \uparrow \beta$ and $\beta \uparrow \alpha$ are tautologies. One shows that under the assumptions of the theorem a $(\to, \overline{0})$-formula Φ not containing the predicate U is a $\Pi\forall$-tautology iff $\Phi^{\#}$ is an $\mathrm{L}_{\mathbb{K}}\forall$-tautology hence if the corresponding two $(\to, \overline{0})$-formulae are $\mathrm{L}_{\mathbb{K}}\forall\!\upharpoonright\!(\to, \overline{0})$-tautologies. This recursively reduces the (non-arithmetical) set $\mathrm{stTAUT}(\Pi\forall\!\upharpoonright\!(\to, \overline{0}))$ to the set $\mathrm{stTAUT}(\mathrm{L}_{\mathbb{K}}\forall\!\upharpoonright\!(\to, \overline{0}))$ which gives a proof of Theorem 5.3.8.

The results are summarized in the Table 6. One may also consult [7].

5.4 Complexity of hoop logics

This section deals with falsity-free first-order fuzzy logics. They can be obtained in two ways: given a ($\triangle$-)core fuzzy logic L, (1) one can consider $\mathrm{L}\forall^-$, the falsity-free fragment of $\mathrm{L}\forall$, or (2) one can first take L^-, the falsity-free fragment of L, and then consider its first-order version $\mathrm{L}^-\forall$. We are going to prove that in many important cases both logics coincide.

DEFINITION 5.4.1. *Let $\boldsymbol{A}$ be an L^--chain and let $a \in A$. By $\boldsymbol{A} \searrow [a,1]$ we mean the algebra whose domain is $[a,1]$, whose lattice operations and whose implication are the restrictions to $[a,1]$ of the corresponding operations of $\boldsymbol{A}$, and whose monoid operation $*$ is $x * y = (x \cdot y) \vee a$. Moreover by $\boldsymbol{A}^+$ we denote the ordinal sum of the two element MV-chain and $\boldsymbol{A}$. By $\boldsymbol{A}'$ we denote $\boldsymbol{A}^+$ if $\boldsymbol{A}$ has no minimum, and $\boldsymbol{A}$ with 0 interpreted as the minimum of $\boldsymbol{A}$ otherwise.*

The following lemma is easy to demonstrate.

LEMMA 5.4.2.

1. *Let L be one of MTL, BL, Ł, or G, let $\boldsymbol{A}$ be an L^--chain and $a \in A$. Then $\boldsymbol{A} \searrow [a,1]$ is an L-chain.*
2. *Let L be one of SMTL or SBL, and let $\boldsymbol{A}$ be an L^--chain. Then $\boldsymbol{A}^+$ is an L-chain.*
3. *Let L be one of Π, ΠMTL, and let $\boldsymbol{A}$ be an L^--chain. Then $\boldsymbol{A}'$ is an L-chain.*

PROPOSITION 5.4.3. *Let L be an axiomatic extension of MTL such that at least one of the following conditions hold:*

1. *For every L^--chain $\boldsymbol{A}$, and for every $a \in A$, $\boldsymbol{A} \searrow [a,1]$ is an L-chain.*
2. *For every L^--chain $\boldsymbol{A}$, $\boldsymbol{A}^+$ is an L-chain.*
3. *L extends SMTL and for every L^--chain $\boldsymbol{A}$, $\boldsymbol{A}'$ is an L-chain.*

Then $\mathrm{L}\forall^- = \mathrm{L}^-\forall$.

Proof. Assume that 1 holds. Since both logics are finitary it is enough to prove that for every finite set $T \cup \{\phi\}$ of formulae of, $T \vdash_{\mathrm{L}\forall^-} \phi$ iff $T \vdash_{\mathrm{L}^-\forall} \phi$.

For the non-trivial direction, suppose that ϕ is invalidated in a model $\mathbf{M}$ of T on a totally ordered L-semihoop $\boldsymbol{H}$. Now let γ be the conjunction of all universal closures of subformulae of $T \cup \{\phi\}$ and let $a = \|\gamma\|^{\boldsymbol{H}}_{\mathbf{M}}$. Consider $\boldsymbol{H} \searrow [a,1]$ and define a new interpretation $\mathbf{M} \searrow [a,1]$ letting for every n-ary predicate P, $P^{\mathbf{M}\searrow[a,1]}(a_1,\ldots,a_n) = P^{\mathbf{M}}(a_1,\ldots,a_n) \vee a$. It is enough to prove that $\mathbf{M} \searrow [a,1]$ is a safe structure on $\boldsymbol{H} \searrow [a,1]$ which is a model of T but not of φ. Thus, we want to see that for any first-order formula χ in the language with $\overline{0}$, the value $\|\chi\|^{\boldsymbol{H}\searrow[a,1]}_{\mathbf{M}\searrow[a,1]}$ exists for any evaluation. To this end, we define a translation ^{0} by induction as:

$$(P(x_1,\ldots,x_n))^0 = P(x_1,\ldots,x_n) \vee \gamma$$

$$(\varphi \mathbin{\&} \psi)^0 = (\varphi^0 \mathbin{\&} \psi^0) \vee \gamma \qquad (\varphi \to \psi)^0 = \varphi^0 \to \psi^0$$

$$(\varphi \wedge \psi)^0 = \varphi^0 \wedge \psi^0 \qquad (\varphi \vee \psi)^0 = \varphi^0 \vee \psi^0$$

$$\overline{0}^0 = \gamma \qquad ((\forall x)\varphi)^0 = (\forall x)\varphi^0 \qquad ((\exists x)\varphi)^0 = (\exists x)\varphi^0.$$

One can check, by an easy induction, that for every first-order formula χ in the language with $\overline{0}$ and any evaluation v, $\|\chi\|^{\boldsymbol{H}\searrow[a,1]}_{\mathbf{M}\searrow[a,1],v} = \|\chi^0\|^{\boldsymbol{H}}_{\mathbf{M},v}$. Therefore, we have a safe structure. On the other hand, another easy induction shows that for every ψ subformula of $T \cup \{\phi\}$, $\|\psi^0\|^{\boldsymbol{H}}_{\mathbf{M},v} = \|\psi\|^{\boldsymbol{H}}_{\mathbf{M},v}$. Thus, for every $\psi \in T$, $\|\psi\|^{\boldsymbol{H}\searrow[a,1]}_{\mathbf{M}\searrow[a,1],v} = \|\psi^0\|^{\boldsymbol{H}}_{\mathbf{M},v} = \|\psi\|^{\boldsymbol{H}}_{\mathbf{M},v} = 1$ and $\|\phi\|^{\boldsymbol{H}\searrow[a,1]}_{\mathbf{M}\searrow[a,1],v} = \|\phi^0\|^{\boldsymbol{H}}_{\mathbf{M},v} = \|\phi\|^{\boldsymbol{H}}_{\mathbf{M},v} < 1$.

The proof with assumptions 2 and 3 is rather similar, and we only point out the differences with the proof from assumption 1. In the case of 2, we must take $\boldsymbol{H}^+$ and in the case of 3 we must take $\boldsymbol{H}'$. Moreover in both cases we need not change $\mathbf{M}$ and we need not use the translation. Safety follows from the fact that negation is strict and hence negated formulae are evaluated either in 0 or in 1. □

COROLLARY 5.4.4. *Let* L *be any of* MTL*,* BL*,* Ł*,* G*,* SMTL*,* SBL*,* Π *or* ΠMTL*. Then* $\mathrm{L}\forall^- = \mathrm{L}^-\forall$.

We focus on the falsity-free fragments of the four most important fuzzy logics—BL∀, Ł∀, G∀, and Π∀; they are denoted by BLH∀, ŁH∀, GH∀, ΠH∀ respectively. According to the previous corollary, their general semantics is given by bounded hoops, Wajsberg hoops, Gödel hoops and product hoops respectively. The *standard semantics* are the $\overline{0}$-free reducts of the corresponding standard semantics of the original logic, i.e. for BLH∀ all t-norm algebras, for ŁH∀, GH∀, ΠH∀ the corresponding t-norm algebra with the same name.

As proved above, BL∀ is a conservative expansion of BLH∀ and the similarly for the other logics. Thus general tautologies of BLH∀ are just falsity-free general tautologies of BL∀ and the same for Ł, G, Π. For trivial reasons, the same holds for standard tautologies: standard tautologies of BLH∀ are, by definition, just falsity-free standard tautologies of BLH∀ and similarly for Ł, G, Π.

For any logic L∀ of our logics, let stTAUT(L∀) denote the set of all its standard tautologies; for L∀ having $\overline{0}$ let $\mathrm{stTAUT}^+(\mathrm{L}\forall)$ denote the set of all its standard falsity-free tautologies (i.e. not containing $\overline{0}$). Recall that the sets stTAUT(G∀), stTAUT(Ł∀), stTAUT(Π∀), stTAUT(BL∀) are respectively $\mathbf{\Sigma_1}$-complete, $\mathbf{\Pi_2}$-complete, not arithmetical, not arithmetical. Our question is what is the arithmetical complexity of standard tautologies of the hoop logic, or equivalently the complexity of $\mathrm{stTAUT}^+(\mathrm{G}\forall)$, $\mathrm{stTAUT}^+(\text{Ł}\forall)$, $\mathrm{stTAUT}^+(\Pi\forall)$, $\mathrm{stTAUT}^+(\mathrm{BL}\forall)$. We will show that it is the same as the complexity of the corresponding set stTAUT.

To this end, we need to consider again, now for real-valued chains, the constructions used above. Given a t-norm algebra $\boldsymbol{B}$ and $0 \leq a < 1$, we consider the BL-algebra $\boldsymbol{B} \searrow [a, 1]$. Clearly there is a monotone $1-1$ mapping of $[a, 1]$ onto $[0, 1]$ transforms $B \searrow [a, 1]$ into a t-norm algebra. $\boldsymbol{B} \searrow [1, 1]$ is the degenerated one element algebra. On the other hand, given a predicate language $\mathcal{P}$ and $Q(d)$, a closed atomic formula with Q, d not belonging to $\mathcal{P}$, we define $\gamma = Q(d)$ and for each formula φ of the language

$\mathcal{P}$ define the formula φ^0 as in the proof of Proposition 5.4.3. If $\mathbf{M}$ is a safe standard interpretation of $\mathcal{P}$ and $a \in [0,1]$, then $\langle \mathbf{M}, a\rangle$ is any expansion of $\mathbf{M}$ with $\|\gamma\| = a$; $\mathbf{M} \searrow [a,1]$ results from $\mathbf{M}$ by replacing the interpretation $P_{\mathbf{M}}$ of each predicate P (of $\mathcal{P}$) by $P'_{\mathbf{M}}(u_1,\dots,u_n) = \max\{P_{\mathbf{M}}(u_1,\dots,u_n), a\}$. The following lemma is easily verified by induction:

LEMMA 5.4.5. *For each formula φ of $\mathcal{P}$, $\mathbf{M}$-evaluation* v *of variables, and $a \in [0,1]$,*

$$\|\varphi^0\|^{\boldsymbol{B}}_{\langle \mathbf{M},a\rangle,v} = \|\varphi\|^{\boldsymbol{B}\searrow[a,1]}_{\mathbf{M}\searrow[a,1],v}.$$

THEOREM 5.4.6. *The sets* $\mathrm{stTAUT}(\mathrm{BLH}\forall)$, $\mathrm{stTAUT}^+(\mathrm{BL}\forall)$ *and* $\mathrm{stTAUT}(\mathrm{BL}\forall)$ *are mutually recursively reducible (and hence have the same Turing degree). Similarly for* Ł∀, ŁH∀ *and* G∀, GH∀.

Proof. Our lemma recursively reduces $\mathrm{stTAUT}(\mathrm{BL}\forall)$ to $\mathrm{stTAUT}^+(\mathrm{BL}\forall)$ in the following way: $\varphi \in \mathrm{stTAUT}(\mathrm{BL}\forall)$ iff $\varphi^0 \in \mathrm{stTAUT}^+(\mathrm{BL}\forall)$. Conversely, a falsity-free formula φ belongs to $\mathrm{stTAUT}^+(\mathrm{BL}\forall)$ iff it belongs to $\mathrm{stTAUT}(\mathrm{BL}\forall)$, thus the identity mapping of falsity-free formulae reduces $\mathrm{stTAUT}^+(\mathrm{BL}\forall)$ to $\mathrm{stTAUT}(\mathrm{BL}\forall)$. Further recall that $\mathrm{stTAUT}(\mathrm{BLH}\forall) = \mathrm{stTAUT}^+(\mathrm{BL}\forall)$. The same for Ł, G. Since if $\boldsymbol{B}$ is the Łukasiewicz or Gödel t-norm algebra and $a < 1$ then $\boldsymbol{B} \searrow [a,1]$ is isomorphic to $\boldsymbol{B}$. □

THEOREM 5.4.7.

(1) $\mathrm{stTAUT}(\Pi\mathrm{H}\forall)$, $\mathrm{stTAUT}^+(\Pi\forall)$, $\mathrm{stTAUT}(\Pi\forall) \cap \mathrm{stTAUT}(\text{Ł}\forall)$ *are mutually recursively reducible (have the same Turing degree).*

(2) The set $\mathrm{stTAUT}(\Pi\forall) \cap \mathrm{stTAUT}(\text{Ł}\forall)$ *is not arithmetical.*

Proof. (1) The mapping sending each φ to φ^0 reduces $\mathrm{stTAUT}(\Pi\forall) \cap \mathrm{stTAUT}(\text{Ł}\forall)$ to $\mathrm{stTAUT}^+(\Pi\forall)$ since for $a = 0$ and $\boldsymbol{B}$ being the standard product algebra, the algebra $\boldsymbol{B} \searrow [a,1]$ is just $\boldsymbol{B}$ and for $0 < a < 1$ we know that $\boldsymbol{B} \searrow [a,1]$ is isomorphic to the standard Łukasiewicz algebra.

(2) Consult the proof of non-arithmeticity of $\mathrm{TAUT}(\Pi\forall)$ in above: there is a closed formula Ψ and for each formula Φ of Peano arithmetic a formula Φ' of $\Pi\forall$ such that Φ is true in the standard model of arithmetic iff $\Psi \to \Phi'$ is in $\mathrm{stTAUT}(\Pi\forall)$. Observe that Ψ implies $\neg(\forall x)U(x)\ \&\ (\forall x)\neg\neg U(x)$ and this formula has in each interpretation the value 0 when computed under standard Łukasiewicz semantics; thus $\Psi \to \Phi'$ is in $\mathrm{stTAUT}(\text{Ł}\forall)$ for any Φ. Hence the mapping sending each Φ to $\Psi \to \Phi'$ reduces truth in $\mathbf{N}$ to $\mathrm{stTAUT}(\Pi\forall) \cap \mathrm{stTAUT}(\text{Ł}\forall)$, which shows that the latter set is not arithmetical. □

COROLLARY 5.4.8.

(1) The set $\mathrm{stTAUT}^+(\mathrm{G}\forall) = \mathrm{stTAUT}(\mathrm{GH}\forall)$ *is* $\boldsymbol{\Sigma_1}$*-complete.*

(2) The set $\mathrm{stTAUT}^+(\text{Ł}\forall) = \mathrm{stTAUT}(\text{ŁH}\forall)$ *is* $\boldsymbol{\Pi_2}$*-complete.*

(3) $\mathrm{stTAUT}^+(\Pi\forall) = \mathrm{stTAUT}(\Pi\mathrm{H}\forall)$ *and* $\mathrm{stTAUT}^+(\mathrm{BL}\forall) = \mathrm{stTAUT}(\mathrm{BLH}\forall)$ *are not arithmetical.*

REMARK 5.4.9. Every *formula in the language of hoops is satisfiable and hence positively satisfiable (interpret every predicate into* 1*).*

5.5 Monadic BL logics

It is well-known that monadic classical predicate logic, i.e. classical predicate logic without function symbols and with only unary predicate symbols, is decidable. It is a natural question to ask for which sets $\mathbb{K}$ of standard BL-algebras the set $\mathrm{TAUTm}(\mathbb{K})$ of monadic predicate formulae valid in all algebras in $\mathbb{K}$ is decidable. An almost complete answer to this question is provided by the next theorem.

THEOREM 5.5.1. *If* $\mathbb{K}$ *is a class of standard* BL*-chains containing an algebra not isomorphic to* $[0,1]_{\text{Ł}}$ *or to* $[0,1]_\Pi$*, then* $\mathrm{TAUTm}(\mathbb{K})$ *is undecidable.*

Proof. We will recursively reduce the classical theory T of two equivalence relations to $\mathrm{TAUTm}(\mathbb{K})$. Since T is undecidable (see e.g. [28]) the claim will follow. Let P, Q and H be unary predicate symbols, and let $R = (\exists x)H(x)$. Let E and S denote the binary predicate symbols of T representing the two equivalence relations. We define for every monadic formula ξ (possibly with parameters) of T, a formula ξ^+ of monadic BL logic in the following inductive way:

If $\xi = E(a,b)$ (where a and b are either variables or parameters) then $\xi^+ = (P(a) \leftrightarrow P(b)) \vee R$.

If $\xi = S(a,b)$, then $\xi^+ = (Q(a) \leftrightarrow Q(b)) \vee R$.

If $\xi = \overline{0}$, then $\xi^+ = R$.

If $\xi = \sigma \mathbin{\&} \gamma$, then $\xi^+ = (\sigma^+ \mathbin{\&} \gamma^+) \vee R$.

If $\xi = \sigma \to \gamma$, then $\xi^+ = \sigma^+ \to \gamma^+$.

If $\xi = (\exists x)\sigma$, then $\xi^+ = (\exists x)\sigma^+$.

If $\xi = (\forall x)\sigma$, then $\xi^+ = (\forall x)\sigma^+$.

Now for every ψ of T, we consider the formula $\psi^\star = ((\forall x)(R \uparrow (P(x) \vee Q(x))) \to (\psi^+ \vee (\exists x)(P(x) \vee Q(x))$. We are going to prove that for every sentence ψ of T one has: $T \vdash \psi$ iff $\psi^\star \in \mathrm{TAUTm}(\mathbb{K})$. Since * is recursive, this will give the desired result.

LEMMA 5.5.2. *Let* $\boldsymbol{A}$ *be any* BL*-chain and* $\mathbf{M}$ *an* $\boldsymbol{A}$*-structure such that* $\|\psi^\star\|^{\boldsymbol{A}}_{\mathbf{M}} \neq 1$*. Let us write* $\|\dots\|$ *instead of* $\|\dots\|^{\boldsymbol{A}}_{\mathbf{M}}$*. Then for all* $d \in M$*,* $\|P(d) \vee Q(d)\| < \|R\| < 1$*, and* $\|P(d) \vee Q(d)\|$ *and* $\|R\|$ *do not belong to the same Wajsberg component.*

Proof. If for some $d \in M$, $\|P(d) \vee Q(d)\| \geq \|R\|$, then by Lemma 3.0.19 we obtain $\|R \uparrow (P(d) \vee Q(d))\| = \|P(d) \vee Q(d)\|$. Hence, $\|(\forall x)(R \uparrow (P(x) \vee Q(x)))\| \leq \|(\exists x)(P(x) \vee Q(x))\|$, and finally $\|\psi^\star\| = 1$, a contradiction. If $\|P(d) \vee Q(d)\| < \|R\|$, but $\|P(d) \vee Q(d)\|$ and $\|R\|$ are in the same Wajsberg component, so again by

Lemma 3.0.19 $\|R \uparrow (P(d) \vee Q(d))\| = \|R\|$. Thus $\|(\forall x)(R \uparrow (P(x) \vee Q(x)))\| \leq \|R\| \leq \|\psi^+\|$, and $\|\psi^\star\| = 1$, which is a contradiction. Finally, if $\|R\| = 1$, then $\|\psi^+\| = 1$ and $\|\psi^\star\| = 1$, which is impossible. □

LEMMA 5.5.3. *With reference to the notation of Lemma 5.5.2, let for all $d \in M$, $\|P(d) \vee Q(d)\| < \|R\|$, and $\|P(d) \vee Q(d)\|$ and $\|R\|$ are not in the same Wajsberg component of $\boldsymbol{A}$. Then for every formula ξ of T, either $\|\xi^+\| = 1$ or $\|\xi^+\| = \|R\|$.*

Proof. Clearly, $\|\overline{0}^+\| = \|R\|$. Now let $a, b \in M$, and let $\boldsymbol{W}$ and $\boldsymbol{U}$ be the Wajsberg components of $\boldsymbol{A}$ such that $\|P(a)\| \in W$ and $\|P(b)\| \in U$ (possibly, $\boldsymbol{W} = \boldsymbol{U}$). If $\|P(a)\| = \|P(b)\|$, then $\|E(a,b)^+\| = \|(P(a) \leftrightarrow P(b)) \vee R\| = 1$. Otherwise, using the definition of ordinal sum, it is readily seen that $\|P(a) \leftrightarrow P(b)\| \in (W \cup U) \setminus \{1\}$. Hence $\|P(a) \leftrightarrow P(b)\| < \|R\|$, and $\|E(a,b)^+\| = \|(P(a) \leftrightarrow P(b)) \vee R\| = \|R\|$. Similarly, either $\|S(a,b)^+\| = 1$ or $\|S(a,b)^+\| = \|R\|$. Hence, the claim holds if ξ is atomic. The induction steps corresponding to $\to$, $\exists$ and $\forall$ are immediate. Finally, suppose $\xi = \sigma \mathbin{\&} \gamma$. Then $\|\xi^+\| = \|(\sigma^+ \mathbin{\&} \gamma^+) \vee R\|$, and the claim follows from the induction hypothesis. □

Suppose that the hypotheses of Lemma 5.5.3 are satisfied. Define a model $\mathbf{M}^{\neg\neg}$ of T from $\mathbf{M}$ as follows: the domain of $\mathbf{M}^{\neg\neg}$ is M, and for $a, b \in M$, let $\mathbf{M}^{\neg\neg} \models E(a,b)$ if $\|P(a) \leftrightarrow P(b)\| = 1$, and $\mathbf{M}^{\neg\neg} \models S(a,b)$ iff $\|Q(a) \leftrightarrow Q(b)\| = 1$. Clearly, $\mathbf{M}^{\neg\neg} \models T$.

LEMMA 5.5.4. *Under the same assumptions as in Lemma 5.5.3, for every sentence ξ of T with parameters in M, one has: $\mathbf{M}^{\neg\neg} \models \xi$ iff $\|\xi^+\| = 1$.*

Proof. We proceed by induction on ξ. If $\xi = E(a,b)$, then $\mathbf{M}^{\neg\neg} \models \xi$ iff $\|P(a) \leftrightarrow P(b)\| = 1$ iff $\|(P(a) \leftrightarrow P(b)) \vee R\| = 1$ (because by Lemma 5.5.4 $\|R\| < 1$). The case where $\xi = S(a,b)$ is treated similarly. If $\xi = \overline{0}$, then $\mathbf{M}^{\neg\neg} \not\models \xi$ and $\|\xi^+\| = \|R\| < 1$. Thus the claim holds for ψ atomic.

Suppose $\xi = \lambda \mathbin{\&} \sigma$. Then $\mathbf{M}^{\neg\neg} \models \xi$ iff $\mathbf{M}^{\neg\neg} \models \lambda$ and $\mathbf{M}^{\neg\neg} \models \sigma$ iff (by the induction hypothesis) $\|\lambda^+\| = \|\sigma^+\| = 1$ iff $\|(\lambda^+ \mathbin{\&} \sigma^+) \vee R\| = 1$ iff $\|\xi^+\| = 1$.

For the induction steps corresponding to $\to$, to $\exists$ and to $\forall$ we use the fact that by Lemmata 5.5.2 and 5.5.4 for every sentence γ of T, $\|\gamma^+\| \in \{\|R\|, 1\}$, i.e., $\|\gamma^+\|$ can assume only two truth values. Thus the semantic interpretations of $\to$, $\exists$ and $\forall$ are the same as in classical logic (with 0 replaced by $\|R\|$), and the proof proceeds in a straightforward way. □

We conclude the proof of Theorem 5.5.1. Suppose that $\psi^\star \notin \mathrm{TAUTm}(\mathbb{K})$. Let $\boldsymbol{A} \in \mathbb{K}$ and $\mathbf{M}$ an $\boldsymbol{A}$-structure such that $\|\psi^\star\|^{\boldsymbol{A}}_{\mathbf{M}} \neq 1$. Then, *a fortiori*, $\|\psi^+\|^{\boldsymbol{A}}_{\mathbf{M}} \neq 1$. By Lemmata 5.5.2, 5.5.3 and 5.5.4, we have a model $\mathbf{M}^{\neg\neg}$ of T such that for every sentence γ of T, $\mathbf{M}^{\neg\neg} \models \gamma$ iff $\|\gamma^+\|^{\boldsymbol{A}}_{\mathbf{M}} = 1$. Since $\|\psi^+\|^{\boldsymbol{A}}_{\mathbf{M}} \neq 1$, $\mathbf{M}^{\neg\neg} \not\models \psi$, and $T \nvdash \psi$.

Conversely, suppose $T \nvdash \psi$. Let $\mathbf{M}_T$ be a finite or countable model of T such that $\mathbf{M}_T \not\models \psi$. Take $\boldsymbol{A} \in \mathbb{K}$ not isomorphic to any of $[0,1]_{Ł}$ and $[0,1]_{\Pi}$. Then $\boldsymbol{A}$ must have an idempotent element (i.e., an element a such that $a \star a = a$) which is different from 0 and from 1. Hence if $0 \leq x < a < y < 1$, then x and y do not belong to the same

Wajsberg component. Now we can obtain an $\boldsymbol{A}$-structure $\mathbf{M}$ such that the following conditions hold:

(i) The domain M of $\mathbf{M}$ coincides with the domain M_T of $\mathbf{M}_T$.

(ii) $a < R_{\mathbf{M}} < 1$ (since $R = (\exists x)H(x)$, it suffices to take $H_{\mathbf{M}}(d) = \frac{a+1}{2}$ for all $d \in M$).

(ii) For all $d \in M$, $P_{\mathbf{M}}(d) < a$ and $Q_{\mathbf{M}}(d) < a$.

(iii) For $a, b \in M$, $P_{\mathbf{M}}(a) = P_{\mathbf{M}}(b)$ iff $\mathbf{M}_T \models E(a, b)$.

(iv) For $a, b \in M$, $Q_{\mathbf{M}}(a) = Q_{\mathbf{M}}(b)$ iff $\mathbf{M}_T \models S(a, b)$.

Then we can easily check that the model $\mathbf{M}^{\neg\neg}$ built from $\mathbf{M}$ as in Lemma 5.5.4 is isomorphic to $\mathbf{M}_T$. Thus, by Lemma 5.5.4, for every sentence γ of T, one has: $\mathbf{M}_T \models \gamma$ iff $\|\gamma^+\|^{\boldsymbol{A}}_{\mathbf{M}} = 1$. Hence, $\|\psi^+\|^{\boldsymbol{A}}_{\mathbf{M}} \neq 1$. Moreover by conditions (i) and (ii) and by Lemma 3.0.19, for all $d \in M$, $\|R \uparrow (P(d) \vee Q(d))\|^{\boldsymbol{A}}_{\mathbf{M}} = 1$, hence $\|(\forall x)(R \uparrow (P(x) \vee Q(x)))\|^{\boldsymbol{A}}_{\mathbf{M}} = 1$. Finally, $\|(\exists x)(P(x) \vee Q(x))\|^{\boldsymbol{A}}_{\mathbf{M}} \leq a < 1$. Hence, $\|\psi^\star\|^{\boldsymbol{A}}_{\mathbf{M}} < 1$. □

An inspection on the proof of Theorem 5.5.1 yields that if $\psi^\star$ can be invalidated in *some arbitrarily given* linearly ordered BL-algebra $\boldsymbol{A}$, then $T \nvdash \psi$. On the other hand, if $T \nvdash \psi$, then by the proof of Theorem 5.5.1 we obtain a standard BL-algebra (hence a fortiori a linearly ordered BL-algebra) $\boldsymbol{A}$ such that $\psi^\star$ is not valid in $\boldsymbol{A}$. In other words, $T \vdash \psi$ iff $\psi^\star$ is valid in all linearly ordered BL-algebras. Since BL∀ is complete with respect to the interpretations in linearly ordered BL-algebras, and since of course the same property holds for its monadic version BLm∀, we have that for every sentence ψ of the language of T, T $\vdash \psi$ iff BLm∀ $\vdash \psi^\star$. It follows:

THEOREM 5.5.5. *The monadic predicate logic* BLm∀ *is undecidable.*

5.6 Extensions of Łukasiewicz logic

We survey the main results on complexity of logics properly extending Łukasiewicz predicate logic Ł∀ (for details see [6]). Recall the definitions of finite Ł-algebras and Komori algebras (see Chapter VI):

- $\mathbf{Ł}_{n+1}$ is the subalgebra of $[0,1]_{Ł}$ with the domain $\{0, \frac{1}{n}, \ldots, \frac{n-1}{n}, 1\}$,
- $\mathbf{K}_{n+1} = \langle \{\langle i, a\rangle \in \mathsf{N} \times_{\text{lex}} \mathsf{Z} \mid \langle 0,0\rangle \leq \langle i, a\rangle \leq \langle n, 0\rangle\}, \oplus_{\mathbf{K}_{n+1}}, \neg_{\mathbf{K}_{n+1}}, \langle 0,0\rangle\rangle$, $\langle i, x\rangle \oplus_{\mathbf{K}_{n+1}} \langle j, y\rangle = \min\{\langle n, 0\rangle, \langle i+j, x+y\rangle\}$ and $\neg_{\mathbf{K}_{n+1}} \langle i, x\rangle = \langle n-i, -x\rangle$.

Recall Komori's result saying that for each consistent propositional logic which is a proper axiomatic extension of Łukasiewicz propositional logic there exist finite subsets A, B of the set of natural numbers bigger than 1 such that $A \cup B \neq \emptyset$ and the set of standard tautologies of this logic is defined as $\bigcap_{i \in A} \mathrm{stTAUT}(\mathbf{K}_i) \cap \bigcap_{j \in B} \mathrm{stTAUT}(\mathbf{Ł}_j)$. ($\mathbf{Ł}_j$ is the finite Łukasiewicz algebra with j-elements). Thus the logic can be denoted by $Ł_{A,B}$; its predicate version is denoted $Ł_{A,B}$∀.

The algebras $\mathbf{K}_i$ for $i \in A$ and $\mathbf{Ł}_j$ for $j \in B$ are the standard algebras of $Ł_{A,B}$; they generate the variety of general $Ł_{A,B}$-algebras. It can be shown that any Komori algebra and any finite Łukasiewicz algebra is in the variety of general $Ł_{A,B}$-algebras iff it is a subalgebra of a generic $Ł_{A,B}$-algebra. Each safe $\mathbf{K}_n$-structure is witnessed.

THEOREM 5.6.1. *Let $Ł_{A,B}$ be any consistent axiomatic extension of Ł. Then:*

(1) $\mathrm{genTAUT}(Ł_{A,B}\forall)$ *is $\mathbf{\Sigma_1}$-complete.*

(2) $\mathrm{stTAUT}(Ł_{A,B}\forall)$ *is $\mathbf{\Sigma_1}$-complete iff A is empty.*

(3) $\mathrm{stTAUT}(Ł_{A,B}\forall)$ *is $\mathbf{\Pi_2}$-complete iff A is non-empty.*

(4) $\mathrm{genSAT}(Ł_{A,B}\forall) = \mathrm{stSAT}(Ł_{A,B}\forall)$ *and this set is $\mathbf{\Pi_1}$-complete.*

COROLLARY 5.6.2. *Let $Ł_{A,B}$ be any consistent axiomatic extension of Ł. Then:*

(1) $\mathrm{genSAT_{pos}}(Ł_{A,B}\forall)$ *is $\mathbf{\Pi_1}$-complete.*

(2) $\mathrm{stSAT_{pos}}(Ł_{A,B}\forall)$ *is $\mathbf{\Pi_1}$-complete iff A is empty.*

(3) $\mathrm{stSAT_{pos}}(Ł_{A,B}\forall)$ *is $\mathbf{\Sigma_2}$-complete iff A is non-empty.*

(4) $\mathrm{genTAUT_{pos}}(Ł_{A,B}\forall) = \mathrm{stTAUT_{pos}}(Ł_{A,B}\forall)$ *and this set is $\mathbf{\Sigma_1}$-complete.*

5.7 Open problems

Some unsolved problems in complexity of predicate fuzzy logics:

- What is the exact complexity of $\mathrm{TAUTm}(\mathbb{K})$, when $\mathbb{K}$ is the class of *all* standard BL-algebras?
- Is any of $\mathrm{TAUTm}([0,1]_Ł)$ or $\mathrm{TAUTm}([0,1]_\Pi)$ decidable?[4]
- What is the complexity (outside the arithmetical hierarchy) of the non arithmetic sets, such as standard tautologies of product logic, contained in this chapter?[5]
- If instead of the full vocabulary we consider one which has a countable number of relational symbols of all arities (but not functional symbols), is the general semantics of first-order fuzzy logics still undecidable?
- What are the complexities of sets of standard (positive) tautologies and satisfiable sentences when $\triangle$ is added to the language?
- For which ($\triangle$-)core fuzzy logics L, does $\mathrm{L}\forall^- = \mathrm{L}^-\forall$ hold?
- What is the complexity of the monadic fragments in the case of the general semantics?
- What are the complexities of first-order fuzzy logics when one considers the supersound semantics in the sense of [2]?

[4] Very recently Félix Bou has presented a proof of the undecidability of the monadic fragments of Łukasiewicz and product predicate logics at a conference and at some seminars. As far as the authors know, these results are still not in a written form at the moment of the publication of the present handbook.

[5] The paper [24] gives a rather tight bound to for the complexity of standard tautologies of BL$\forall$ and $\Pi\forall$, showing that in both cases it is between $\emptyset^\omega$ (the degree of true arithmetic) and $\emptyset^{\omega+1}$ (the degree of the halting problem with oracle on $\emptyset^\omega$).

6 Historical remarks and further reading

For a general reference on recursion theory and arithmetical hierarchy see e.g. [28]. Arguably, the first work considering the arithmetical complexity of a particular first-order fuzzy logic is Scarpellini's paper [29] published on 1962 which shows that the set of (standard) tautologies of Łukasiewicz predicate logic is not recursively axiomatizable. Ragaz proved in his PhD Thesis [27] from 1981 that this set is actually $\mathbf{\Pi_2}$-complete (alternative proofs of the same fact have been obtained by Goldstern in [8] and by Hájek in [11]; we have presented here the latter). When first-order versions (in full language) for other propositional fuzzy logics started being systematically studied by Hájek in his works during the nineties, their undecidability appeared as a general problem. Actually, Montagna and Ono proved in [26] that all first-order versions of consistent axiomatic extensions of MTL are undecidable and thus the issue of their arithmetical complexity became a crucial item in the agenda of Mathematical Fuzzy Logic. Various results concerning the position in the arithmetical hierarchy of the sets of tautologies, positive tautologies, satisfiable sentences, and positively satisfiable sentences w.r.t. the standard semantics of the three main fuzzy logics are in Hájek's papers [9, 10] and in Chapter 6 of his monograph [11]. The next natural step was the study of complexity problems for first-order logics based on other continuous t-norms: this was done again by Hájek in [12, 13] and by Montagna in [22, 23]. Note that [12] contains the first result of non-arithmeticity in the present context (non-arithmeticity of $\mathrm{stSAT}(\Pi\forall)$). In 2005, the survey paper [14], besides collecting the mentioned results, provides a new study where the standard semantics is replaced by the general semantics, i.e. the one given by models over arbitrary linearly ordered BL-algebras. Recent works have extended the scope of the studies on arithmetical complexity issues to: witnessed semantics over product logic [16], logics of complete BL-chains [21], monadic fragments [23], logics with less propositional connectives [17] and extensions of Łukasiewicz logic [6]. Moreover, the recent survey [18] collects information (with references) on sets of standard/general tautologies, satisfiable sentences, also for positive tautologies and satisfiable sentences of important logics (extending $\mathrm{BL}\forall$), logics with Baaz's $\triangle$, with truth constants, logics extending Łukasiewicz logic and Baaz-Gödel logics $\mathrm{G}(V)$ whose set of truth values V is a subset of the real unit interval. The latter are also studied in [1, 15]. Recently, in the paper [25] Montagna and Noguera have presented a general approach to complexity problems extending the scope to core and $\triangle$-core fuzzy logics and to arbitrary semantics (Sections 2 and 4 of the present chapter are based on this paper). Theorem 2.0.20 has been obtained by Bou and Noguera in [3]. Finally, it is worth mentioning the recent paper [19] which gives new important insight into undecidability issues in fuzzy logics with an example of a decidable theory T over Łukasiewicz (propositional) logic and a formula φ such that the theory $T \cup \{\varphi\}$ is undecidable.

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