Equality-free Logic: The Method of Diagrams and Preservation Theorems

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Abstract

In this article I prove preservation theorems for the positive and for the universal-existential fragment of equality-free logic. I give a systematic presentation of the method of diagrams for first-order languages without equality.

Keywords: equality-free logic, model theory

1 Introduction

The interest for the study of languages without equality has its origin in the works of W. Blok and D. Pigozzi, [1], [2] and [3]. Equality-free Logic can be seen as a bridge between two disciplines: Model Theory and Algebraic Logic. It was first observed by S. L. Bloom in [4], that to any propositional deductive system we can associate an equality-free strict universal Horn theory. For this reason, in order to study the algebraic aspects of deductive systems, it is useful to learn about the model-theoretic properties of this fragment of first-order logic. There are two main concepts which lay in the background of both disciplines and that allow the development of this study, the notion of Leibniz congruence and the notion of relative relation. W. Blok and D. Pigozzi introduced the concept of relative relation for the special case of logical matrices in [1], and in [3] they made an extensive use of what they named the Leibniz congruence. Motivated by their works a general classical model-theoretical study of this logic was carried on in [5], [8], [9], [10] and [11]. In [5] we developed back-and-forth methods for equality-free languages and different characterizations of the elementary equivalence for this logic were given, using back-and-forth systems, elementary extensions and ultrapowers. As a consequence of these theorems elementary classes in Equality-free Logic were characterized. In [9] was studied the equality-free universal Horn fragment of the infinitary languages $L_{\kappa\kappa}$. We gave some characterization and preservation theorems for this fragment, drawing as consequences, interpolation, joint-consistency and definability theorems. Independently, in [11], preservation results were given for the universal, universal-atomic and universal Horn fragment of first-order logic without equality.

The method of diagrams, due to L. A. Henkin and A. Robinson, has proved to be a useful tool for Model Theory. Nevertheless, if we want to have its advantages working in Equality-free Logic, we can not use this technique as it stands. In the first part of this article I present different results that will allow us to work with diagrams. Some of these results are used in the proofs of some propositions in [9] and [11]. Here I give a systematic presentation of these techniques, proving new equivalencies. Proposition 2.4 and the equivalencies between i) and vii) in Proposition 2.8 are taken from [11]. The second part of the article is devoted to the study of preservation results for the universal-existential fragment of Equality-free Logic. Using an extended version of Lyndon's Interpolation Theorem (see [14]), a preservation theorem for the positive fragment of this logic is given.

First of all I introduce some notation and basic notions. From now on L will be a similarity type with at least one relation symbol. The set of equality-free first-order formulas of type L is denoted by $L^-$and by L_0^- the set of quantifier-free formulas of L^-. For any L-structure $\mathfrak{A}$ and any set $B \subseteq A$, by $L(B)$ is denoted the similarity type obtained from L by adding a new constant symbol for each element of B and $\mathfrak{A}_B$ denotes the natural expansion of $\mathfrak{A}$ to $L(B)$, where every new constant denotes its corresponding element. For the sake of clarity I use the same symbol for the constant and for the element that is denoted by the constant, with the exceptions of Lemma 3.1 and Theorem 3.2, where it can lead to confusion. $|A|$ denotes the power of the set A. Given a structure $\mathfrak{A}$, for any $B \subseteq A$, $\langle B \rangle$ denotes the substructure of $\mathfrak{A}$ generated by B. $\mathfrak{A} \equiv^- \mathfrak{B}$ means that $\mathfrak{A}$ and $\mathfrak{B}$ satisfy the same sentences of L^-, and $\mathfrak{A} \equiv_0^- \mathfrak{B}$ that $\mathfrak{A}$ and $\mathfrak{B}$ satisfy the same sentences of L_0^-. Finally $\mathfrak{A} \subseteq \mathfrak{B}$ means that $\mathfrak{A}$ is a substructure of $\mathfrak{B}$ and $\mathfrak{A} \overset{\subseteq}{\sim} \mathfrak{B}$ that $\mathfrak{A}$ is isomorphic to a substructure of $\mathfrak{B}$.

Definition 1.1 If $\mathfrak{A}$ and $\mathfrak{B}$ are L-structures, it is said that an homomorphism $h : \mathfrak{A} \to \mathfrak{B}$ is *strict* if for any n-adic relation symbol $R \in L$ and any $a_1, \ldots, a_n \in A$,

$$\langle a_1, \ldots, a_n \rangle \in R^{\mathfrak{A}} \iff \langle h(a_1), \ldots, h(a_n) \rangle \in R^{\mathfrak{B}}.$$

It is a well-known fact that if $h : \mathfrak{A} \to \mathfrak{B}$ is a strict homomorphism onto $\mathfrak{B}$, then $\mathfrak{A} \equiv \mathfrak{B}$ and the kernel of h is a congruence of $\mathfrak{A}$. Moreover, for any congruence θ of $\mathfrak{A}$, the canonical homomorphism from $\mathfrak{A}$ onto $\mathfrak{A}/\theta$ is strict.

The notion of elementary substructure can be generalized to Equality-free Logic in a natural way.

Definition 1.2 If $\mathfrak{A}$ and $\mathfrak{B}$ are L-structures, $\mathfrak{A}$ is an L^--*substructure* of $\mathfrak{B}$, in symbols $\mathfrak{A} \preceq^- \mathfrak{B}$, if $\mathfrak{A} \subseteq \mathfrak{B}$ and for any $\phi(x_1, \ldots, x_n) \in L^-$ and any $a_1, \ldots, a_n \in A$,

$$\mathfrak{A} \models \phi\,[a_1, \ldots, a_n] \iff \mathfrak{B} \models \phi\,[a_1, \ldots, a_n]\,.$$

If $\mathfrak{A}$ is an L^--substructure of $\mathfrak{B}$, it is said that $\mathfrak{B}$ is an L^--extension of $\mathfrak{A}$. $\mathfrak{A} \precsim^- \mathfrak{B}$ means that $\mathfrak{A}$ is isomorphic to an L^--substructure of $\mathfrak{B}$.

Given a class K of L-structures I define the following classes of L-structures:

$\mathbf{S}(K)$—the class of all substructures of members of K.
$\mathbf{S}^{\preceq^-}(K)$—the class of all L^--substructures of members of K.
$\mathbf{H}(K)$—the class of all homomorphic images
of members of K.

$\mathbf{H}^{-1}(K)$—the class of all homomorphic pre-images of members of K.
$\mathbf{H_S}(K)$—the class of all strict homomorphic images of members of K.
$\mathbf{H_S^{-1}}(K)$—the class of all strict homomorphic pre-images of members of K.

It is known that given an L-structure $\mathfrak{A}$, we can construct a strict homomorphic pre-image of $\mathfrak{A}$ that has as algebraic reduct an algebra of terms. Let us recall this construction. Unless otherwise stated, from now on enumerations of sets are allowed to have repetitions.

Definition 1.3 Let $\mathfrak{A}$ be an L-structure. Given an enumeration $\overline{a} = (a_i : i \in I)$ of A, L-structure $Ter^{\mathfrak{A}}_{\overline{a}}$ is defined in the following way:

- The algebraic reduct of $Ter^{\mathfrak{A}}_{\overline{a}}$ is Ter_{V_I}, that is, the algebra of terms of type L generated by the set $V_I = \{x_i : i \in I\}$.
- In order to define the interpretation of the relation symbols, consider the function $h_0 : V_I \to A$ defined by:
$$h_0(x_i) = a_i,$$
for any $i \in I$. h_0 extends to an homomorphism h from Ter_{V_I} onto the algebraic reduct of $\mathfrak{A}$. For any n-adic relation symbol $R \in L$, $R^{Ter^{\mathfrak{A}}_{\overline{a}}}$ *is defined* as follows: for any $t_1, \ldots, t_n \in Ter_{V_I}$,
$$\langle t_1, \ldots, t_n \rangle \in R^{Ter^{\mathfrak{A}}_{\overline{a}}} \iff \langle h(t_1), \ldots, h(t_n) \rangle \in R^{\mathfrak{A}}.$$

Observe that given an L-structure $\mathfrak{A}$ and an enumeration $\overline{a} = (a_i : i \in I)$ of A, then $\mathfrak{A} \in \mathbf{H_S}(Ter^{\mathfrak{A}}_{\overline{a}})$. Let us recall now the notions of Leibniz congruence and of relative relation. For references on these two concepts see [1], [3] and [8].

Definition 1.4 Given an L-structure $\mathfrak{A}$, the relation $\Omega(\mathfrak{A})$ on $\mathfrak{A}$ is defined as follows: $\langle a, b \rangle \in \Omega(\mathfrak{A})$ if and only if for any atomic formula $\phi(x, y_1, \ldots, y_n) \in L^-$, and any $d_1, \ldots, d_n \in A$,
$$\mathfrak{A} \models \phi\,[a, d_1, \ldots, d_n] \iff \mathfrak{A} \models \phi\,[b, d_1, \ldots, d_n]\,.$$
for any $a, b \in A$. $\Omega(\mathfrak{A})$ is called the *Leibniz congruence* of $\mathfrak{A}$.

For any $a \in A$, $[a]_{\Omega(\mathfrak{A})}$ denotes the equivalence class of a modulo the Leibniz congruence. The Leibniz congruence of $\mathfrak{A}$ always exists and it is the greatest congruence relation on $\mathfrak{A}$. We denote by $\mathfrak{A}^*$ the quotient structure $\mathfrak{A}/\Omega(\mathfrak{A})$, with algebraic reduct defined as usual and with the interpretation of the relation symbols as follows: for any n-adic relation symbol $S \in L$, any $a_1, \ldots, a_n \in A$,
$$\langle [a_1]_{\Omega(\mathfrak{A})}, \ldots, [a_n]_{\Omega(\mathfrak{A})} \rangle \in S^{\mathfrak{A}^*} \iff \langle a_1, \ldots, a_n \rangle \in S^{\mathfrak{A}}$$
It is said that a structure is *reduced* if its Leibniz congruence is the identity. It is easy to see that $\mathfrak{A}^*$ is reduced, $\mathfrak{A}^*$ is called *the reduction of* $\mathfrak{A}$. Since the canonical

homomorphism from $\mathfrak{A}$ onto $\mathfrak{A}^*$ is strict, $\mathfrak{A}$ and $\mathfrak{A}^*$ satisfy exactly the same equality-free sentences

Now we introduce the notion of relative relation, which is an equivalence relation between structures that plays in languages without equality the same role that the isomorphism relation plays in languages with equality. The definition and the proof of the equivalences below are taken from [5].

Definition 1.5 Let $\mathfrak{A}$ and $\mathfrak{B}$ be L-structures. A relation $R \subseteq A \times B$ is a *relative correspondence* between $\mathfrak{A}$ and $\mathfrak{B}$ if $\mathrm{dom}(R) = A$, $\mathrm{rg}(R) = B$ and

(1) for any constant $c \in L, c^{\mathfrak{A}} R c^{\mathfrak{B}}$,
(2) for any n-adic function symbol $f \in L$, any $a_1, \ldots, a_n \in A$ and any $b_1, \ldots, b_n \in B$ such that $a_i R b_i$ for each $i = 1, \ldots, n$,

$$f^{\mathfrak{A}}(a_1, \ldots, a_n) R f^{\mathfrak{B}}(b_1, \ldots, b_n),$$

(3) for any n-adic relation symbol $S \in L$, any $a_1, \ldots, a_n \in A$ and any $b_1, \ldots, b_n \in B$ such that $a_i R b_i$ for each $i = 1, \ldots, n$,

$$\langle a_1, \ldots, a_n \rangle \in S^{\mathfrak{A}} \iff \langle b_1, \ldots, b_n \rangle \in S^{\mathfrak{B}}.$$

Two L-structures $\mathfrak{A}$ and $\mathfrak{B}$ are *relatives*, in symbols $\mathfrak{A} \sim \mathfrak{B}$, if there is a relative correspondence between them.

Proposition 1.6 *Let $\mathfrak{A}$ and $\mathfrak{B}$ be L-structures. The following are equivalent:*

i) $\mathfrak{A} \sim \mathfrak{B}$.
ii) *There are $n \in \omega$ and L-structures $\mathfrak{C}_0, \ldots, \mathfrak{C}_n$ such that $\mathfrak{A} = \mathfrak{C}_0$, $\mathfrak{B} = \mathfrak{C}_n$ and for any $i < n$, $\mathfrak{C}_{i+1} \in \mathbf{H_S}(\mathfrak{C}_i)$ or $\mathfrak{C}_{i+1} \in \mathbf{H_S^{-1}}(\mathfrak{C}_i)$.*
iii) $\mathfrak{A}$, $\mathfrak{B} \in \mathbf{H_S}(\mathfrak{C})$, *for some* $\mathfrak{C}$.
iv) $\mathfrak{A}$, $\mathfrak{B} \in \mathbf{H_S^{-1}}(\mathfrak{C})$, *for some* $\mathfrak{C}$.
v) $\mathfrak{A}^* \cong \mathfrak{B}^*$.
vi) $\mathfrak{A}^* \sim \mathfrak{B}^*$.
vii) *There are enumerations of A and B, $\overline{a} = (a_i : i \in I)$ and $\overline{b} = (b_i : i \in I)$ respectively, such that $(\mathfrak{A}, \overline{a}) \equiv_0^- (\mathfrak{B}, \overline{b})$.*
viii) *There are enumerations of A and B, $\overline{a} = (a_i : i \in I)$ and $\overline{b} = (b_i : i \in I)$ respectively, such that $(\mathfrak{A}, \overline{a}) \equiv^- (\mathfrak{B}, \overline{b})$.*

Proof. viii) $\Rightarrow$ vii), v) $\Rightarrow$ vi), iii) $\Rightarrow$ ii), v) $\Rightarrow$ iv) and iv) $\Rightarrow$ ii) are clear and vii) $\Rightarrow$ viii) is proved by induction on the formulas of L^-.

vii) $\Rightarrow$ v) Suppose that there are enumerations of A and B, $\overline{a} = (a_i : i \in I)$ and $\overline{b} = (b_i : i \in I)$ respectively, such that $(\mathfrak{A}, \overline{a}) \equiv_0^- (\mathfrak{B}, \overline{b})$. We define $h : \mathfrak{A}^* \to \mathfrak{B}^*$ as follows:

$$h([a_i]_{\Omega(\mathfrak{A})}) = [b_i]_{\Omega(\mathfrak{B})}\,,$$

for any $i \in I$. First of all we see that for any term $t(y_1, \ldots, y_n)$ of L and any $i_1, \ldots, i_n, j \in I$,

$$(1) \quad \left[t^{\mathfrak{A}}\left[a_{i_1}, \ldots, a_{i_n}\right]\right]_{\Omega(\mathfrak{A})} = [a_j]_{\Omega(\mathfrak{A})} \quad \text{iff} \quad \left[t^{\mathfrak{B}}\left[b_{i_1}, \ldots, b_{i_n}\right]\right]_{\Omega(\mathfrak{B})} = [b_j]_{\Omega(\mathfrak{B})}\,.$$

Assume that $\left[t^{\mathfrak{A}}\left[a_{i_1},\ldots,a_{i_n}\right]\right]_{\Omega(\mathfrak{A})}=[a_j]_{\Omega(\mathfrak{A})}$ and that

$$\left[t^{\mathfrak{B}}\left[b_{i_1},\ldots,b_{i_n}\right]\right]_{\Omega(\mathfrak{B})}\neq[b_j]_{\Omega(\mathfrak{B})}.$$

Then, there is a quantifier-free formula $\phi(z,x_1,\ldots,x_m)\in L^-$ (where the variables $z,x_1,\ldots,x_m$ do not occur in t) and a sequence $d_1,\ldots,d_m$ of elements of B such that

$$\mathfrak{B}\models\phi(z,x_1,\ldots,x_m)\left[t^{\mathfrak{B}}\left[b_{i_1},\ldots,b_{i_n}\right],d_1,\ldots,d_m\right],$$

but

$$\mathfrak{B}\not\models\phi(z,x_1,\ldots,x_m)\left[b_j,d_1,\ldots,d_m\right].$$

Then, for any $1\leq k\leq m$, we choose $j_k\in I$ such that $d_k=b_{j_k}$ in the enumeration $\overline{b}$ of B. Hence,

$$\mathfrak{B}\models\phi(z,x_1,\ldots,x_m)\left[t^{\mathfrak{B}}\left[b_{i_1},\ldots,b_{i_n}\right],b_{j_1},\ldots,b_{j_m}\right]$$

but

$$\mathfrak{B}\not\models\phi(z,x_1,\ldots,x_m)\left[b_j,b_{j_1},\ldots,b_{j_m}\right].$$

Let ϕ' be obtained from ϕ by substituting the term t for the variable z. We have

$$\mathfrak{B}\models\phi'(y_1,\ldots,y_n,x_1,\ldots,x_m)\left[b_{i_1},\ldots,b_{i_n},b_{j_1},\ldots,b_{j_m}\right].$$

Since

$$(\mathfrak{A},\overline{a})\equiv_0^-(\mathfrak{B},\overline{b}),$$

we have

$$\mathfrak{A}\models\phi'(y_1,\ldots,y_n,x_1,\ldots,x_m)\left[a_{i_1},\ldots,a_{i_n},a_{j_1},\ldots,a_{j_m}\right]$$

and

$$\mathfrak{A}\not\models\phi(z,x_1,\ldots,x_m)\left[a_j,a_{j_1},\ldots,a_{j_m}\right].$$

But then,

$$\mathfrak{A}\models\phi(z,x_1,\ldots,x_m)\left[t^{\mathfrak{A}}\left[a_{i_1},\ldots,a_{i_n}\right],a_{j_1},\ldots,a_{j_m}\right],$$

which is absurd. Therefore, we conclude that

$$\left[t^{\mathfrak{B}}\left[b_{i_1},\ldots,b_{i_n}\right]\right]_{\Omega(\mathfrak{B})}=[b_j]_{\Omega(\mathfrak{B})}.$$

Analogously, we can prove the other direction of (1). By (1) we have that h is well-defined and injective. Moreover, since $\overline{b}$ is an enumeration of B, we have that h is surjective. Let us see now that h is a strict homomorphism: for any n-adic relation symbol $R\in L$ and any $a_{i_1},\ldots,a_{i_n}\in A$,

$$\begin{aligned}\langle[a_{i_1}]_{\Omega(\mathfrak{A})},\ldots,[a_{i_n}]_{\Omega(\mathfrak{A})}\rangle\in R^{\mathfrak{A}^*} \quad & \text{iff} \quad \langle a_{i_1},\ldots,a_{i_n}\rangle\in R^{\mathfrak{A}}\\ & \text{iff} \quad \mathfrak{A}\models Rx_1\ldots x_n\left[a_{i_1},\ldots,a_{i_n}\right]\\ & \text{iff} \quad \mathfrak{B}\models Rx_1\ldots x_n\left[b_{i_1},\ldots,b_{i_n}\right]\\ & \text{iff} \quad \langle b_{i_1},\ldots,b_{i_n}\rangle\in R^{\mathfrak{B}}\\ & \text{iff} \quad \langle[b_{i_1}]_{\Omega(\mathfrak{B})},\ldots,[b_{i_n}]_{\Omega(\mathfrak{B})}\rangle\in R^{\mathfrak{B}^*}.\end{aligned}$$

Suppose now that $f\in L$ is an n-adic function symbol and $a_{i_1},\ldots,a_{i_n}\in A$. We have

$$h(f^{\mathfrak{A}^*}([a_{i_1}]_{\Omega(\mathfrak{A})},\ldots,[a_{i_n}]_{\Omega(\mathfrak{A})}))=h(\left[f^{\mathfrak{A}}(a_{i_1},\ldots,a_{i_n})\right]_{\Omega(\mathfrak{A})}).$$

Let $j \in I$ be such that $f^{\mathfrak{A}}(a_{i_1}, \ldots, a_{i_n}) = a_j$ in the enumeration $\overline{a}$ of A, then

$$h(\left[f^{\mathfrak{A}}(a_{i_1}, \ldots, a_{i_n})\right]_{\Omega(\mathfrak{A})}) = h([a_j]_{\Omega(\mathfrak{A})}) = [b_j]_{\Omega(\mathfrak{B})},$$

and by (1)

$$[b_j]_{\Omega(\mathfrak{B})} = \left[f^{\mathfrak{B}}(b_{i_1}, \ldots, b_{i_n})\right]_{\Omega(\mathfrak{B})} = f^{\mathfrak{B}^*}([b_{i_1}]_{\Omega(\mathfrak{B})}, \ldots, [b_{i_n}]_{\Omega(\mathfrak{B})}).$$

In an analogous way we prove that for any constant symbol $c \in L$, $h(c^{\mathfrak{A}^*}) = c^{\mathfrak{B}^*}$. Therefore, we can conclude that h is an isomorphism.

ii) $\Rightarrow$ v) It suffices to show that if $\mathfrak{A}$ and $\mathfrak{B}$ are L-structures and $h : \mathfrak{A} \to \mathfrak{B}$ is a strict homomorphism from $\mathfrak{A}$ onto $\mathfrak{B}$, then $\mathfrak{A}^* \cong \mathfrak{B}^*$. Since h is onto $\mathfrak{B}$, we have that $\overline{a} = (a : a \in A)$ and $\overline{b} = (h(a) : a \in A)$ are enumerations of A and B. And since h is a strict homomorphism, $(\mathfrak{A}, \overline{a}) \equiv_0^- (\mathfrak{B}, \overline{b})$. Therefore, by the implication vii) $\Rightarrow$ v) already proved, we conclude that $\mathfrak{A}^* \cong \mathfrak{B}^*$.

vii) $\Rightarrow$ iii) Suppose that there are enumerations of A and B, $\overline{a} = (a_i : i \in I)$ and $\overline{b} = (b_i : i \in I)$ respectively, such that $(\mathfrak{A}, \overline{a}) \equiv_0^- (\mathfrak{B}, \overline{b})$. Let $\mathfrak{C} = Ter^{\mathfrak{A}}_{\overline{a}}$ (see Definition 1.3) then we have $\mathfrak{A} \in \mathbf{H_S}(\mathfrak{C})$. Since $(\mathfrak{A}, \overline{a}) \equiv_0^- (\mathfrak{B}, \overline{b})$, the function $f_0 : Var_I \to B$ such that for any $i \in I$, $f_0(x_i) = b_i$, can be extended to a strict homomorphism f from $\mathfrak{C}$ onto $\mathfrak{B}$. Therefore, $\mathfrak{A}$, $\mathfrak{B} \in \mathbf{H_S}(\mathfrak{C})$.

vi) $\Rightarrow$ i) Assume that $\mathfrak{A}^* \sim \mathfrak{B}^*$ and let $R^* \subseteq A^* \times B^*$ be a relative correspondence. Define the relation $R \subseteq A \times B$ by

$$aRb \quad \text{iff} \quad [a]_{\Omega(\mathfrak{A})} \, R^* \, [b]_{\Omega(\mathfrak{B})},$$

for any $a \in A$ and any $b \in B$. It is easy to check that R is a relative correspondence between $\mathfrak{A}$ and $\mathfrak{B}$.

i) $\Rightarrow$ vii) Let $R = \{\langle a_i, b_i \rangle : i \in I\}$ be a relative correspondence between $\mathfrak{A}$ and $\mathfrak{B}$. By induction it is easy to prove that for any $\phi(x_1, \ldots, x_n) \in L_0^-$ and any $i_1, \ldots, i_n \in I$,

$$\mathfrak{A} \models \phi\,[a_{i_1}, \ldots, a_{i_n}] \quad \text{iff} \quad \mathfrak{B} \models \phi\,[b_{i_1}, \ldots, b_{i_n}]\,.$$

Then $\overline{a} = (a_i : i \in I)$ and $\overline{b} = (b_i : i \in I)$ are enumerations of A and B respectively, such that $(\mathfrak{A}, \overline{a}) \equiv_0^- (\mathfrak{B}, \overline{b})$. ■

The following result is an straightforward corollary to Proposition 1.6.

Corollary 1.7 *For any class K of L-structures,*

$$\mathbf{H_S}\mathbf{H_S^{-1}}(K) = \mathbf{H_S^{-1}}\mathbf{H_S}(K).$$

The last result of this section shows that, given two L-structures $\mathfrak{A}$ and $\mathfrak{B}$, if we consider sequences of elements $\overline{a} = (a_i : i \in I)$ and $\overline{b} = (b_i : i \in I)$ of A and B respectively, that are not necessarily enumerations, then the substructure $\langle \overline{a} \rangle$ of $\mathfrak{A}$ generated by $\overline{a}$ and the substructure $\langle \overline{b} \rangle$ of $\mathfrak{B}$ generated by $\overline{b}$ are relatives.

Corollary 1.8 *Let $\mathfrak{A}$ and $\mathfrak{B}$ be L-structures and $\overline{a} = (a_i : i \in I)$ and $\overline{b} = (b_i : i \in I)$ sequences of elements of A and B respectively. Then the following are equivalent:*

i) $(\mathfrak{A}, \overline{a}) \equiv_0^- (\mathfrak{B}, \overline{b})$.

ii) $\langle \overline{a} \rangle^* \cong \langle \overline{b} \rangle^*$ *and there is an isomorphism* $h : \langle \overline{a} \rangle^* \to \langle \overline{b} \rangle^*$ *such that for any* $i \in I$, $h([a_i]_{\Omega(\langle \overline{a} \rangle)}) = [b_i]_{\Omega(\langle \overline{b} \rangle)}$.

iii) $\langle \overline{a} \rangle \sim \langle \overline{b} \rangle$ *and there is a relative correspondence* R *between* $\langle \overline{a} \rangle$ *and* $\langle \overline{b} \rangle$ *such that for any* $i \in I$, $a_i R b_i$.

Proof. i) $\Rightarrow$ ii) Assume that $(\mathfrak{A}, \overline{a}) \equiv_0^- (\mathfrak{B}, \overline{b})$. Since $\langle \overline{a} \rangle$ is a substructure of $\mathfrak{A}$ and $\langle \overline{b} \rangle$ is a substructure of $\mathfrak{B}$, it is clear that $(\mathfrak{A}, \overline{a}) \equiv_0^- (\langle \overline{a} \rangle, \overline{a})$ and $(\mathfrak{B}, \overline{b}) \equiv_0^- (\langle \overline{b} \rangle, \overline{b})$. Consequently,

$$(\langle \overline{a} \rangle, \overline{a}) \equiv_0^- (\langle \overline{b} \rangle, \overline{b}).$$

Let Ter be the set of terms of L. We define enumerations $\overline{c}$ and $\overline{d}$, of $\langle \overline{a} \rangle$ and $\langle \overline{b} \rangle$ respectively, by:

$$\overline{c} = \langle t^{\langle \overline{a} \rangle} [a_{i_1}, \ldots, a_{i_n}] : t(x_1, \ldots, x_n) \in Ter, i_1, \ldots, i_n \in I, n \in \omega \rangle$$

$$\overline{d} = \langle t^{\langle \overline{b} \rangle} [b_{i_1}, \ldots, b_{i_n}] : t(x_1, \ldots, x_n) \in Ter, i_1, \ldots, i_n \in I, n \in \omega \rangle.$$

Then,

$$(\langle \overline{a} \rangle, \overline{c}) \equiv_0^- (\langle \overline{b} \rangle, \overline{d}),$$

and by the proof of vii) $\Rightarrow$ v) of Proposition 1.6, $\langle \overline{a} \rangle^* \cong \langle \overline{b} \rangle^*$ and there is an isomorphism $h : \langle \overline{a} \rangle^* \to \langle \overline{b} \rangle^*$ such that for any $i \in I$,$h([a_i]_{\Omega(\langle \overline{a} \rangle)}) = [b_i]_{\Omega(\langle \overline{b} \rangle)}$.

ii) $\Rightarrow$ iii) By the proof of v) $\Rightarrow$ i) of Proposition 1.6.

iii) $\Rightarrow$ i) By the proof of i) $\Rightarrow$ vi) of Proposition 1.6 and the fact that $(\mathfrak{A}, \overline{a}) \equiv_0^- (\langle \overline{a} \rangle, \overline{a})$ and $(\mathfrak{B}, \overline{b}) \equiv_0^- (\langle \overline{b} \rangle, \overline{b})$. ■

2 The method of diagrams

Given an L-structure $\mathfrak{A}$, I define in the natural way, *the equality-free diagram of* $\mathfrak{A}$, in symbols $\mathsf{diag}^-(\mathfrak{A})$, as the set of all equality-free sentences of the diagram of $\mathfrak{A}$. In an analogous way I define *the equality-free elementary diagram of* $\mathfrak{A}$, *the equality-free positive diagram of* $\mathfrak{A}$ and *the equality-free negative diagram of* $\mathfrak{A}$, that will be denoted respectively by $\mathsf{eldiag}^-(\mathfrak{A})$, $\mathsf{posdiag}^-(\mathfrak{A})$ and $\mathsf{negdiag}^-(\mathfrak{A})$. For references on the method of diagrams see [12].

Let us recall some definitions. Given two L-structures $\mathfrak{A}$ and $\mathfrak{B}$ and a map $f : A \to B$, a formula $\phi(\overline{x}) \in L$ is *preserved by* f if for every tuple $\overline{a}$ of elements of A, if $\mathfrak{A} \models \phi[\overline{a}]$, then $\mathfrak{B} \models \phi[f(\overline{a})]$. Given a set of formulas $\Phi \subseteq L$, f is a Φ-*map* if f preserves all the formulas in Φ. Observe that an L^--map need not preserve the values of terms and need not be injective.

Proposition 2.1 *Let* $\mathfrak{A}$ *and* $\mathfrak{B}$ *be* L*-structures. Then the following are equivalent:*

i) *There is an expansion of* $\mathfrak{B}$ *that satisfies* $\mathsf{diag}^-(\mathfrak{A})$.

ii) *There is a* L_0^-*-map* $h : A \to B$.

iii) *There is an enumeration* $\overline{a} = (a_i : i \in I)$ *of* A *and a sequence* $\overline{b} = (b_i : i \in I)$ *of elements of* B *such that* $(\mathfrak{A}, \overline{a}) \equiv_0^- (\mathfrak{B}, \overline{b})$.

iv) $\mathfrak{A} \sim \mathfrak{C}$, *for some* $\mathfrak{C} \subseteq \mathfrak{B}$.

v) $\mathfrak{A} \in \mathbf{H_S^{-1} H_S S}(\mathfrak{B})$.

vi) $\mathfrak{A} \in \mathbf{H_S H_S^{-1} S}(\mathfrak{B})$.

PROOF. Clearly i) ⇔ ii) ⇔ iii). By Proposition 1.6 and Corollary 1.7,
iv) ⇔ v) ⇔ vi) is also clear. iii) ⇒ iv) By Corollary 1.8 we have $\langle\overline{a}\rangle \sim \langle\overline{b}\rangle$ and, since $\overline{a}$ is an enumeration of A, $\mathfrak{A} = \langle\overline{a}\rangle$. Therefore, $\mathfrak{A} \sim \langle\overline{b}\rangle$ and $\langle\overline{b}\rangle \subseteq \mathfrak{B}$.

iv) ⇒ iii) Since $\mathfrak{A} \sim \mathfrak{C}$, for some $\mathfrak{C} \subseteq \mathfrak{B}$, by Proposition 1.6, there are enumerations of A and C, $\overline{a} = (a_i : i \in I)$ and $\overline{b} = (b_i : i \in I)$ respectively, such that $(\mathfrak{A}, \overline{a}) \equiv_0^- (\mathfrak{C}, \overline{b})$. Therefore, since $\mathfrak{C} \subseteq \mathfrak{B}$, $(\mathfrak{C}, \overline{b}) \equiv_0^- (\mathfrak{B}, \overline{b})$ and consequently, $(\mathfrak{A}, \overline{a}) \equiv_0^- (\mathfrak{B}, \overline{b})$ ∎

We will see that for relational similarity types we can improve Proposition 2.1 by seeing that conditions i) – vi) are equivalent to $\mathfrak{A}^* \underset{\sim}{\subset} \mathfrak{B}^*$.

Lemma 2.2 *Let L be relational. For any L-structures $\mathfrak{A}$ and $\mathfrak{B}$, if $\mathfrak{A} \underset{\sim}{\subset} \mathfrak{B}$, then $\mathfrak{A}^* \underset{\sim}{\subset} \mathfrak{B}^*$.*

PROOF. Assume that $\mathfrak{A} \underset{\sim}{\subset} \mathfrak{B}$ and let $h : \mathfrak{A} \to \mathfrak{B}$ be an embedding. Then choose for any equivalence class $x \in \mathfrak{A}^*$ a representative $a_x \in A$. Let X be the set of these representatives. Let $f : \mathfrak{A}^* \to \mathfrak{B}^*$ be defined by: $f([a]_{\Omega(\mathfrak{A})}) = [h(a)]_{\Omega(\mathfrak{B})}$, for any $a \in X$. Since L is relational, it is easy to prove that it is an embedding. ∎

Corollary 2.3 *Let L be relational. For any L-structures $\mathfrak{A}$ and $\mathfrak{B}$ the following are equivalent:*

i) *There is an expansion of $\mathfrak{B}$ that satisfies* $\mathsf{diag}^-(\mathfrak{A})$.

ii) $\mathfrak{A}^* \underset{\sim}{\subset} \mathfrak{B}^*$.

PROOF. i) ⇒ ii) We see that condition iv) of Proposition 2.1 implies $\mathfrak{A}^* \underset{\sim}{\subset} \mathfrak{B}^*$. If $\mathfrak{A} \sim \mathfrak{C}$, for some $\mathfrak{C} \subseteq \mathfrak{B}$, then since L is relational, by Lemma 2.2, $\mathfrak{C}^* \underset{\sim}{\subset} \mathfrak{B}^*$. Therefore, $\mathfrak{A}^* \underset{\sim}{\subset} \mathfrak{B}^*$. ii) ⇒ i) Assume that $\mathfrak{A}^* \underset{\sim}{\subset} \mathfrak{B}^*$. Let $f : A^* \to B^*$ be an embedding. For any $a \in A$, choose an element $b_a \in B$ such that $f([a]_{\Omega(\mathfrak{A})}) = [b_a]_{\Omega(\mathfrak{B})}$. Then using the fact that $\mathfrak{A}^* \underset{\sim}{\subset} \mathfrak{B}^*$ it is easy to check that $(\mathfrak{B}, b_a)_{a \in A}$ satisfies $\mathsf{diag}^-(\mathfrak{A})$. ∎

I introduce now the *Leibniz diagram* of a model. This diagram will allow us to obtain a characterization of when $\mathfrak{A}^* \underset{\sim}{\subset} \mathfrak{B}^*$, for two arbitrary L-structures $\mathfrak{A}$ and $\mathfrak{B}$. The definition of Leibniz diagram was first introduced in [10]. Given an L-structure $\mathfrak{A}$, the *Leibniz diagram* of $\mathfrak{A}$, in symbols $\mathsf{Ldiag}(\mathfrak{A})$, is the set $\mathsf{diag}^-(\mathfrak{A}) \cup \Delta$, where Δ is the set of sentences of $L^-(A)$ of the form $\forall \overline{z}\, [\phi(t_1(a_1, \ldots, a_n), \overline{z}) \leftrightarrow \phi(t_2(b_1, \ldots, b_n), \overline{z})]$ such that

$$t_1^{\mathfrak{A}}[a_1, \ldots, a_n] \equiv t_2^{\mathfrak{A}}[b_1, \ldots, b_n] \qquad \mathsf{mod}(\Omega(\mathfrak{A})),$$

where $\phi(x, \overline{z}) \in L^-$ is an atomic formula, $t_1(x_1, \ldots, x_n)$ and $t_2(y_1, \ldots, y_n)$ are terms of L and $a_1, \ldots, a_n, b_1, \ldots, b_n \in A$.

Proposition 2.4 *Let $\mathfrak{A}$ and $\mathfrak{B}$ be L-structures. Then the following are equivalent:*

i) *There is an expansion of $\mathfrak{B}$ that satisfies* $\mathsf{Ldiag}(\mathfrak{A})$.

ii) $\mathfrak{A}^* \underset{\sim}{\subset} \mathfrak{B}^*$.

Proof. i) $\Rightarrow$ ii) Suppose that there is an expansion of $\mathfrak{B}$ that satisfies $\mathsf{Ldiag}(\mathfrak{A})$. Let $\overline{a} = (a : a \in A)$. Then there is a sequence of elements of $\mathfrak{B}$, $\overline{b} = (b_a : a \in A)$, such that $(\mathfrak{B}, \overline{b}) \models \mathsf{Ldiag}(\mathfrak{A})$. Since $\mathsf{diag}^-(\mathfrak{A}) \subseteq \mathsf{Ldiag}(\mathfrak{A})$, $(\mathfrak{A}, \overline{a}) \equiv_0^- (\mathfrak{B}, \overline{b})$. By Corollary 1.8, there is an isomorphism $f : \mathfrak{A}^* \to \langle\overline{b}\rangle^*$ such that for any $a \in A$, $f([a]_{\Omega(\mathfrak{A})}) = [b_a]_{\Omega(\langle\overline{b}\rangle)}$. Define $g : \langle\overline{b}\rangle^* \to \mathfrak{B}^*$ by: $g([c]_{\Omega(\langle\overline{b}\rangle)}) = [c]_{\Omega(\mathfrak{B})}$, for any $c \in \langle\overline{b}\rangle$. I will show that g is an embedding. Observe that g is well-defined: assume that $c, c' \in \langle\overline{b}\rangle$ and $[c]_{\Omega(\langle\overline{b}\rangle)} = [c']_{\Omega(\langle\overline{b}\rangle)}$. Let $b_{a_1}, \ldots, b_{a_n} \in \mathrm{rg}(\overline{b})$ and $t(\bar{x})$ and $t'(\bar{x})$ terms of L, where $\bar{x} = x_1, \ldots, x_n$, such that $t^{\langle\overline{b}\rangle}[b_{a_1}, \ldots, b_{a_n}] = c$ and $t'^{\langle\overline{b}\rangle}[b_{a_1}, \ldots, b_{a_n}] = c'$. Suppose, searching for a contradiction, that $[c]_{\Omega(\mathfrak{B})} \neq [c']_{\Omega(\mathfrak{B})}$. Then there is an equality-free atomic formula $\phi = \phi(y, \bar{w})$ such that

$$\mathfrak{B} \not\models \forall \bar{w}(\phi(y, \bar{w}) \leftrightarrow \phi(y', \bar{w}))\,[c, c'],$$

where y, y' and the variables in $\bar{w}$ are different from the variables in $\bar{x}$. Let ϕ_1 be the formula obtained by substituting in ϕ the term t for the variable y. And let ϕ_2 be the formula obtained by substituting in ϕ the term t' for the variable y'. Then

$$\mathfrak{B} \not\models \forall \bar{w}(\phi_1(\bar{x}, \bar{w}) \leftrightarrow \phi_2(\bar{x}, \bar{w}))\,[b_{a_1}, \ldots, b_{a_n}]. \tag{2.1}$$

But $[c]_{\Omega(\langle\overline{b}\rangle)} = [c']_{\Omega(\langle\overline{b}\rangle)}$, therefore we have

$$\langle\overline{b}\rangle \models \forall \bar{w}(\phi_1(\bar{x}, \bar{w}) \leftrightarrow \phi_2(\bar{x}, \bar{w}))\,[b_{a_1}, \ldots, b_{a_n}],$$

then

$$\langle\overline{b}\rangle^* \models \forall \bar{w}(\phi_1(\bar{x}, \bar{w}) \leftrightarrow \phi_2(\bar{x}, \bar{w}))\left[[b_{a_1}]_{\Omega(\langle\overline{b}\rangle)}, \ldots, [b_{a_n}]_{\Omega(\langle\overline{b}\rangle)}\right],$$

and since $f : \mathfrak{A}^* \to \langle\overline{b}\rangle^*$ is an isomorphism such that for any $a \in A$, $f([a]_{\Omega(\mathfrak{A})}) = [b_a]_{\Omega(\langle\overline{b}\rangle)}$, we have that

$$\mathfrak{A} \models \forall \bar{w}(\phi_1(\bar{x}, \bar{w}) \leftrightarrow \phi_2(\bar{x}, \bar{w}))\,[a_1, \ldots, a_n]$$

and since $(\mathfrak{B}, \overline{b}) \models \mathsf{Ldiag}(\mathfrak{A})$,

$$\mathfrak{B} \models \forall \bar{w}(\phi_1(\bar{x}, \bar{w}) \leftrightarrow \phi_2(\bar{x}, \bar{w}))\,[b_{a_1}, \ldots, b_{a_n}],$$

but this contradicts (2.1). Therefore, we can conclude that g is well-defined. Moreover, g is clearly injective and it is a strict homomorphism because $\langle\overline{b}\rangle \subseteq \mathfrak{B}$. Therefore, $\mathfrak{A}^* \stackrel{\subseteq}{\sim} \mathfrak{B}^*$. ii) $\Rightarrow$ i) Let $f : \mathfrak{A}^* \to \mathfrak{B}^*$ be an embedding. For any $a \in A$, choose an element $b_a \in B$ such that $f([a]_{\Omega(\mathfrak{A})}) = [b_a]_{\Omega(\mathfrak{B})}$. Then $(\mathfrak{B}, b_a)_{a \in A}$ satisfies $\mathsf{Ldiag}(\mathfrak{A})$. ■

Corollary 2.5 *Let L be relational. For any L-structures $\mathfrak{A}$ and $\mathfrak{B}$ the following are equivalent:*

i) *There is an expansion of* $\mathfrak{B}$ *that satisfies* $\mathsf{diag}^-(\mathfrak{A})$.

ii) *There is an expansion of* $\mathfrak{B}$ *that satisfies* $\mathsf{Ldiag}(\mathfrak{A})$.

iii) $\mathfrak{A}^* \stackrel{\subseteq}{\sim} \mathfrak{B}^*$.

I introduce now the *equality-free elementary diagram* of a model. Given two structures $\mathfrak{A}$ and $\mathfrak{B}$, by means of this diagram I will present a characterization of when $\mathfrak{A}^* \precsim^- \mathfrak{B}^*$. Let us see before some preliminary lemmas.

Lemma 2.6 *For any L-structures $\mathfrak{A}$ and $\mathfrak{B}$, if $\mathfrak{A} \precsim^- \mathfrak{B}$, then $\mathfrak{A}^* \precsim^- \mathfrak{B}^*$.*

Proof. Assume that $\mathfrak{A} \precsim^- \mathfrak{B}$ and let $h : \mathfrak{A} \to \mathfrak{B}$ be an embedding that preserves all the equality-free formulas. Let $f : \mathfrak{A}^* \to \mathfrak{B}^*$ be defined by: $f([a]_{\Omega(\mathfrak{A})}) = [h(a)]_{\Omega(\mathfrak{B})}$, for any $a \in A$. Since $\mathfrak{A} \precsim^- \mathfrak{B}$, it is easy to prove with the usual arguments that given $a, a' \in A$,

$$[a]_{\Omega(\mathfrak{A})} = [a']_{\Omega(\mathfrak{A})} \Longleftrightarrow [h(a)]_{\Omega(\mathfrak{B})} = [h(a')]_{\Omega(\mathfrak{B})} .$$

We conclude from this fact that f is well-defined and it is injective. Using the fact that $\mathfrak{A} \precsim^- \mathfrak{B}$, it is easy to show that f is an embedding that preserves all the equality-free formulas. ■

Lemma 2.7 *For any class K of L-structures,*

i) $\mathbf{S}^{\preceq^-}\mathbf{H}_\mathbf{S}^{-1}(K) \subseteq \mathbf{H}_\mathbf{S}^{-1}\mathbf{S}^{\preceq^-}(K)$.
ii) $\mathbf{S}^{\preceq^-}\mathbf{H}_\mathbf{S}(K) \subseteq \mathbf{H}_\mathbf{S}\mathbf{S}^{\preceq^-}(K)$.

Proof. See [11], Lemma 4.2. ■

Proposition 2.8 *Let $\mathfrak{A}$ and $\mathfrak{B}$ be L-structures. Then the following are equivalent:*

i) *There is an expansion of $\mathfrak{B}$ that satisfies* $\mathsf{eldiag}^-(\mathfrak{A})$.
ii) *There is a L^--map $h : A \to B$.*
iii) *There is an enumeration of A, $\overline{a} = (a_i : i \in I)$, and a sequence of elements of B, $\overline{b} = (b_i : i \in I)$, such that $(\mathfrak{A}, \overline{a}) \equiv^- (\mathfrak{B}, \overline{b})$.*
iv) $\mathfrak{A} \sim \mathfrak{C}$, *for some* $\mathfrak{C} \preceq^- \mathfrak{B}$.
v) $\mathfrak{A} \in \mathbf{H}_\mathbf{S}^{-1}\mathbf{H}_\mathbf{S}\mathbf{S}^{\preceq^-}(\mathfrak{B})$
vi) $\mathfrak{A} \in \mathbf{H}_\mathbf{S}\mathbf{H}_\mathbf{S}^{-1}\mathbf{S}^{\preceq^-}(\mathfrak{B})$.
vii) $\mathfrak{A}^* \precsim^- \mathfrak{B}^*$.

Proof. Clearly i) ⇔ ii) ⇔ iii). And iv) ⇔ v) ⇔ vi) is also clear by Proposition 1.6 and Corollary 1.7. iv) ⇒ iii) Since $\mathfrak{A} \sim \mathfrak{C}$, for some $\mathfrak{C} \preceq^- \mathfrak{B}$, by Proposition 1.6, there are enumerations of A and C, $\overline{a} = (a_i : i \in I)$ and $\overline{b} = (b_i : i \in I)$ respectively, such that $(\mathfrak{A}, \overline{a}) \equiv^- (\mathfrak{C}, \overline{b})$. Therefore, since $\mathfrak{C} \preceq^- \mathfrak{B}$, $(\mathfrak{C}, \overline{b}) \equiv^- (\mathfrak{B}, \overline{b})$ and consequently, $(\mathfrak{A}, \overline{a}) \equiv^- (\mathfrak{B}, \overline{b})$.

iii) ⇒ iv) Since $(\mathfrak{A}, \overline{a}) \equiv^- (\mathfrak{B}, \overline{b})$, by Corollary 1.8, $\mathfrak{A}^* \cong \langle\overline{b}\rangle^*$ and there is an isomorphism $h : \mathfrak{A}^* \to \langle\overline{b}\rangle^*$ such that for any $i \in I$, $h([a_i]_{\Omega(\mathfrak{A})}) = [b_i]_{\Omega(\langle\overline{b}\rangle)}$. Thus, $\mathfrak{A} \sim \langle\overline{b}\rangle$ and for any $\phi(x_1, \ldots, x_n) \in L^-$, any $i_1, \ldots, i_n \in I$,

$$\begin{aligned}
\langle\overline{b}\rangle \models \phi\,[b_{i_1}, \ldots, b_{i_n}] \quad &\Leftrightarrow \quad \langle\overline{b}\rangle^* \models \phi\left[[b_{i_1}]_{\Omega(\langle\overline{b}\rangle)}, \ldots, [b_{i_n}]_{\Omega(\langle\overline{b}\rangle)}\right] \\
&\Leftrightarrow \quad \mathfrak{A}^* \models \phi\left[[a_{i_1}]_{\Omega(\mathfrak{A})}, \ldots, [a_{i_n}]_{\Omega(\mathfrak{A})}\right] \\
&\Leftrightarrow \quad \mathfrak{A} \models \phi\,[a_{i_1}, \ldots, a_{i_n}] \\
&\Leftrightarrow \quad \mathfrak{B} \models \phi\,[b_{i_1}, \ldots, b_{i_n}] .
\end{aligned}$$

Therefore $\langle\overline{b}\rangle \preceq^- \mathfrak{B}$, so condition iv) holds.

iv) ⇒ vii) by Lemma 2.6. vii) ⇒ v) Since $\mathfrak{A}^* \precsim^- \mathfrak{B}^*$ implies that $\mathfrak{A} \in \mathbf{H}_\mathbf{S}^{-1}\mathbf{S}^{\preceq^-}\mathbf{H}_\mathbf{S}(\mathfrak{B})$, by Lemma 2.7, $\mathfrak{A} \in \mathbf{H}_\mathbf{S}^{-1}\mathbf{H}_\mathbf{S}\mathbf{S}^{\preceq^-}(\mathfrak{B})$. ■

I end this section with the *equality-free positive diagram* and the *equality-free negative diagram* of a model.

Proposition 2.9 *Let $\mathfrak{A}$ and $\mathfrak{B}$ be L-structures and* At^- *the set of atomic formulas of L^-. Then the following are equivalent:*

i) *There is an expansion of $\mathfrak{B}$ that satisfies* $\mathsf{posdiag}^-(\mathfrak{A})$.

ii) *There is an* At^-*-map* $h : A \to B$.

iii) $\mathfrak{A} \in \mathbf{H_S H^{-1} S}(\mathfrak{B})$.

Proof. i) $\Leftrightarrow$ ii) is clear. ii) $\Rightarrow$ iii) Assume that $h : A \to B$ is an At^--map. Let $\mathfrak{C}$ be the substructure of $\mathfrak{B}$ generated by $h[A]$. Let $\kappa = |A|$ and let $\overline{a} = (a_\alpha : \alpha \in \kappa)$ be an enumeration of A without repetitions. Consider the L-structure $Ter^{\mathfrak{A}}_{\overline{a}}$ of Definition 1.3 and the function $g : V_\kappa \to C$, defined by: $g(x_\alpha) = h(a_\alpha)$, for any $\alpha \in \kappa$. The function g can be extended to an homomorphism g' from Ter_{V_κ} into the algebraic reduct of $\mathfrak{C}$. Since $\mathfrak{C}$ is generated by $h[A]$, g' is onto C. Using the fact that h preserves equality-free atomic formulas it is routine to show that g' is an homomorphism from $Ter^{\mathfrak{A}}_{\overline{a}}$ onto $\mathfrak{C}$. Therefore, since $\mathfrak{A} \in \mathbf{H_S}(Ter^{\mathfrak{A}}_{\overline{a}})$, $\mathfrak{A} \in \mathbf{H_S H^{-1} S}(\mathfrak{B})$.

iii) $\Rightarrow$ ii) Assume that $\mathfrak{A} \in \mathbf{H_S H^{-1} S}(\mathfrak{B})$. Then there are L-structures $\mathfrak{C}$ and $\mathfrak{D}$ with the following properties: (1) $\mathfrak{C} \subseteq \mathfrak{B}$, (2) there is a strict homomorphism f from $\mathfrak{D}$ onto $\mathfrak{A}$ and (3) there is an homomorphism g from $\mathfrak{D}$ onto $\mathfrak{C}$. Choose for any $a \in A$, an element $d_a \in f^{-1}[a]$. Let $h : \mathfrak{A} \to \mathfrak{B}$ be defined by: $h(a) = g(d_a)$, for any $a \in A$. So defined h is clearly an At^--map. ∎

Proposition 2.10 *Let $\mathfrak{A}$ and $\mathfrak{B}$ be L-structures and* Negat^- *the set of negations of atomic formulas of L^-. Then the following are equivalent:*

i) There is an expansion of $\mathfrak{B}$ that satisfies $\mathsf{negdiag}^-(\mathfrak{A})$.

ii) There is a Negat^-*-map* $h : A \to B$.

iii) $\mathfrak{A} \in \mathbf{H H_S^{-1} S}(\mathfrak{B})$.

Proof. i) $\Leftrightarrow$ ii) is clear. ii) $\Rightarrow$ iii) Assume that $h : A \to B$ is a Negat^--map. Let $\mathfrak{C}$ be the substructure of $\mathfrak{B}$ generated by $h[A]$, $\overline{c} = (h(a) : a \in A)$ be an enumeration of $h[A]$ and Ter_{V_A} be the algebra of terms of type L generated by the set $V_A = \{x_a : a \in A\}$. Consider the function $g : V_A \to A$, defined by: $g(x_a) = a$, for any $a \in A$. The function g can be extended to an homomorphism g' from Ter_{V_A} into the algebraic reduct of $\mathfrak{A}$. Now let $f : V_A \to C$ be defined by: $f(x_a) = h(a)$. The function f can be extended to an homomorphism f' from Ter_{V_A} onto the algebraic reduct of $\mathfrak{C}$. Let now $\mathfrak{D}$ be the L-structure with algebraic reduct Ter_{V_A} and with the following interpretation for the relation symbols: For any n-adic relation symbol $R \in L$, for any $t_1, \ldots, t_n \in Ter_{V_A}$

$$\langle t_1, \ldots, t_n \rangle \in R^{\mathfrak{D}} \Longleftrightarrow \langle f'(t_1), \ldots, f'(t_n) \rangle \in R^{\mathfrak{C}}.$$

Then f' is a strict homomorphism from $\mathfrak{D}$ onto $\mathfrak{C}$. Using the fact that h preserves equality-free negated atomic formulas it is easy to show that g' is an homomorphism from $\mathfrak{D}$ onto $\mathfrak{A}$. Therefore, since $\mathfrak{A} \in \mathbf{H}(\mathfrak{D})$, $\mathfrak{A} \in \mathbf{H H_S^{-1} S}(\mathfrak{B})$.

iii) $\Rightarrow$ ii) Assume that $\mathfrak{A} \in \mathbf{H H_S^{-1} S}(\mathfrak{B})$. Then there are L-structures $\mathfrak{C}$ and $\mathfrak{D}$ with the following properties: (1) $\mathfrak{C} \subseteq \mathfrak{B}$, (2) there is an homomorphism f from $\mathfrak{D}$ onto $\mathfrak{A}$ and (3) there is a strict homomorphism g from $\mathfrak{D}$ onto $\mathfrak{C}$. Choose for any $a \in A$, an element $d_a \in f^{-1}[a]$. Let $h : \mathfrak{A} \to \mathfrak{B}$ be defined by: $h(a) = g(d_a)$, for any $a \in A$. So defined h is clearly an Negat^--map. ∎

3 Universal-existential equality-free classes

Theories preserved by unions of chains were characterized by J. Łoś and R. Suszko in [13], C. C. Chang improved this result in [6]. I give an analogous theorem for Equality-free Logic. A formula $\phi \in L$ is *universal-existential* if $\phi = \forall \overline{y} \exists \overline{x} \psi$, for some quantifier-free formula ψ. Given a class K of L-structures, let

$$\mathrm{Th}^{\forall\exists^-}(K) = \left\{ \sigma \in \mathrm{Th}^-(K) : \sigma \text{ is universal-existential} \right\}.$$

Given an L-structure $\mathfrak{A}$, expand the language by adding a new constant symbol for each element of A. $\mathsf{univdiag}^-(\mathfrak{A})$ will denote the following set of sentences in the expanded language

$$\mathsf{univdiag}^-(\mathfrak{A}) = \left\{ \sigma \in L^-(A) : \sigma \text{ is a universal sentence and } \mathfrak{A}_A \models \sigma \right\}.$$

Lemma 3.1 *Suppose that $\mathfrak{A}$ and $\mathfrak{B}$ are L-structures such that any equality-free universal-existential sentence true in $\mathfrak{B}$ is also true in $\mathfrak{A}$. Then there are L-structures $\mathfrak{C}$ and $\mathfrak{D}$ such that:*

i) $\mathfrak{A}^* \subseteq \mathfrak{C} \subseteq \mathfrak{D}^*$.

ii) $\mathfrak{A}^* \preceq^- \mathfrak{D}^*$.

iii) $\mathfrak{B} \equiv^- \mathfrak{C}$.

Proof. Expand the language introducing a set $S_A = \{c_a : a \in A\}$ of new constant symbols and consider the set $\Gamma = \mathrm{Th}^-(\mathfrak{B}) \cup \mathsf{univdiag}^-(\mathfrak{A})$ of sentences in the expanded language. By standard arguments one can see that Γ is consistent. Let $(\mathfrak{B}', c^{\mathfrak{B}'})_{c \in S_A}$ be a model of Γ. Then $\mathfrak{B} \equiv^- \mathfrak{B}'$. Now expand the language further by adding a set

$$S_{B'} = \left\{ c_b : b \in B' - \left\{ c^{\mathfrak{B}'} : c \in S_A \right\} \right\}$$

of new constant symbols disjoint from $S_A \cup L$. Consider $\mathsf{eldiag}^-(\mathfrak{A})$, the equality-free elementary diagram of $\mathfrak{A}$ in the expanded language $L \cup S_A$ and $\mathsf{diag}^-(\mathfrak{B}')$, the equality-free diagram of $\mathfrak{B}'$ in the expanded language $L \cup S_A \cup S_{B'}$. Let Σ be the union of these two diagrams. Σ is consistent, let $(\mathfrak{D}, c^{\mathfrak{D}})_{c \in S_A \cup S_{B'}}$ be a model of Σ.

On the one hand, since $(\mathfrak{D}, c^{\mathfrak{D}})_{c \in S_A \cup S_{B'}}$ is a model of $\mathsf{eldiag}^-(\mathfrak{A})$, by Proposition 2.8, $\mathfrak{A}^* \precsim^- \mathfrak{D}^*$ and using the usual arguments, it is easy to check that the map $h : \mathfrak{A}^* \to \mathfrak{D}^*$ defined by: $h([a]_{\Omega(\mathfrak{A})}) = \left[c_a^{\mathfrak{D}}\right]_{\Omega(\mathfrak{D})}$, for every $a \in A$, is an embedding that preserves equality-free formulas. We may assume without loss of generality that $[a]_{\Omega(\mathfrak{A})} = \left[c_a^{\mathfrak{D}}\right]_{\Omega(\mathfrak{D})}$, for every $a \in A$. Let $\mathfrak{C}$ be the substructure of $\mathfrak{D}^*$ generated by the set

$$\left\{ \left[c^{\mathfrak{D}}\right]_{\Omega(\mathfrak{D})} : c \in S_A \cup S_{B'} \right\}.$$

Then we have that $\mathfrak{A}^* \subseteq \mathfrak{C} \subseteq \mathfrak{D}^*$ and $\mathfrak{A}^* \preceq^- \mathfrak{D}^*$.

On the other hand, since all the sentences in $\mathsf{diag}^-(\mathfrak{B}')$ are equality-free, $(\mathfrak{D}^*, \left[c^{\mathfrak{D}}\right]_{\Omega(\mathfrak{D})})_{c \in S_A \cup S_{B'}}$ is also a model of $\mathsf{diag}^-(\mathfrak{B}')$. Thus, by the proof of iii) $\Rightarrow$ iv) in Proposition 2.1, $\mathfrak{B}' \sim \mathfrak{C}$ and consequently, $\mathfrak{B} \equiv^- \mathfrak{C}$. So defined, the models $\mathfrak{C}$ and $\mathfrak{D}$ satisfy the required conditions. ∎

Given a class K of structures, a subset $X \subseteq K$ is an *upward directed subset* if for every two structures $\mathfrak{A}, \mathfrak{B} \in X$ there is a structure $\mathfrak{C} \in X$ such that $\mathfrak{A} \cup \mathfrak{B} \subseteq \mathfrak{C}$. A class K is *closed under unions of upward directed subsets* if for every upward directed subset $X \subseteq K$, $\bigcup X \in K$.

Theorem 3.2 *If K is a class of L-structures, the following are equivalent:*

i) *K is axiomatizable by a set of equality-free universal-existential sentences.*
ii) *K is closed under ultraproducts, $\mathbf{H_S}$, $\mathbf{H_S^{-1}}$ and unions of upward directed subsets and for every L-structure $\mathfrak{A}$, if some ultrapower of $\mathfrak{A}$ lies in K, then $\mathfrak{A} \in K$.*
iii) *K is closed under ultraproducts, $\mathbf{H_S}$, $\mathbf{H}_S^{-1}$ and unions of countable chains and for every L-structure $\mathfrak{A}$, if some ultrapower of $\mathfrak{A}$ lies in K, then $\mathfrak{A} \in K$.*
iv) *K is closed under ultraproducts, $\mathbf{H_S}$, $\mathbf{H_S^{-1}}$ and unions of chains and for every L-structure $\mathfrak{A}$, if some ultrapower of $\mathfrak{A}$ lies in K, then $\mathfrak{A} \in K$.*

Proof. i) $\Rightarrow$ ii), i) $\Rightarrow$ iii), ii) $\Rightarrow$ iii), iv) $\Rightarrow$ iii) and i) $\Rightarrow$ iv) are clear. iii) $\Rightarrow$ i) Since K is closed under ultraproducts, $\mathbf{H_S}$ and $\mathbf{H_S^{-1}}$ and the complement of K is closed under ultrapowers, by Theorem 4.3, K is axiomatizable by a set T of equality-free sentences. I prove that, for every L-structure $\mathfrak{A}$, if $\mathfrak{A} \models \mathrm{Th}^{\forall\exists^-}(K)$, then $\mathfrak{A} \models T$ and consequently, $\mathfrak{A} \in K$. Assume that $\mathfrak{A} \models \mathrm{Th}^{\forall\exists^-}(K)$.

First, I show that there is an L-structure $\mathfrak{B} \in K$ such that any equality-free universal-existential sentence true in $\mathfrak{B}$ is also true in $\mathfrak{A}$. Consider the set of sentences $\Sigma = T \cup \{\neg\sigma : \sigma \text{ an } \forall\exists \text{ sentence}, \mathfrak{A} \not\models \sigma\}$. Σ is constistent, otherwise, there will be universal-existential sentences $\sigma_1, \ldots, \sigma_n$ with $\mathfrak{A} \not\models \sigma_i$, for each i, and such that $T \models \sigma_1 \vee \ldots \vee \sigma_n$. But $\sigma_1 \vee \ldots \vee \sigma_n$ is equivalent to an universal-existential sentence, say σ. As $T \models \sigma$, we have $\sigma \in \mathrm{Th}^{\forall\exists^-}(K)$ and then $\mathfrak{A} \models \sigma$, contradiction. Now, any model $\mathfrak{B}$ of Σ will do.

Now define by induction two sequences of models, $(\mathfrak{A}_n : n \in \omega)$ and $(\mathfrak{C}_{n+1} : n \in \omega)$ with the following properties: for every $n \in \omega$,

a) Any equality-free universal-existential sentence true in $\mathfrak{B}$ is also true in $\mathfrak{A}_n$.
b) $\mathfrak{A}_n^* \subseteq \mathfrak{C}_{n+1} \subseteq \mathfrak{A}_{n+1}^*$.
c) $\mathfrak{A}_n^* \preceq^- \mathfrak{A}_{n+1}^*$.
d) $\mathfrak{B} \equiv^- \mathfrak{C}_{n+1}$.

Let $\mathfrak{A}_0 = \mathfrak{A}$ and apply Lemma 3.1 to obtain $\mathfrak{A}_1$ and $\mathfrak{C}_1$. Let $n > 0$ and suppose inductively that $\mathfrak{A}_n$ and $\mathfrak{C}_n$ have been defined with properties a) – d). Apply again Lemma 3.1 to obtain $\mathfrak{A}_{n+1}$ and $\mathfrak{C}_{n+1}$Now let $\mathfrak{E} = \bigcup_{n\in\omega} \mathfrak{C}_{n+1} = \bigcup_{n\in\omega} \mathfrak{A}_n^*$. Since $\mathfrak{B} \models T$ and for every $n \in \omega$, $\mathfrak{B} \equiv^- \mathfrak{C}_{n+1}$, for every $n \in \omega$, $\mathfrak{C}_{n+1} \models T$. Therefore, since by assumption, K is closed under unions of countable chains, $\mathfrak{E} \models T$. And since $\mathfrak{A}_n^* \preceq^- \mathfrak{A}_{n+1}^*$, for every $n \in \omega$, $\mathfrak{A}^* = \mathfrak{A}_0^* \preceq^- \mathfrak{E}$. Then, since T is a set of equality-free sentences, $\mathfrak{A}^* \models T$ and consequently, $\mathfrak{A} \models T$. ■

Corollary 3.3 *Let $T \cup \{\sigma\}$ be a set of sentences of L. Then:*

i) *T is axiomatizable by a set of equality-free universal-existential sentences if and only if T is preserved under $\mathbf{H_S^{-1}}$ and $\mathbf{H_S}$ and unions of chains.*

ii) *σ is logically equivalent to an equality-free universal-existential sentence if and only if σ is preserved under $\mathbf{H_S^{-1}}$ and $\mathbf{H_S}$ and unions of chains.*

PROOF. By Theorem 3.2, using the fact that any finite conjunction of universal-existential sentences is logically equivalent to one universal-existential sentence. ∎

4 Positive equality-free classes

The characterization theorem for the positive fragment of first-order logic with equality is due to R. C. Lyndon, see [14]. Now I obtain a version of it for equality-free languages. We shall consider now only formulas built up from atomic and negation of atomic formulas using only the connectives $\wedge$, $\vee$ and the quantifiers $\forall$, $\exists$. Given a symbol s (s can belong to L or s can be the identity symbol) and a formula ϕ of L, we say that s occurs *positively* in ϕ if and only if s has an occurrence in ϕ, which is not within the scope of a negation symbol. And we say that s occurs *negatively* in ϕ if and only if s has an occurrence in ϕ, which is within the scope of a negation symbol. Given a formula $\phi \in L$, let $Rel(\phi)$ ($Fun(\phi)$) be the set of relation (function) symbols of L that occur in ϕ, and let $Rel^+(\phi)$ ($Rel^-(\phi)$) be the set of symbols of $Rel(\phi)$ that occur positively (negatively) in ϕ.

In [15] we can find the following extended version of Lyndon's interpolation theorem, due to A. Oberschelp and T. Fujiwara.

Theorem 4.1 *Suppose that ϕ and ψ are sentences of L such that $\phi \models \psi$, $\not\models \neg\phi$ and $\not\models \psi$. Then there is a sentence θ of L such that:*

i) $\phi \models \theta$ *and* $\theta \models \psi$.

ii) $Rel^+(\theta) \subseteq Rel^+(\phi) \cap Rel^+(\psi)$ *and* $Rel^-(\theta) \subseteq Rel^-(\phi) \cap Rel^-(\psi)$.

iii) $Fun(\theta) \subseteq Fun(\phi) \cap Fun(\psi)$.

iv) *If θ has at least one positive (negative) occurrence of the identity symbol, then ϕ (ψ) has at least one positive (negative) occurrence of the identity symbol.*

A formula $\phi \in L$ is *positive* if ϕ is built up from atomic formulas using only the connectives $\wedge$, $\vee$ and the quantifiers $\forall$, $\exists$. Let us recall Lyndon's characterization theorem for the positive fragment of logic with equality (see [14]):

Theorem 4.2 *If K is a class of L-structures, the following are equivalent:*

i) *K is axiomatizable by a set of positive sentences.*

ii) *K is closed under ultraproducts, $\mathbf{H}$ and for every L-structure $\mathfrak{A}$, if some ultrapower of $\mathfrak{A}$ lies in K, then $\mathfrak{A} \in K$.*

Now I prove its version for Equality-free Logic. In the proof I will use the following algebraic characterization of the elementary equality-free classes, given in [5].

Theorem 4.3 *Let K be a class of L-structures. The following are equivalent:*

i) *K is axiomatizable by a set of equality-free sentences.*

ii) *K is closed under ultraproducts, $\mathbf{H_S}$ and $\mathbf{H_S^{-1}}$ and for any L-structure $\mathfrak{A}$ the following holds: if some ultrapower of $\mathfrak{A}$ lies in K, then $\mathfrak{A} \in K$.*

Theorem 4.4 *If K is a non-empty class of L-structures, the following are equivalent:*

i) *K is axiomatizable by a set of equality-free positive sentences.*

ii) *K is closed under ultraproducts, $\mathbf{H}$ and $\mathbf{H_S^{-1}}$ and for every L-structure $\mathfrak{A}$, if some ultrapower of $\mathfrak{A}$ lies in K, then $\mathfrak{A} \in K$.*

Proof. i) $\Rightarrow$ ii) is clear. ii) $\Rightarrow$ i) Since K is closed under $\mathbf{H}$, K is also closed under $\mathbf{H_S}$. Therefore, by ii) and Theorem 4.3, K is axiomatizable by a set Γ of equality-free sentences. Moreover, by ii) and Theorem 4.2, K is axiomatizable by a set Σ of positive sentences (possibly with equality). Since $\Gamma \models \Sigma$, for every $\sigma \in \Sigma$ such that $\not\models \sigma$ we can choose $\alpha_1, \ldots, \alpha_n \in \Gamma$ such that

$$\alpha_1, \ldots, \alpha_n \models \sigma.$$

Let $\alpha_\sigma = \alpha_1 \wedge \ldots \wedge \alpha_n$; we have that $\alpha_\sigma \models \sigma$ and α_σ is an equality-free sentence. Furthermore, $\not\models \neg\alpha_\sigma$, because K is non-empty. Therefore, there is a sentence $\theta_\sigma \in L$ that satisfies conditions i) $-$ iv) of Theorem 4.1. Observe that, by condition iv), since in α_σ does not occur the identity symbol, in θ_σ the equality symbol does not occur positively. Moreover, since σ is positive, in θ_σ the equality symbol does not occur negatively. Then we can conclude that θ_σ is an equality-free sentence and by condition ii), since σ is positive,

$$Rel^-(\theta_\sigma) \subseteq Rel^-(\alpha_\sigma) \cap Rel^-(\sigma) = Rel^-(\alpha_\sigma) \cap \emptyset = \emptyset.$$

Then θ_σ is a positive sentence. Thus, $\{\theta_\sigma : \sigma \in \Sigma \text{ and } \not\models \sigma\}$ is a set of equality-free positive sentences that axiomatizes K. ∎

Corollary 4.5 *Let T be a consistent set of sentences of L and σ a consistent sentence of L. Then:*

i) *T is axiomatizable by a set of equality-free positive sentences if and only if T is preserved under $\mathbf{H_S^{-1}}$ and $\mathbf{H}$.*

ii) *σ is logically equivalent to an equality-free positive sentence if and only if σ is preserved under $\mathbf{H_S^{-1}}$ and $\mathbf{H}$.*

Proof. By Theorem 4.4, using the fact that any finite conjunction of positive sentences is a positive sentence. ∎

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References

[1] W. J. Blok, D. Pigozzi (1986): "Protoalgebraic logics", *Studia Logica* 45:337–369.

[2] W. J. Blok, D. Pigozzi (1992): "Algebraic semantics for universal Horn logic without equality". In: Romanowska, A.; Smith, J. D. H. (Eds.), *Universal Algebra and Quasigroups.* Heldermann Verlag.

[3] W. J. Blok, D. Pigozzi (1989): *Algebraizable Logics*, Memoirs of the Am. Math. Soc. 396.

[4] S. L. Bloom (1975): "Some theorems on structural consequence operations", *Studia Logica* 34:1–9.

[5] E. Casanovas, P. Dellunde, R. Jansana (1996): "On elementary equivalence for Equality-free Logic", *Notre Dame Journal of Formal Logic* 37:506–522.

[6] C. C. Chang (1959): "On unions of chains of models", *Proc. Amer. Math. Soc.* 10:120–127.

[7] C. C. Chang, J. Keisler (1991): *Model theory*. Amsterdam: North-Holland. Studies in Logic and the Fundations of Mathematics, 73.

[8] P. Dellunde (1996): *Contributions to the Model Theory of Equality-Free Logic*, Ph.D. Thesis, Universitat de Barcelona.

[9] P. Dellunde, R. Jansana, (1996): Some Characterization Theorems for Infinitary Universal Horn Logic without equality, *The Journal of Symbolic Logic* 61:1242–1260.

[10] R. Elgueta (1994): *Algebraic Model Theory for Languages without equality*, Ph.D. Thesis, Universitat de Barcelona.

[11] R. Elgueta (1997): "Characterizing classes defined without equality", *Studia Logica* 58:357–394.

[12] W. Hodges (1993): *Model Theory.* Cambridge University Press.

[13] J. Łos, R. Suszko (1955): "On the extending of models II", *Fundamenta Math.* 44:52–60.

[14] R. C. Lyndon (1959): "An interpolation theorem in the predicate calculus", *Pacific J. Math.* 9:129–142.

[15] N. Motohashi (1984): "Equality and Lyndon's interpolation theorem", *The Journal of Symbolic Logic* 49:123–128.

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