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SOME CHARACTERIZATION THEOREMS FOR INFINITARY UNIVERSAL HORN LOGIC WITHOUT EQUALITY

PILAR DELLUNDE AND RAMON JANSANA

In this paper we mainly study preservation theorems for two fragments of the infinitary languages $L_{\kappa\kappa}$, with κ regular, without the equality symbol: the universal Horn fragment and the universal strict Horn fragment. In particular, when κ is ω , we obtain the corresponding theorems for the first-order case.

The universal Horn fragment of first-order logic (with equality) has been extensively studied; for references see [10], [7] and [8]. But the universal Horn fragment without equality, used frequently in logic programming, has received much less attention from the model theoretic point of view. At least to our knowledge, the problem of obtaining preservation results for it has not been studied before by model theorists. In spite of this, in the field of abstract algebraic logic we find a theorem which, properly translated, is a preservation result for the strict universal Horn fragment of infinitary languages without equality which, apart from function symbols, have only a unary relation symbol. This theorem is due to J. Czelakowski; see [5], Theorem 6.1, and [6], Theorem 5.1. A. Torrens [12] also has an unpublished result dealing with matrices of sequent calculi which, properly translated, is a preservation result for the strict universal Horn fragment of a first-order language. And in [2] of W. J. Blok and D. Pigozzi we find Corollary 6.3 which properly translated corresponds to our Corollary 19, but for the case of a first-order language that apart from its function symbols has only one k -ary relation symbol, and for strict universal Horn sentences. The study of these results is the basis for the present work. In the last part of the paper, Section 4, we will make these connections clear and obtain some of these results from our theorems. In this way we hope to make clear two things: (1) The field of abstract algebraic logic can be seen, in part, as a disguised study of universal Horn logic without equality and so has an added interest. (2) A general study of universal Horn logic without equality from a model theoretic point of view can be of help in the field of abstract algebraic logic.

The main result of the paper is Theorem 6, a theorem that establishes the closure conditions that a class of structures of a given similarity type has to satisfy in order to be the class of models of a set of universal Horn sentences without equality of the infinitary language $L_{\kappa\kappa}$, for κ an infinite regular cardinal. Theorem 9 gives a better characterization when κ is strongly compact or is ω . These theorems are proved using Lemma 5. This lemma can also be used to obtain a characterization theorem for the first-order universal classes without equality, and an analogous

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lemma (obtained by minor modifications) can be used to prove characterization theorems for the first-order logic without equality and some of its fragments such as the positive one. However, in this paper we restrict ourselves to the universal Horn case. This lemma also permits us to prove two more theorems similar to Theorem 6, one for κ -local classes of structures and the other for the infinitary language $L_{\infty\infty}$ that has a proper class of variables and allows conjunctions of any set of formulas and quantification of any set of variables.

From the main theorems we draw several consequences. A first group is to do with the analogous theorems for the case of universal strict Horn sentences. A second group (see Section 2) deals with characterization theorems for the classes of reduced structures (see Definition 3) that are the classes of reduced models of a universal Horn theory without equality in $L_{\kappa\kappa}$. And a third group of consequences (see Section 3) deals with joint consistency, interpolation, and definability theorems.

§1. Universal Horn classes without equality. Let κ be a regular cardinal $\geq \omega$ and let τ be a similarity type with at least one relation symbol. $L_{\kappa\kappa}$ denotes the infinitary language of type τ and $L_{\kappa\kappa}^-$ denotes the set of formulas of $L_{\kappa\kappa}$ that do not contain the equality symbol. We say that a formula ϕ of $L_{\kappa\kappa}$ is a *basic Horn formula* provided that ϕ is a disjunction of less than κ formulas, at most one of which is atomic and all the others are negations of atomic formulas. A basic Horn formula is *strict* if exactly one of its disjuncts is atomic. A *universal Horn formula* ϕ of $L_{\kappa\kappa}$ is a formula of the form:

$$\forall \{ x_\xi : \xi < \mu \} \bigwedge_{\rho < \nu} \psi_\rho$$

where $\nu, \mu < \kappa$, and for each $\rho < \nu$, ψ_ρ is a basic Horn formula. When for all $\rho < \nu$, ψ_ρ is a strict basic Horn formula, we say that ϕ is a *strict universal Horn formula*.

Let \mathfrak{A} and \mathfrak{B} be τ -structures; we say that a function $h: A \rightarrow B$ is a *strict homomorphism* from \mathfrak{A} into \mathfrak{B} if for every constant symbol $c \in \tau$,

$$h(c^{\mathfrak{A}}) = c^{\mathfrak{B}},$$

for every n -ary function symbol $f \in \tau$ and every $a_1, \dots, a_n \in A$,

$$h(f^{\mathfrak{A}}(a_1, \dots, a_n)) = f^{\mathfrak{B}}(h(a_1), \dots, h(a_n)),$$

and for every n -ary relation symbol $R \in \tau$ and every $a_1, \dots, a_n \in A$,

$$\langle a_1, \dots, a_n \rangle \in R^{\mathfrak{A}} \quad \text{if and only if} \quad \langle h(a_1), \dots, h(a_n) \rangle \in R^{\mathfrak{B}}.$$

Strict homomorphisms are called in [11] two-way homomorphisms and they must not be confused with the strong homomorphisms of [4]. It is easy to prove that if a structure is a strict homomorphic image of another one then both structures satisfy exactly the same sentences without equality of $L_{\kappa\kappa}$, for every κ .

Given a τ -structure \mathfrak{A} , a *congruence* of \mathfrak{A} is an equivalence relation θ on A with the property that for every $a_1, \dots, a_n, b_1, \dots, b_n \in A$ such that $\langle a_i, b_i \rangle \in \theta$, for each $i = 1, \dots, n$, every n -ary function symbol $f \in \tau$ and every n -ary relation symbol $R \in \tau$,

$$\langle f^{\mathfrak{A}}(a_1, \dots, a_n), f^{\mathfrak{A}}(b_1, \dots, b_n) \rangle \in \theta,$$

and

$$\text{if } \langle a_1, \dots, a_n \rangle \in R^{\mathfrak{A}} \text{ then } \langle b_1, \dots, b_n \rangle \in R^{\mathfrak{B}}.$$

If h is a strict homomorphism from \mathfrak{A} into \mathfrak{B} then its kernel is a congruence of \mathfrak{A} . Moreover, if θ is any congruence of \mathfrak{A} we can consider the quotient structure \mathfrak{A}/θ ; then the canonic homomorphism from \mathfrak{A} onto \mathfrak{A}/θ is a strict homomorphism.

Let K be a class of τ -structures. We use the following terminology:

- $S(K)$ the class of all substructures of members of K .
- $P(K)$ the class of all direct products of systems of members of K .
- $P_{R_\kappa}(K)$ the class of all reduced products, over proper κ -complete filters, of systems of members of K .
- $P_{U_\kappa}(K)$ the class of all ultraproducts, over proper κ -complete ultrafilters, of systems of members of K .
- $H(K)$ the class of all strict homomorphic images of members of K .
- $H^{-1}(K)$ the class of all strict homomorphic pre-images of members of K .

We suppose that every one of the above classes is closed under isomorphic images and that the reduced and direct products are of non-empty systems.

For every τ -structure \mathfrak{B} and every $X \subseteq B$ let $L_{\kappa\kappa}^-(X)$ be the infinitary language without equality of the similarity type τ' obtained by adding to τ a constant symbol for each member of X . In the next definition we introduce the notion of a type without equality over a set of parameters.

DEFINITION 1. Let \mathfrak{B} be a τ -structure and $X \subseteq B$, we define for all $b \in B$,

$$\text{tp}_{\mathfrak{B}}^-(b/X) = \{ \phi(x) \in L_{\kappa\kappa}^-(X) : (\mathfrak{B}, d)_{d \in X} \models \phi[b] \}.$$

This set of formulas of $L_{\kappa\kappa}^-(X)$ is the *type without equality of b over X in \mathfrak{B}* . It should be observed that for all $a, b \in B$, $\text{tp}_{\mathfrak{B}}^-(a/B) = \text{tp}_{\mathfrak{B}}^-(b/B)$ if and only if for every atomic formula $\phi(x) \in L_{\kappa\kappa}^-(B)$

$$(\mathfrak{B}, d)_{d \in B} \models \phi[a] \iff (\mathfrak{B}, d)_{d \in B} \models \phi[b].$$

That is, the type without equality of an object over a structure is determined by its atomic type without equality (Definition 1 restricted to atomic formulas).

Now we define the fundamental congruence relation of having the same type without equality over a structure. The quotient structures obtained by this congruence play a crucial role in the paper.

DEFINITION 2. Given a τ -structure \mathfrak{B} , we define the following relation \sim_* on \mathfrak{B} by: For all $a, b \in B$,

$$a \sim_* b \iff \text{tp}_{\mathfrak{B}}^-(a/B) = \text{tp}_{\mathfrak{B}}^-(b/B).$$

It is straightforward to see that \sim_* is a congruence relation on \mathfrak{B} . In fact, it is the greatest congruence on \mathfrak{B} (i.e., every congruence of \mathfrak{B} is included in it). Let \mathfrak{B}^* denote the quotient structure \mathfrak{B}/\sim_* , and g^* the canonic homomorphism from \mathfrak{B} onto \mathfrak{B}^* , that is defined by: for all $b \in B$, $g^*(b) = [b]_{\sim_*}$. Moreover, as is easy to check, g^* is a strict homomorphism.

The next notion, the notion of reduced structure, is central to the paper. A structure is reduced if there are no different elements with the same type without equality over the structure.

DEFINITION 3. Given a τ -structure \mathfrak{B} , we say that \mathfrak{B} is *reduced* if and only if the relation \sim_* on \mathfrak{B} is the identity on \mathfrak{B} .

It can easily be shown that for every τ -structure \mathfrak{A} , the quotient structure \mathfrak{A}^* is reduced. \mathfrak{A}^* is called *the reduction of \mathfrak{A}* .

LEMMA 4. *If \mathfrak{A} and \mathfrak{B} are τ -structures and there is a strict homomorphism from \mathfrak{A} onto \mathfrak{B} , then $\mathfrak{A}^* \cong \mathfrak{B}^*$.*

PROOF. Suppose that $h: \mathfrak{A} \rightarrow \mathfrak{B}$ is a strict homomorphism from \mathfrak{A} onto \mathfrak{B} . We define the following function $f: \mathfrak{A}^* \rightarrow \mathfrak{B}^*$ by: for all $a \in A$,

$$f([a]_{\sim_*}) = [h(a)]_{\sim_*}.$$

We have that, for all $a, c \in A$,

$$(1) \quad \text{tp}_{\mathfrak{A}}^-(a/A) = \text{tp}_{\mathfrak{A}}^-(c/A) \iff \text{tp}_{\mathfrak{B}}^-(h(a)/B) = \text{tp}_{\mathfrak{B}}^-(h(c)/B)$$

because h is a strict homomorphism onto \mathfrak{B} . So by (1) f is a well defined function. We have also that f is onto \mathfrak{B}^* , because h is onto \mathfrak{B} , and by (1) we have that f is one-to-one. It is routine to see that f is a strict homomorphism. Therefore $\mathfrak{A}^* \cong \mathfrak{B}^*$. -1

The next lemma is the main tool in our proofs of the preservation theorems.

LEMMA 5 (Fundamental Lemma). *If \mathfrak{A} and \mathfrak{D} are τ -structures and $h: \mathfrak{A} \rightarrow \mathfrak{D}$ is a function such that, for all atomic $\phi(x_1, \dots, x_n) \in L_{\kappa\kappa}$ and all $a_1, \dots, a_n \in A$*

$$\mathfrak{A} \models \phi[a_1, \dots, a_n] \iff \mathfrak{D} \models \phi[h(a_1), \dots, h(a_n)]$$

then, if \mathfrak{B} is the substructure of \mathfrak{D} generated by $h[\mathfrak{A}]$ and $g^: \mathfrak{B} \rightarrow \mathfrak{B}^*$ is the canonic homomorphism, we have that*

- (i) $g^* \circ h$ is a strict homomorphism from \mathfrak{A} onto \mathfrak{B}^* .
- (ii) $\mathfrak{A} \in \mathbf{H}^{-1}\mathbf{HS}(\mathfrak{D})$.

PROOF. Before starting the proof it is worth noting that since we work in languages without equality the assumption on the function h is not enough to guarantee that it is a homomorphism.

Let us now begin the proof. We only prove (i) because (ii) follows directly from (i). In order to show that $g^* \circ h$ is a strict homomorphism from \mathfrak{A} onto \mathfrak{B}^* , it is enough to prove the following: for every term of $L_{\kappa\kappa}$, $t(x_1, \dots, x_n)$, and every $a_1, \dots, a_n \in A$,

$$(2) \quad \text{tp}_{\mathfrak{B}}^-(t^{\mathfrak{B}}(h(a_1), \dots, h(a_n))/B) = \text{tp}_{\mathfrak{B}}^-(h(t^{\mathfrak{A}}(a_1, \dots, a_n))/B),$$

because from (2) it follows that for each $b \in B$ there exists $c \in h[\mathfrak{A}]$ such that $\text{tp}_{\mathfrak{B}}^-(b/B) = \text{tp}_{\mathfrak{B}}^-(c/B)$, obtaining that $g^* \circ h$ is onto \mathfrak{B}^* . And from (2) it follows also that for every n -ary function symbol $f \in \tau$ and all $a_1, \dots, a_n \in A$,

$$(g^* \circ h)(f^{\mathfrak{A}}(a_1, \dots, a_n)) = f^{\mathfrak{B}^*}((g^* \circ h)(a_1), \dots, (g^* \circ h)(a_n)),$$

and for every constant symbol $c \in \tau$,

$$(g^* \circ h)(c^{\mathfrak{A}}) = c^{\mathfrak{B}^*}.$$

So we have the function and the constant symbols clause of the definition of homomorphism, and because the relation symbols clause follows directly from the definition of h , we have that $g^* \circ h$ is a strict homomorphism from \mathfrak{A} onto \mathfrak{B}^* . Let us check (2): let $t(x_1, \dots, x_n)$ be a term of $L_{\kappa\kappa}$, $a_1, \dots, a_n \in A$, $\phi(y_1, \dots, y_k, z)$ an atomic formula of $L_{\kappa\kappa}^-$ and $b_1, \dots, b_k \in B$. We will prove that

$$\begin{aligned} \mathfrak{B} \models \phi[b_1, \dots, b_k, t^{\mathfrak{B}}(h(a_1), \dots, h(a_n))] \\ \iff \mathfrak{B} \models \phi[b_1, \dots, b_k, h(t^{\mathfrak{A}}(a_1, \dots, a_n))]. \end{aligned}$$

We choose $d_1, \dots, d_j \in h[\mathfrak{A}]$, and for each $1 \leq i \leq k$ a term of $L_{\kappa\kappa}$, $t_i(v_1, \dots, v_j)$, such that $t_i^{\mathfrak{B}}(d_1, \dots, d_j) = b_i$. For each $1 \leq i \leq j$, we also choose $a_{d_i} \in A$ such that $h(a_{d_i}) = d_i$. Then, we have:

$$\begin{aligned} \mathfrak{B} \models \phi[b_1, \dots, b_k, t^{\mathfrak{B}}(h(a_1), \dots, h(a_n))] \\ \iff \mathfrak{B} \models \phi[t_1^{\mathfrak{B}}(d_1, \dots, d_j), \dots, t_k^{\mathfrak{B}}(d_1, \dots, d_j), t^{\mathfrak{B}}(h(a_1), \dots, h(a_n))] \\ \iff \mathfrak{B} \models \phi'[d_1, \dots, d_j, h(a_1), \dots, h(a_n)] \end{aligned}$$

(where ϕ' is obtained from ϕ by replacing, for all $1 \leq i \leq k$, the variable y_i by the term t_i and the variable z by the term t , that is, $\phi' = \phi(y_1/t_1, \dots, y_n/t_n, z/t)$)

$$\begin{aligned} \iff \mathfrak{D} \models \phi'[d_1, \dots, d_j, h(a_1), \dots, h(a_n)] \\ \iff \mathfrak{A} \models \phi'[a_{d_1}, \dots, a_{d_j}, a_1, \dots, a_n] \\ \iff \mathfrak{A} \models \phi^\diamond[a_{d_1}, \dots, a_{d_j}, t^{\mathfrak{A}}(a_1, \dots, a_n)] \end{aligned}$$

(where ϕ^\diamond is obtained from ϕ by replacing, for all $1 \leq i \leq k$, the variable y_i by the term t_i , that is, $\phi^\diamond = \phi(y_1/t_1, \dots, y_n/t_n, z)$)

$$\begin{aligned} \iff \mathfrak{D} \models \phi^\diamond[d_1, \dots, d_j, h(t^{\mathfrak{A}}(a_1, \dots, a_n))] \\ \iff \mathfrak{B} \models \phi^\diamond[d_1, \dots, d_j, h(t^{\mathfrak{A}}(a_1, \dots, a_n))] \\ \iff \mathfrak{B} \models \phi[b_1, \dots, b_k, h(t^{\mathfrak{A}}(a_1, \dots, a_n))], \end{aligned}$$

so we have (2). -

Let us recall that if we have a non-empty set I , a filter over I is a κ -complete filter over I if it is closed under intersections of less than κ elements. It is well known that if I is a non-empty set and κ an infinite regular cardinal and J is a set of subsets of I that has the κ -intersection property (i.e., the intersection of less than κ elements of J is non-empty) then there exists a κ -complete proper filter F over I which contains J . We need this fact in the proof of the next theorem.

Given a class K of τ -structures, we say that K is a $L_{\kappa\kappa}^-$ -universal Horn class if and only if K can be axiomatized by a set of universal Horn sentences of $L_{\kappa\kappa}^-$, and we say that is a $L_{\kappa\kappa}^-$ -strict universal Horn class if and only if it can be axiomatized by a set of strict universal Horn sentences of $L_{\kappa\kappa}^-$. The following theorem is one of the two main characterization theorems of the paper; it characterizes when a class of structures is an $L_{\kappa\kappa}^-$ -universal Horn class.

THEOREM 6. *If K is a class of τ -structures, then the following are equivalent:*

- (i) K is a $L_{\kappa\kappa}^-$ -universal Horn class.
- (ii) K is closed under the operators \mathbf{H}^{-1} , \mathbf{H} , \mathbf{S} and $\mathbf{P}_{\mathbf{R}_\kappa}$.
- (iii) $K = \mathbf{H}^{-1}\mathbf{HSP}_{\mathbf{R}_\kappa}(M)$, for some class M of τ -structures.

PROOF. (i) \implies (ii) and (ii) \implies (iii) are easily checked. In order to prove (iii) \implies (i) suppose that $K = \mathbf{H}^{-1}\mathbf{HSP}_{\mathbf{R}_\kappa}(M)$, for some class M of τ -structures. Let

$$T = \{ \sigma \in L_{\kappa\kappa}^- : \sigma \text{ is a universal Horn sentence and for all } \mathfrak{D} \in M, \mathfrak{D} \models \sigma \}$$

and let \mathfrak{A} be a τ -structure such that $\mathfrak{A} \models T$. We will prove that $\mathfrak{A} \in K$. Let T_0 be the set of all atomic and negations of atomic sentences of $L_{\kappa\kappa}^-(\mathcal{A})$ which hold in the model $(\mathfrak{A}, a)_{a \in \mathcal{A}}$. For each $\Gamma \subseteq T_0$ with $|\Gamma| < \kappa$, if the constants which occur in the sentences of Γ are in $\{ \bar{a}_\xi : \xi < \mu \}$ we consider the set of variables $\{ x_\xi : \xi < \mu \}$ and the set of formulas Γ' obtained from Γ by substituting for each $\xi < \mu$ the variable x_ξ for the constant \bar{a}_ξ . Then we consider the sentence $\sigma = \exists \{ x_\xi : \xi < \mu \} \wedge \Gamma'$. We now claim that there exists $\mathfrak{D} \in \mathbf{P}(M)$ such that $\mathfrak{D} \models \sigma$. To prove that claim we suppose the contrary and search for a contradiction; to do so we distinguish two cases:

CASE I. There is at most one sentence in Γ which is a negation of an atomic sentence.

In this case, $\neg\sigma$ is logically equivalent to a universal Horn sentence $\gamma \in L_{\kappa\kappa}^-$. We have supposed that for all $\mathfrak{D} \in P(M)$, $\mathfrak{D} \models \neg\sigma$, and in particular we have that for all $\mathfrak{D} \in M$, $\mathfrak{D} \models \neg\sigma$, so $\gamma \in T$. But that is impossible, because $\mathfrak{A} \models T$ and $\mathfrak{A} \models \sigma$.

CASE II. There is more than one sentence in Γ that is a negation of an atomic sentence.

In that case, let $\Gamma_0 \subseteq \Gamma$ be the set of all atomic sentences of Γ , and let $\{ \psi_v : v < \lambda \}$ enumerate all sentences of Γ which are negations of atomic sentences. If for all $v < \lambda$,

$$\sigma_v = \exists \{ x_\xi : \xi < \mu \} \left(\bigwedge \Gamma'_0 \wedge \psi'_v \right)$$

then $\neg\sigma_v$ is logically equivalent to a universal Horn sentence of $L_{\kappa\kappa}^-$ and by an analogous argument to the one given in Case I, we can obtain $\mathfrak{D} \in M$ such that

$$\mathfrak{D} \models \sigma_v.$$

We choose, for all $v < \lambda$ a structure $\mathfrak{D}_v \in M$ and elements of \mathfrak{D}_v , $\{ \bar{a}_\xi(v) : \xi < \mu \}$, such that

$$\mathfrak{D}_v \models \left(\bigwedge \Gamma'_0 \wedge \psi'_v \right) [\bar{a}_\xi(v) : \xi < \mu].$$

Then we have

$$\prod_{v < \lambda} \mathfrak{D}_v \models \bigwedge \Gamma' [\bar{a}_\xi : \xi < \mu]$$

and we have also that $\prod_{v < \lambda} \mathfrak{D}_v \in \mathbf{P}(M)$, which is impossible because we have supposed just the contrary.

Now to prove the theorem let $I = \{ \Gamma \subseteq T_0 : |\Gamma| < \kappa \}$. For all $\Gamma \in I$, using the claim just proved we choose $\mathfrak{D}_\Gamma \in \mathbf{P}(M)$ and elements $\{ a_\xi^\Gamma : \xi < \mu \}$ of \mathfrak{D}_Γ such that

$$\mathfrak{D}_\Gamma \models \bigwedge \Gamma' [a_\xi^\Gamma : \xi < \mu].$$

For all $\Gamma \in I$, let $J_\Gamma = \{\Delta \in I : \Gamma \subseteq \Delta\}$. Let J be the set $\{J_\Gamma : \Gamma \in I\}$. It is easy to see that J has the κ -intersection property because κ is regular. Thus, as a consequence, J can be extended to a κ -complete proper filter F . Now we construct the following reduced product $\mathfrak{D} = \prod_{\Gamma \in I} \mathfrak{D}_\Gamma / F$. Observe that $\mathfrak{D} \in \mathbf{P}_{\mathbf{R}_\kappa} \mathbf{P}(M)$. Let us define, for all $a \in A$, an element $\hat{a} \in \prod_{\Gamma \in I} D_\Gamma$ by:

$$\text{for all } \Gamma \in I, \quad \hat{a}(\Gamma) = \begin{cases} a_{\xi_0}^\Gamma, & \text{if } \bar{a} \in \{\bar{a}_\xi : \xi < \mu\} \text{ and } \bar{a} = \bar{a}_{\xi_0}, \\ \text{arbitrary,} & \text{otherwise.} \end{cases}$$

Then, for all $\psi \in T_0$ we have that

$$J_{\{\psi\}} \subseteq \{\Delta \in I : \mathfrak{D}_\Delta \models \psi' [a_\xi^\Delta : \xi < \mu]\} \in F,$$

so for all $\phi \in L_{\kappa\kappa}^-$ atomic, and for all $a_1, \dots, a_n \in A$,

$$\mathfrak{A} \models \phi[a_1, \dots, a_n] \iff \mathfrak{D} \models \phi[[\hat{a}_1]_F, \dots, [\hat{a}_n]_F].$$

Let $h: \mathfrak{A} \rightarrow \mathfrak{D}$ be the function defined by: for all $a \in A$, $h(a) = [\hat{a}]_F$. Then we have that for each atomic $\phi \in L_{\kappa\kappa}^-$ and for every $a_1, \dots, a_n \in A$,

$$\mathfrak{A} \models \phi[a_1, \dots, a_n] \iff \mathfrak{D} \models \phi[h(a_1), \dots, h(a_n)].$$

Then applying the Fundamental Lemma we have that $\mathfrak{A} \in \mathbf{H}^{-1} \mathbf{HSP}_{\mathbf{R}_\kappa} \mathbf{P}(M)$. But, since

$$\mathbf{P}_{\mathbf{R}_\kappa} \mathbf{P}(M) \subseteq \mathbf{P}_{\mathbf{R}_\kappa} \mathbf{P}_{\mathbf{R}_\kappa}(M) \subseteq \mathbf{P}_{\mathbf{R}_\kappa}(M),$$

we can conclude that $\mathfrak{A} \in \mathbf{H}^{-1} \mathbf{HSP}_{\mathbf{R}_\kappa}(M)$. -1

It is worth making explicit that as a consequence of the theorem for every class of τ -structures M , $\mathbf{H}^{-1} \mathbf{HSP}_{\mathbf{R}_\kappa}(M)$ is just the class of models of the universal Horn theory without equality of M in $L_{\kappa\kappa}$.

REMARK 1. Observe that Theorem 6 is not true if we delete the operator \mathbf{H} in its formulation: Let $\tau = \{P, c_1, c_2\}$, where P is a unary relation symbol and c_1 and c_2 are constant symbols, let $M = \text{Mod}(\forall x Px \wedge c_1 \neq c_2)$ and $K = \mathbf{H}^{-1} \mathbf{SP}_{\mathbf{R}_\kappa}(M)$. It is easy to check that K is not closed under \mathbf{H} .

And we also have that Theorem 6 is not true if we delete the operator \mathbf{H}^{-1} in its formulation: Let τ be the same similarity type as the previous example, let $M = \text{Mod}(\forall x Px \wedge c_1 = c_2)$ and $K = \mathbf{HSP}_{\mathbf{R}_\kappa}(M)$, it is easy to check that K is not closed under \mathbf{H}^{-1} .

REMARK 2. Observe that if λ and μ are infinite regular cardinals such that $\lambda < \kappa$, $\mu \leq \kappa$, the following is not true: For every class K of τ -structures,

K is a $L_{\kappa\lambda}^-$ -universal Horn class if and only if $K = \mathbf{H}^{-1} \mathbf{HSP}_{\mathbf{R}_\mu}(M)$ for some class M of τ -structures.

PROOF. For $\lambda < \mu$, take $\tau = \{R\}$, where R is a binary relation symbol. Let the τ -structure $\mathfrak{A} = (\lambda + 1, <)$, where $<$ is the usual well-ordering of the ordinal $\lambda + 1$ and let \mathfrak{B} be the substructure of \mathfrak{A} with domain λ . It is easy to check that \mathfrak{A} and \mathfrak{B} satisfy exactly the same universal sentences of $L_{\kappa\lambda}^-$ and, therefore, they satisfy exactly

the same universal Horn sentences of $L_{\kappa\lambda}^-$. But clearly they do not satisfy the same universal Horn sentences of $L_{\mu\mu}^-$ because the sentence

$$\forall \{x_\xi : \xi \leq \lambda\} \bigvee_{\rho < \xi \leq \lambda} \neg R x_\rho x_\xi$$

is true in \mathfrak{B} but not in \mathfrak{A} . The point is that every subset of λ of cardinality λ is unbounded in λ but not in $\lambda + 1$. Therefore $K = \mathbf{H}^{-1}\mathbf{HSP}_{\mathbf{R}_\mu}(\mathfrak{B})$ is not axiomatized by a set of universal Horn sentences of $L_{\kappa\lambda}^-$.

For $\mu \leq \lambda$, we can find a $L_{\kappa\lambda}^-$ -universal Horn class which is not a $L_{\mu\mu}^-$ -universal Horn class: Take $\tau = \{P_\xi : \xi \in \mu\}$, where for each $\xi \in \mu$, P_ξ is a unary relation symbol. Let σ be the following sentence of $L_{\kappa\lambda}^-$,

$$\forall x \bigvee_{\xi \in \mu} \neg P_\xi x.$$

We have that $\text{Mod}(\sigma)$ is a $L_{\kappa\lambda}^-$ -universal Horn class, but it is not a $L_{\mu\mu}^-$ -universal Horn class. It is straightforward to show this: Let $\mathfrak{A} = (\mu, P_\xi^{\mathfrak{A}})_{\xi \in \mu}$, where for each $\xi \in \mu$, $P_\xi^{\mathfrak{A}} = \{\alpha \in \mu : \xi \leq \alpha\}$. And let $\mathfrak{B} = (\mu + 1, P_\xi^{\mathfrak{B}})_{\xi \in \mu}$, where for each $\xi \in \mu$, $P_\xi^{\mathfrak{B}} = \{\alpha \in \mu + 1 : \xi \leq \alpha\}$. We have that \mathfrak{A} and \mathfrak{B} satisfy exactly the same universal Horn sentences of $L_{\mu\mu}^-$. If $\text{Mod}(\sigma)$ were a $L_{\mu\mu}^-$ -universal Horn class, we would have $\mathfrak{B} \in \text{Mod}(\sigma)$, because $\mathfrak{A} \in \text{Mod}(\sigma)$, but it is clear that $\mathfrak{B} \notin \text{Mod}(\sigma)$. ⊣

COROLLARY 7. *Let κ be either a strongly compact cardinal or $\kappa = \omega$. For every sentence $\sigma \in L_{\kappa\kappa}$ the following holds: σ is preserved under \mathbf{H}^{-1} , \mathbf{H} , \mathbf{S} and $\mathbf{P}_{\mathbf{R}_\kappa}$ if and only if σ is logically equivalent to a universal Horn sentence of $L_{\kappa\kappa}^-$.*

PROOF. The direction from right to left is clear. To prove the other direction suppose that $\sigma \in L_{\kappa\kappa}$ is preserved under \mathbf{H}^{-1} , \mathbf{H} , \mathbf{S} and $\mathbf{P}_{\mathbf{R}_\kappa}$. Then we have that $\mathbf{H}^{-1}\mathbf{HSP}_{\mathbf{R}_\kappa}(\text{Mod}(\sigma)) = \text{Mod}(\sigma)$. Therefore, by Theorem 6, $\text{Mod}(\sigma)$ is axiomatized by a set T of universal Horn sentences of $L_{\kappa\kappa}^-$. Since κ is strongly compact or $\kappa = \omega$, by the corresponding compactness theorem, there is $T_0 \subseteq T$ such that $|T_0| < \kappa$ and $T_0 \models \sigma$. So we have that $\bigwedge_{\phi \in T_0} \phi$ is logically equivalent to a universal Horn sentence of $L_{\kappa\kappa}^-$ logically equivalent to σ . ⊣

Now, as a consequence of Theorem 6 we obtain the characterization theorem for the strict universal Horn fragment. Remember that a *trivial structure* of similarity type τ is a one-element structure of type τ in which the interpretations of the relation symbols are non-empty, and therefore a structure where the universal closure of every atomic formula is true.

COROLLARY 8. *If K is a class of τ -structures, then the following are equivalent:¹*

- (i) *K is a $L_{\kappa\kappa}^-$ -strict universal Horn class.*
- (ii) *K is closed under the operators \mathbf{H}^{-1} , \mathbf{H} , \mathbf{S} and $\mathbf{P}_{\mathbf{R}_\kappa}$ and contains a trivial structure.*

¹During the process of writing this paper we have been informed that R. Elgueta has proved independently this Corollary and Theorem 9 for the first-order case using methods similar to ours.

- (iii) $K = \mathbf{H}^{-1}\mathbf{HSP}_{\mathbf{R}_\kappa}(M)$, for some class M of τ -structures that contains a trivial structure.

PROOF. From Theorem 6, using the fact that in a trivial structure every strict universal Horn sentence of $L_{\kappa\kappa}^-$ is true, and every universal Horn sentence of $L_{\kappa\kappa}^-$ that is not strict is false. ⊣

In the next theorem we improve the Characterization Theorem 6 for $L_{\kappa\kappa}^-$ when κ is strongly compact or is ω in terms of \mathbf{H}^{-1} , \mathbf{H} , \mathbf{P} and \mathbf{P}_U .

THEOREM 9. For every class K of τ -structures, if κ is strongly compact or ω , then the following are equivalent:

- (i) K is an $L_{\kappa\kappa}^-$ -universal Horn class ($L_{\kappa\kappa}^-$ -strict universal Horn class).
- (ii) K is closed under the operators \mathbf{H}^{-1} , \mathbf{H} , \mathbf{S} , \mathbf{P} and \mathbf{P}_{U_κ} (and contains a trivial structure).
- (iii) $K = \mathbf{H}^{-1}\mathbf{HSPP}_{U_\kappa}(M)$, for some class M of τ -structures (that contains a trivial structure).

PROOF. We just prove the non strict case. The strict case is obtained from it in the same way as Corollary 8 has been obtained from Theorem 6. (i) \implies (ii) and (ii) \implies (iii) are easily checked.

(iii) \implies (i) Suppose that $K = \mathbf{H}^{-1}\mathbf{HSPP}_{U_\kappa}(M)$, for some class M of τ -structures. Then

$$K = \mathbf{H}^{-1}\mathbf{HSPP}_{U_\kappa}(M) = \mathbf{H}^{-1}\mathbf{HSP}_{\mathbf{R}_\kappa}(M),$$

since it is easy to see that

$$\mathbf{PP}_{U_\kappa}(M) \subseteq \mathbf{P}_{\mathbf{R}_\kappa}\mathbf{P}_{\mathbf{R}_\kappa}(M) \subseteq \mathbf{P}_{\mathbf{R}_\kappa}(M),$$

and one can prove with the usual arguments, taking into account that κ is strongly compact or ω , that $\mathbf{P}_{\mathbf{R}_\kappa}(M) \subseteq \mathbf{SPP}_{U_\kappa}(M)$. By Theorem 6, we can conclude that K is a $L_{\kappa\kappa}^-$ -universal Horn class. ⊣

Now we will use the Fundamental Lemma to prove a theorem similar to Theorem 6 but for the infinitary language without equality $L_{\infty\infty}^-$ that has a well-ordered proper class of variables and admits conjunctions of sets of formulas of any cardinality and quantification of sets of variables of arbitrary cardinality. The characterization of the $L_{\infty\infty}^-$ -universal Horn classes can be obtained in terms of the operators \mathbf{S} , \mathbf{H} , \mathbf{H}^{-1} and \mathbf{P} , thus dispensing with the reduced products and the ultraproducts, but at the price of considering theories that are proper classes. A $L_{\infty\infty}^-$ -universal Horn class is a class of structures axiomatized by a class (possibly proper) of universal Horn sentences of $L_{\infty\infty}^-$.

THEOREM 10. If K is a class of τ -structures, then the following are equivalent:

- (i) K is a $L_{\infty\infty}^-$ -universal Horn class.
- (ii) K is closed under the operators \mathbf{H}^{-1} , \mathbf{H} , \mathbf{S} and \mathbf{P} .
- (iii) $K = \mathbf{H}^{-1}\mathbf{HSP}(M)$, for some class M of τ -structures.

PROOF. The implications (i) \implies (ii) \implies (iii) are as always routine to check. To prove that (iii) \implies (i), suppose that M is a class of τ -structures such that $K = \mathbf{H}^{-1}\mathbf{HSP}(M)$. Let T be the $L_{\infty\infty}^-$ -universal Horn theory of M . If $\mathfrak{A} \models T$ then let Γ be the set of all atomic and negations of atomic sentences of $L_{\infty\infty}^-(\mathfrak{A})$ which hold

in the model $(\mathfrak{A}, a)_{a \in A}$. Suppose that λ is the cardinality of \mathfrak{A} and that $\{\bar{a}_\xi : \xi < \lambda\}$ are the new constants of $L_{\infty\infty}^-(A)$ (being $\{a_\xi : \xi < \lambda\}$ an enumeration of the domain of \mathfrak{A}). Let us substitute the variables $\{x_\xi : \xi < \lambda\}$ for the constants $\{\bar{a}_\xi : \xi < \lambda\}$ in the sentences of Γ obtaining Γ' . Then arguing as in the proof of Theorem 6 we can conclude that there exists $\mathfrak{D} \in \mathbf{P}(M)$ such that

$$\mathfrak{D} \models \exists\{x_\xi : \xi < \lambda\} \bigwedge \Gamma'.$$

Then let $\{b_\xi : \xi < \lambda\}$ be elements of D such that satisfy Γ' . Then the function h from \mathfrak{A} into \mathfrak{D} such that for every $\xi < \lambda$ $h(a_\xi) = b_\xi$ satisfies the condition of the Fundamental Lemma. Therefore $\mathfrak{A} \in \mathbf{H}^{-1}\mathbf{HSP}(M)$. \dashv

To conclude the section let us prove a characterization theorem using instead of reduced products the operation of direct product and the notion of κ -local class of structures. This theorem, given the previous one, permits us to conclude that if a class of τ -structures is closed under the operators \mathbf{H}^{-1} , \mathbf{H} , \mathbf{S} and \mathbf{P} , then it is axiomatized by a set of universal Horn sentences of $L_{\infty\infty}^-$ if and only if it is κ -local for some regular cardinal $\kappa > \omega$ bigger than the cardinality of τ .

DEFINITION 11. Let κ be an infinite cardinal. A class K of τ -structures is κ -local if and only if every τ -structure \mathfrak{A} with the property that all its substructures generated by less than κ elements belong to K also belongs to K .

THEOREM 12. Let κ be a regular infinite cardinal $> \omega$ and assume that τ has cardinality less than κ . Let K be a class of τ -structures. Then the following are equivalent:

- (i) K is a $L_{\kappa\kappa}^-$ -universal Horn class.
- (ii) K is κ -local and it is closed under the operators \mathbf{H}^{-1} , \mathbf{H} , \mathbf{S} and \mathbf{P} .

PROOF. (i) \implies (ii) is easy since if a universal Horn sentence of $L_{\kappa\kappa}$ is not true in a structure it is not true in a substructure of it generated by less than κ elements.

To prove that (ii) \implies (i) let T be the universal Horn theory in $L_{\kappa\kappa}^-$ of K . Let \mathfrak{A} be a model of T . We will see that $\mathfrak{A} \in K$. Since K is κ -local we only have to prove that every substructure of \mathfrak{A} generated by less than κ elements belongs to K . Let \mathfrak{B} be a substructure of \mathfrak{A} generated by less than κ elements. Since κ is regular and $> \omega$ and $|\tau| < \kappa$ we have that $|\mathfrak{B}| < \kappa$. Let Γ be the set of all atomic and negations of atomic sentences of $L_{\kappa\kappa}^-(\mathfrak{B})$ true in $(\mathfrak{B}, b)_{b \in B}$. Clearly Γ is a set of cardinality less than κ . Therefore we can argue as in the proof of Theorem 10 to conclude that $\mathfrak{B} \in \mathbf{H}^{-1}\mathbf{HSP}(K) = K$. \dashv

One cannot prove the theorem for similarity types of cardinality greater than or equal to κ . With a slightly more complicated argument one can also prove the following theorem:

THEOREM 13. Let κ be an infinite cardinal and assume that τ has cardinality less than κ . Let K be a class of τ -structures. Then the following are equivalent:

- (i) K is a $L_{\kappa^+\kappa}^-$ -universal Horn class.
- (ii) K is κ -local and it is closed under the operators \mathbf{H}^{-1} , \mathbf{H} , \mathbf{S} and \mathbf{P} .

As usual we can obtain the analogous theorems to Theorems 10, 12 and 13 for the strict case.

§2. Reduced universal Horn classes without equality. In this section we will study characterization theorems for the classes of reduced structures that are models of some universal Horn theory without equality in the infinitary language $L_{\kappa\kappa}$.

Since for every structure we have its reduction, we can consider for each operator O that transforms a class of structures K into another one $O(K)$, the corresponding operator O^* that transforms the class of structures K into the class of the structures isomorphic to some reduction of a member of $O(K)$. We will call these operators with a star *reduction operators*, and call O^* the *reduction* of O . The reduction operators were first considered in [2]. We will prove a theorem that characterizes when a class of reduced structures is the class of the reduced models of a universal Horn theory in $L_{\kappa\kappa}^-$. This theorem follows easily from previous results and Lemma 15 which studies the behaviour of some reduction operators. Its formulation is just like the one of the corresponding theorem for the full $L_{\kappa\kappa}$ (with equality) except that the operators \mathbf{S} and $\mathbf{P}_{\mathbf{R}_\kappa}$ are replaced by their reductions.

DEFINITION 14. For every class K of τ -structures let K^* be the following class:

$$K^* = \{ \mathfrak{B} : \text{there is } \mathfrak{A} \in K \text{ such that } \mathfrak{B} \cong \mathfrak{A}^* \}.$$

And, for every operator \mathbf{O} , \mathbf{O}^* is the operator such that for every class K of τ -structures

$$\mathbf{O}^*(K) = (\mathbf{O}(K))^*.$$

The next lemma studies the behaviour of the reduction operators.

LEMMA 15. For every class K of τ -structures and every operator $\mathbf{O} \in \{ \mathbf{S}, \mathbf{P}, \mathbf{P}_{\mathbf{R}_\kappa}, \mathbf{P}_{\mathbf{U}_\kappa} \}$ the following holds:

- (i) $\mathbf{H}^{-1*}(K^*) \subseteq K^*$
- (ii) $\mathbf{H}^*(K^*) \subseteq K^*$
- (iii) $\mathbf{O}^*(K) \subseteq \mathbf{O}^*(K^*)$.

PROOF. (i) Suppose that $\mathfrak{A} \in \mathbf{H}^{-1*}(K^*)$. Let $\mathfrak{B} \in \mathbf{H}^{-1}(K^*)$ such that $\mathfrak{A} \cong \mathfrak{B}^*$ and let h be a strict homomorphism from \mathfrak{B} onto some $\mathfrak{C} \in K^*$. Then by Lemma 4 $\mathfrak{B}^* \cong \mathfrak{C}$. Therefore, $\mathfrak{A} \cong \mathfrak{C}$. Hence, $\mathfrak{A} \in K^*$. The proof of (ii) is similar.

(iii) We prove first the case where \mathbf{O} is \mathbf{S} . Suppose that $\mathfrak{A} \in \mathbf{S}^*(K)$. Let $\mathfrak{B} \in K$ and $\mathfrak{C} \subseteq \mathfrak{B}$ such that $\mathfrak{A} \cong \mathfrak{C}^*$. Let g_B^* be the canonic homomorphism from \mathfrak{B} onto \mathfrak{B}^* . Then $\mathfrak{D} = g_B^*[\mathfrak{C}]$ is a substructure of \mathfrak{B}^* , so $\mathfrak{D} \in \mathbf{S}(K^*)$. Since, by Lemma 4, $\mathfrak{C}^* \cong \mathfrak{D}^*$, we obtain that $\mathfrak{A} \cong \mathfrak{D}^*$. Therefore, $\mathfrak{A} \in \mathbf{S}^*(K^*)$.

Now we prove the case where \mathbf{O} is $\mathbf{P}_{\mathbf{U}_\kappa}$. Suppose that $\mathfrak{A} \in \mathbf{P}_{\mathbf{U}_\kappa}^*(K)$. Let $\mathfrak{B} \in \mathbf{P}_{\mathbf{U}_\kappa}(K)$ be such that $\mathfrak{A} \cong \mathfrak{B}^*$ and let $\{ \mathfrak{B}_i : i \in I \} \subseteq K$ be a family of structures and U a proper κ -complete ultrafilter over a non-empty set I such that $\mathfrak{B} = \prod_{i \in I} \mathfrak{B}_i / U$. Let $\mathfrak{C} = \prod_{i \in I} \mathfrak{B}_i^* / U$. Then, if for each $i \in I$ $g_{B_i}^*$ is the natural homomorphism from \mathfrak{B}_i onto \mathfrak{B}_i^* , we define for each $f \in \prod_{i \in I} B_i$, $f' \in \prod_{i \in I} B_i^*$ by

$$f'(i) = g_{B_i}^*(f(i)),$$

for every $i \in I$. Then we define the function $h : \mathfrak{B} \rightarrow \mathfrak{C}$ by

$$h([f]_U) = [f']_U,$$

for each $f \in \prod_{i \in I} B_i$, where $[f]_U$ is the equivalence class of f in the ultraproduct \mathfrak{B} and f'_U the equivalence class of f' in the ultraproduct \mathfrak{C} . Since the natural homomorphisms $g_{B_i}^*$ are strict, it is straightforward to check that this definition is independent of the representatives chosen and that it is a strict homomorphism from \mathfrak{B} onto \mathfrak{C} . Therefore, by Lemma 4, $\mathfrak{B}^* \cong \mathfrak{C}^*$. Since $\{\mathfrak{B}_i^* : i \in I\} \subseteq K^*$, we conclude that $\mathfrak{B}^* \in \mathbf{P}_{U_\kappa}^*(K^*)$.

The proof for the remaining cases is similar to the last one given. ⊣

Now we prove the promised theorem and the corresponding version when κ is strongly compact or ω .

THEOREM 16. *For every class K of reduced τ -structures, the following are equivalent:*

- (i) K is the class of reduced models of a universal Horn theory of $L_{\kappa\kappa}^-$.
- (ii) K is closed under the operators \mathbf{S}^* and $\mathbf{P}_{R_\kappa}^*$.
- (iii) $K = \mathbf{S}^*\mathbf{P}_{R_\kappa}^*(M)$, for some class M of τ -structures.

PROOF. (i) \implies (ii) \implies (iii) is easy. To prove that (iii) implies (i) let T be the universal Horn theory of M in $L_{\kappa\kappa}^-$. If \mathfrak{A} is a reduced model of T then by Theorem 6 we have that $\mathfrak{A} \in \mathbf{H}^{-1}\mathbf{HSP}_{R_\kappa}(M)$. Therefore, by Lemma 4, since \mathfrak{A} is reduced $\mathfrak{A} \in \mathbf{S}^*\mathbf{P}_{R_\kappa}(M)$ and by Lemma 15 (iii) we have that $\mathfrak{A} \in K$. ⊣

COROLLARY 17. *For every class K of reduced τ -structures, if T is the set of all universal Horn sentences of $L_{\kappa\kappa}^-$ true in every structure of K , then $\text{Mod}^*(T) = \mathbf{S}^*\mathbf{P}_{R_\kappa}^*(K)$.*

THEOREM 18. *If κ is either a strongly compact cardinal or ω , then for every class K of reduced τ -structures the following are equivalent:*

- (i) K is the class of reduced models of a universal Horn theory of $L_{\kappa\kappa}^-$.
- (ii) K is closed under the operators \mathbf{S}^* , \mathbf{P}^* and $\mathbf{P}_{U_\kappa}^*$.
- (iii) $K = \mathbf{S}^*\mathbf{P}^*\mathbf{P}_{U_\kappa}^*(M)$, for some class M of τ -structures.

PROOF. As the proof of Theorem 16, but using Theorem 9, instead of Theorem 6, and (iii) of Corollary 15. ⊣

COROLLARY 19. *If κ is either a strongly compact cardinal or ω , then for every class K of τ -structures, if T is the set of all universal Horn sentences of $L_{\kappa\kappa}^-$ true in every structure of K , $\text{Mod}^*(T) = \mathbf{S}^*\mathbf{P}^*\mathbf{P}_{U_\kappa}^*(K)$.*

In the same way we can prove the following:

THEOREM 20. *For every class K of reduced τ -structures the following are equivalent:*

- (i) K is the class of reduced models of a universal Horn theory of $L_{\infty\infty}^-$.
- (ii) K is closed under the operators \mathbf{S}^* , \mathbf{P}^* .
- (iii) $K = \mathbf{S}^*\mathbf{P}^*(M)$, for some class M of τ -structures.

COROLLARY 21. *For every class K of τ -structures, if T is the class of all universal Horn sentences of $L_{\infty\infty}^-$ true in every structure of K , $\text{Mod}^*(T) = \mathbf{S}^*\mathbf{P}^*(K)$.*

We can also obtain from these results the corresponding results for the case of strict universal Horn sentences by adding that the classes contain a trivial structure.

Now as a by-product we will give a proof, using our results, of the well known theorem that characterizes the class of models of the universal Horn theory in $L_{\kappa\kappa}$

(with equality) of a given class of structures. And with an analogous argument we can obtain the corresponding theorem for languages with equality to Theorem 20 substituting the operators \mathbf{S} and \mathbf{P} for the operators \mathbf{S}^* and \mathbf{P}^* .

THEOREM 22. *Let τ be any similarity type. Then for every class K of τ -structures the following are equivalent:*

- (i) K is the class of models of a universal Horn theory of $L_{\kappa\kappa}$.
- (ii) K is closed under the operators \mathbf{S} and $\mathbf{P}_{\mathbf{R}_\kappa}$.
- (iii) $K = \mathbf{SP}_{\mathbf{R}_\kappa}(M)$, for some class M of τ -structures.

PROOF. (i) \implies (ii) \implies (iii) are easy. To see that (iii) \implies (i), let T be the universal Horn theory of M in $L_{\kappa\kappa}$ (with equality). We will deal with the equality symbol as if it were a usual binary relation symbol (not always interpreted as the identity relation). To be clear about this let us expand the similarity type τ with a new binary relation symbol E and substitute the binary relation symbol E for the equality symbol in every sentence in T . Let us call the resulting theory T' . Given a τ -structure \mathfrak{A} let \mathfrak{A}^E be the structure of type $\tau \cup \{E\}$, $\langle \mathfrak{A}, E^{\mathfrak{A}^E} \rangle$, where $E^{\mathfrak{A}^E}$ is the identity on the domain A of \mathfrak{A} . Obviously for every τ -structure \mathfrak{A} the $\tau \cup \{E\}$ -structure \mathfrak{A}^E is reduced. Now we consider the class of $\tau \cup \{E\}$ -structures $M' = \{ \mathfrak{A}^E : \mathfrak{A} \in M \}$. Obviously the universal Horn theory in the language $L_{\kappa\kappa}$ of type $\tau \cup \{E\}$ of the class M' is precisely the theory T' . To prove what we want, let \mathfrak{A} be a τ -structure that is a model of T . Then \mathfrak{A}^E is a reduced model of T' . Therefore by Corollary 17 we have that $\mathfrak{A}^E \in \mathbf{S}^*\mathbf{P}_{\mathbf{R}_\kappa}^*(M')$. Let \mathfrak{B} and \mathfrak{C} be $\tau \cup \{E\}$ -structures such that $\mathfrak{A}^E \cong \mathfrak{C}^*$, \mathfrak{C} is a substructure of \mathfrak{B} and \mathfrak{B} is isomorphic to the reduction of some reduced product, by a κ -complete proper filter, of some system of structures in M' . Since the interpretation of E is the identity in every structure in M' , the interpretation of E in every reduced product of elements of M' must be the identity. Therefore all these reduced products are reduced structures. Hence $E^{\mathfrak{B}}$ is the identity and so is $E^{\mathfrak{C}}$. Hence \mathfrak{C} is reduced and so $\mathfrak{A}^E \cong \mathfrak{C}$ and therefore $E^{\mathfrak{A}^E}$ is the identity. Hence, as is easily seen, $\mathfrak{A} \in \mathbf{SP}_{\mathbf{R}_\kappa}(M)$. -1

§3. Interpolation and definability. In this section we will use the results of Section 1 to obtain a joint consistency theorem, Theorem 26, an interpolation theorem, Theorem 27, and a definability theorem, Theorem 28.

In order to prove the joint consistency theorem we first state the compactness theorem for the Horn fragment of the infinitary language $L_{\kappa\kappa}$, for κ regular, see [9] Lemma 2.

THEOREM 23 (Compactness). *Let τ be a similarity type, κ a regular cardinal $\geq \omega$ and Γ a set of Horn sentences of $L_{\kappa\kappa}$. If every $\Delta \subseteq \Gamma$ with $|\Delta| < \kappa$ has a model then Γ has a model.*

COROLLARY 24. *Let τ be a similarity type, κ a regular cardinal $\geq \omega$ and $\Gamma \cup \{ \sigma \}$ a set of universal Horn sentences of $L_{\kappa\kappa}$. If $\Gamma \models \sigma$ then there is $\Delta \subseteq \Gamma$ such that $|\Delta| < \kappa$ and $\Delta \models \sigma$.*

PROOF. Suppose that $\sigma = \forall \bar{x} (\bigwedge_{\xi < \mu} \psi_\xi(\bar{x}) \rightarrow \phi(\bar{x}))$ with $\mu < \kappa$. Then $\Gamma \models \sigma$ implies $\Gamma \models \sigma(\bar{c})$ for new constants \bar{c} . Thus $\Gamma \cup \{ \psi_\xi(\bar{c}) \} \cup \{ \neg\phi(\bar{c}) \}$ has no model. Since the sentences $\psi_\xi(\bar{c})$ and $\neg\phi(\bar{c})$ are trivially universal Horn sentences,

Theorem 23 applies and we can conclude that there is some subset Δ of Γ of cardinality less than κ such that $\Delta \cup \{\psi_\xi(\bar{c})\} \models \phi(\bar{c})$ and hence $\Delta \models \sigma$. In case $\sigma = \forall \bar{x} (\bigvee_{\xi < \mu} \neg \psi_\xi(\bar{x}))$ we argue similarly. \dashv

The next lemma is essentially due to W. J. Blok and D. Pigozzi, see [1], and it is necessary for the proof of the next theorem. We present its proof for the interested reader.

LEMMA 25. *If \mathfrak{A} and \mathfrak{B} are τ -structures such that $\mathfrak{A} \in \mathbf{H}^{-1}\mathbf{H}(\mathfrak{B})$, then there is a τ -structure \mathfrak{C} such that $\mathfrak{C} \in \mathbf{H}^{-1}(\mathfrak{A})$ and $\mathfrak{C} \in \mathbf{H}^{-1}(\mathfrak{B})$.*

PROOF. Let \mathfrak{D} , f and g be respectively a τ -structure, a strict homomorphism from \mathfrak{A} onto \mathfrak{D} and a strict homomorphism from \mathfrak{B} onto \mathfrak{D} . Let for each $d \in D$, $X_d = f^{-1}[\{d\}]$ and $Y_d = g^{-1}[\{d\}]$. Consider the algebra Ter of τ -terms in the set of variables $\{x_d^\alpha : d \in D, \alpha < \mu_d\}$, where for each $d \in D$, $\{x_d^\alpha : \alpha < \mu_d\}$ is a set of variables of cardinality μ_d and μ_d is the cardinality of the set $X_d \cup Y_d$. Consider a homomorphism f' from Ter onto the algebraic reduct of \mathfrak{A} such that for all $d \in D$,

$$f'[\{x_d^\alpha : \alpha < \mu_d\}] = X_d.$$

Define now the τ -structure \mathfrak{C} whose algebraic part is the algebra of terms Ter and for every n -ary relation symbol $R \in \tau$ and for all $t_1, \dots, t_n \in \text{Ter}$

$$\langle t_1, \dots, t_n \rangle \in R^{\mathfrak{C}} \quad \text{if and only if} \quad \langle f'(t_1), \dots, f'(t_n) \rangle \in R^{\mathfrak{A}}.$$

Clearly, by definition, f' is a strict homomorphism from \mathfrak{C} onto \mathfrak{A} . Moreover, if we consider a homomorphism g' from Ter onto the algebraic reduct of \mathfrak{B} such that for every $d \in D$,

$$g'[\{x_d^\alpha : \alpha < \mu_d\}] = Y_d,$$

we have that $f \circ f' = g \circ g'$. Therefore it is easy to check that g' is a strict homomorphism from \mathfrak{C} onto \mathfrak{B} . \dashv

Let τ_0 and τ_1 be similarity types with the same function symbols and the same constant symbols and with at least one relation symbol in common. Let κ be a regular cardinal $\kappa \geq \omega$, and let, for $i = 0, 1$, $L_{\kappa\kappa}^i$ be the infinitary language of type τ_i .

THEOREM 26. *For every regular cardinal $\kappa \geq \omega$, if $\Gamma_0 \subseteq L_{\kappa\kappa}^0$ and $\Gamma_1 \subseteq L_{\kappa\kappa}^1$ are sets of universal Horn sentences without the equality symbol, then the following are equivalent:*

- (i) $\Gamma_0 \cup \Gamma_1$ is unsatisfiable.
- (ii) There is a universal Horn sentence without the equality symbol $\theta \in L_{\kappa\kappa}^0 \cap L_{\kappa\kappa}^1$ such that $\Gamma_0 \models \theta$ and $\Gamma_1 \models \neg\theta$.

PROOF. (ii) \implies (i) is clear. (i) \implies (ii) Suppose that $\Gamma_0 \cup \Gamma_1$ is unsatisfiable. We have that, for $i = 0, 1$, $\text{Mod}(\Gamma_i)$ is closed under the operators \mathbf{H}^{-1} , \mathbf{H} , \mathbf{S} and $\mathbf{P}_{\mathbf{R}_\kappa}$, because $\text{Mod}(\Gamma_i)$ is axiomatized by a set of universal Horn sentences without the equality symbol. Let K_i be, for $i = 0, 1$, the following class:

$$K_i = \{\mathfrak{A} : \mathfrak{A} \text{ is the reduct to } \tau_0 \cap \tau_1 \text{ of some } \mathfrak{B} \in \text{Mod}(\Gamma_i)\}.$$

As a consequence of the Expansion Theorem in [4] (Theorem 4.1.8), we have that K_0 and K_1 are closed under \mathbf{P}_{R_κ} and it is easy to see that they are closed under \mathbf{S} . Let $K = \mathbf{H}^{-1}\mathbf{H}(K_0)$. Thus, by Theorem 6, K is axiomatizable by a set of universal Horn sentences without the equality symbol in $L_{\kappa\kappa}^0 \cap L_{\kappa\kappa}^1$.

We claim that $K \cap K_1 = \emptyset$. Assuming the contrary, let $\mathfrak{B}' \in K \cap K_1$. Then let \mathfrak{B} be an expansion of \mathfrak{B}' to τ_1 such that $\mathfrak{B} \models \Gamma_1$, let $\mathfrak{A}' \in K_0$ be such that $\mathfrak{B}' \in \mathbf{H}^{-1}\mathbf{H}(\mathfrak{A}')$ and let \mathfrak{A} be an expansion of \mathfrak{A}' to τ_0 such that $\mathfrak{A} \models \Gamma_0$. By Lemma 25, there is a $\tau_0 \cap \tau_1$ -structure \mathfrak{C}' and strict homomorphisms f and g from \mathfrak{C}' onto \mathfrak{B}' and onto \mathfrak{A}' respectively. We define an expansion of \mathfrak{C}' to a $\tau_0 \cup \tau_1$ -structure \mathfrak{C} in the following way: for each n -ary relation symbol $R \in \tau_0 - \tau_1$, and every $a_1, \dots, a_n \in C$,

$$\langle a_1, \dots, a_n \rangle \in R^{\mathfrak{C}} \quad \text{if and only if} \quad \langle g(a_1), \dots, g(a_n) \rangle \in R^{\mathfrak{A}'},$$

and for each n -ary relation symbol $R \in \tau_1 - \tau_0$ and every $a_1, \dots, a_n \in C$,

$$\langle a_1, \dots, a_n \rangle \in R^{\mathfrak{C}} \quad \text{if and only if} \quad \langle f(a_1), \dots, f(a_n) \rangle \in R^{\mathfrak{B}'}. \quad \dashv$$

Then, g is a strict homomorphism from $\mathfrak{C} \upharpoonright \tau_0$ onto \mathfrak{A} and f is a strict homomorphism from $\mathfrak{C} \upharpoonright \tau_1$ onto \mathfrak{B} . Therefore, $\mathfrak{C} \upharpoonright \tau_0 \models \Gamma_0$ and $\mathfrak{C} \upharpoonright \tau_1 \models \Gamma_1$. From these it follows that $\mathfrak{C} \models \Gamma_0 \cup \Gamma_1$, which is absurd.

From the claim we obtain the theorem. Let Δ be the universal Horn theory without equality of K in $L_{\kappa\kappa}$ of type $\tau_0 \cap \tau_1$. On the one hand, if $\sigma \in \Delta$ then $\Gamma_0 \models \sigma$, for if $\mathfrak{A} \models \Gamma_0$ then $\mathfrak{A} \upharpoonright \tau_0 \cap \tau_1 \in K_0 \subseteq K$. On the other hand, $\Gamma_1 \cup \Delta$ is unsatisfiable, for if $\mathfrak{B} \models \Gamma_1 \cup \Delta$ then $\mathfrak{B} \upharpoonright \tau_0 \cap \tau_1 \in K_1 \cap K$. By Theorem 23, there is $\Delta' \subseteq \Delta$ of cardinality less than κ such that $\Gamma_1 \cup \Delta'$ is unsatisfiable. Then, $\Gamma_0 \models \bigwedge \Delta'$ and $\Gamma_1 \models \neg \bigwedge \Delta'$, and clearly $\bigwedge \Delta'$ is equivalent to a universal Horn sentence without equality of $L_{\kappa\kappa}^0 \cap L_{\kappa\kappa}^1$. \dashv

THEOREM 27. *For every regular cardinal $\kappa \geq \omega$, if ϕ_0 and ϕ_1 are universal Horn sentences of $L_{\kappa\kappa}^-$ such that $\phi_0 \models \neg \phi_1$ and, at least, they have a relation symbol in common, then there is a universal Horn sentence $\theta \in L_{\kappa\kappa}^-$ such that:*

- (i) $\phi_0 \models \theta$ and $\theta \models \neg \phi_1$.
- (ii) For every n -ary relation symbol $R \in \tau$, if R occurs in θ , then R occurs in both, ϕ_0 and ϕ_1 .

PROOF. Let τ' be the set of function and constant symbols of τ and for $i = 0, 1$, let τ'_i be the set of relation symbols that occur in ϕ_i . Finally, for $i = 0, 1$, let $\tau_i = \tau' \cup \tau'_i$. Then, τ_0 and τ_1 have the same function and constant symbols and $\{\phi_0\} \cup \{\phi_1\}$ is unsatisfiable. Therefore, by Theorem 26 we obtain the sentence that satisfies (i) and (ii). \dashv

REMARK 3. We have a counterexample that shows that the following clause:

$$(\diamond) \quad \text{Every symbol of } \tau \text{ that occurs in } \theta, \text{ occurs in both, } \phi_0 \text{ and } \phi_1$$

cannot be substituted for clause (ii) of Theorem 27. Let $\tau = \{<, f\}$, where $<$ is a binary relation symbol and f a monadic function symbol. Let ϕ_0 be the conjunction

of the following sentences:

- (1) $\forall x (\neg x < x)$
- (2) $\forall x (x < f(x))$
- (3) $\forall x \forall y \forall z (x < y \wedge y < z \rightarrow x < z)$

and let ϕ_1 be $\forall x \forall y \forall z (x < z \rightarrow y < z)$. Clearly ϕ_0 and ϕ_1 are equivalent to universal Horn sentences without the equality symbol and $\phi_0 \models \neg\phi_1$. But there is no universal Horn sentence without the equality symbol that satisfies (i) and (\diamond) . Suppose that such a sentence exists, say θ . The similarity type of θ would be $\{<\}$, because of (\diamond) , and every model of θ would have more than one element, since by (i) $\theta \models \neg\phi_1$. Let \mathfrak{A} be a model of θ , of similarity type $\{<\}$, and $a \in A$ arbitrary. If \mathfrak{A}' is the substructure of \mathfrak{A} generated by $\{a\}$, we have that \mathfrak{A}' has only one element in its domain and $\mathfrak{A}' \models \theta$, because θ is a universal sentence. Absurd. So we can conclude that such a sentence does not exist.

Let now τ be an arbitrary similarity type with a relation symbol R and such that $\tau - \{R\}$ has at least one relation symbol. Let κ be a regular cardinal $\geq \omega$, and let $L_{\kappa\kappa}$ be the infinitary language of type τ . Suppose that n is the arity of R . Then:

THEOREM 28. *For every set Γ of universal Horn sentences without the equality symbol of $L_{\kappa\kappa}$ the following are equivalent:*

- (i) Γ implicitly defines R . (i.e., if $\mathfrak{A}, \mathfrak{B} \models \Gamma$ and the reducts of \mathfrak{A} and \mathfrak{B} to $\tau - \{R\}$ are the same, then $\mathfrak{A} = \mathfrak{B}$).
- (ii) There is a universal Horn formula $\phi(x_1, \dots, x_n)$, without the equality symbol, of $L_{\kappa\kappa}$ such that ϕ is an explicit definition of R with respect to Γ (i.e., we have that $\Gamma \models \forall x_1 \dots x_n (\phi(x_1, \dots, x_n) \leftrightarrow Rx_1 \dots x_n)$), and the symbols of ϕ belong to $\tau - \{R\}$.

PROOF. (ii) \implies (i) is clear. To prove (i) \implies (ii) one can use the usual argument using the Interpolation Theorem 27 and Corollary 24. ⊣

§4. Some results on propositional logics. In this section we will prove as corollaries some results on propositional logics due to J. Czelakowski and to W. J. Blok and D. Pigozzi. First of all we introduce the main notions of the algebraic study of propositional logics via matrix semantics.

Given an algebraic similarity type τ , let Fm_κ , with κ an infinite cardinal, denote the absolutely free algebra of type τ with κ generators. This algebra will be called the *formula algebra with κ variables*, and its elements (*propositional formulas*). A *propositional logic* (a logic, for short) S on Fm_κ is a relation \vdash_S between sets of formulas and formulas such that for every $\Delta, \Gamma \subseteq Fm_\kappa$ and every $\phi, \psi \in Fm_\kappa$ it holds that:

- (1) If $\phi \in \Gamma$, then $\Gamma \vdash_S \phi$.
- (2) If $\Gamma \vdash_S \phi$ and for every $\psi \in \Gamma \Delta \vdash_S \psi$, then $\Delta \vdash_S \phi$.
- (3) For every homomorphism σ from Fm_κ into itself (i.e., a substitution), if $\Gamma \vdash_S \phi$, then $\sigma[\Gamma] \vdash_S \sigma(\phi)$.

From (1) and (2) it follows that

- (4) If $\Gamma \vdash_S \phi$ and $\Gamma \subseteq \Delta$ then $\Delta \vdash_S \phi$.

Condition (3) is called *the structurality condition*.

We say that a logic on Fm_κ is λ -consequence compact (λ -compact, for short), with λ an infinite regular cardinal less than or equal to $\max(|\tau|, \kappa)^+$, if for every $\Gamma \subseteq Fm_\kappa$ and every $\phi \in Fm_\kappa$ it holds that

(5) If $\Gamma \vdash_S \phi$ then there is $\Delta \subseteq \Gamma$ of cardinality less than λ such that $\Delta \vdash_S \phi$.

When a logic is ω -consequence compact it is called *finitary*.

A *matrix* of type τ is a pair $\langle \mathbf{A}, F \rangle$ where \mathbf{A} is an algebra of type τ and F is a subset of the domain of \mathbf{A} . Given a logic S on Fm_κ , a matrix $\langle \mathbf{A}, F \rangle$ is a *matrix model* of S (or an *S-matrix*) if for every homomorphism h from Fm_κ into \mathbf{A} (i.e., a valuation) and every $\Gamma \cup \{\phi\} \subseteq Fm_\kappa$, if $\Gamma \vdash_S \phi$ and $h[\Gamma] \subseteq F$ then $h(\phi) \in F$.

For the purpose of this exposition, given an algebra of formulas Fm_κ , we call a *sequent* on Fm_κ any pair $\langle \Gamma, \phi \rangle$ where Γ is a set of formulas of Fm_κ and ϕ is a formula of Fm_κ . The cardinality of a sequent $\langle \Gamma, \phi \rangle$ is the cardinality of Γ . Given a logic S on Fm_κ , we say that a sequent $\langle \Gamma, \phi \rangle$ is an *S-sequent* if $\Gamma \vdash_S \phi$. Thus a logic over Fm_κ is just a set of sequents of Fm_κ that is closed under the conditions (1)–(3) of the definition of logic.

If a logic S on Fm_κ is λ -compact for some $\lambda \leq \max(|\tau|, \kappa)^+$, then the set of *S-sequents*, $\langle \Gamma, \phi \rangle$, with $|\Gamma| < \lambda$, has the properties (1)–(3) of the definition of logics restricted to sets of formulas of cardinality less than λ . This set determines uniquely the logic S , since $\Gamma \vdash_S \phi$ if and only if for some $\Delta \subseteq \Gamma$ such that $|\Delta| < \lambda$, $\Delta \vdash_S \phi$. Therefore we can identify a λ -compact logic with its sequents of cardinality less than λ .

A matrix $\langle \mathbf{A}, F \rangle$ is a *model of a sequent* $\langle \Gamma, \phi \rangle$ on Fm_κ if for every homomorphism h from Fm_κ into \mathbf{A} (i.e., a valuation) such that $h[\Gamma] \subseteq F$ it holds that $h(\phi) \in F$. Hence, a matrix is a model of a logic S over Fm_κ if and only if it is a model of all the *S-sequents*. And if a logic S on Fm_κ is λ -compact then a matrix is a model of S if and only if it is a model of all the *S-sequents* of cardinality less than λ . In the case that a matrix is a model of a sequent we will also say that the sequent holds in the matrix.

Given the algebraic similarity type τ let us consider the extended similarity type $\tau \cup \{P\}$ where P is a unary relation symbol. For every infinite cardinal κ we consider the infinitary language $L_{\kappa^+ \kappa^+}$ of this extended type. To each sequent on Fm_κ we will associate a strict universal Horn formula without equality of this infinitary language. To do so we need to observe that the elements of the algebra Fm_κ of type τ are precisely the terms of the language $L_{\kappa^+ \kappa^+}$. Then, to a sequent on Fm_κ , $\langle \Gamma, \phi \rangle$, we associate the strict universal Horn formula without equality of $L_{\kappa^+ \kappa^+}$

$$\forall X \left(\bigwedge \{ P\psi : \psi \in \Gamma \} \rightarrow P\phi \right),$$

(i.e., the universal closure of $\bigwedge \{ P\psi : \psi \in \Gamma \} \rightarrow P\phi$), and will refer to it as *the translation* of $\langle \Gamma, \phi \rangle$ into $L_{\kappa^+ \kappa^+}$. The idea of translating sequents into universal strict Horn sentences has its origin in [3].

Obviously, each matrix of type τ can be seen as a structure of type $\tau \cup \{P\}$, and it is easy to see that a matrix of type τ is a model of a sequent on Fm_κ if and only if it is a model (in the usual model-theoretic sense) of the translation of the sequent into $L_{\kappa^+ \kappa^+}$. Therefore, we have that a matrix is a model of a logic S on Fm_κ if and only if it is a model of the translations into $L_{\kappa^+ \kappa^+}$ of all the *S-sequents*.

To every class K of matrices of type τ we can associate a logic on Fm_κ , the set of the sequents on Fm_κ that hold in every matrix in K . This logic translates precisely in the universal strict Horn theory without equality of K in the language $L_{\kappa+\kappa^+}$ of type $\tau \cup \{P\}$. And we have that a matrix is a model of this logic if and only if it is a model of that theory. If the logic is λ -compact then, since it can be identified with its sequents of cardinality less than λ , we can restrict ourselves to the universal strict Horn theory without equality of K in the language $L_{\lambda\lambda}$. Therefore using our characterization Theorem 6 we can obtain the following theorem of J. Czelakowski; see [5] and [6].

THEOREM 29 (Czelakowski). *Let K be a class of matrices of type τ and let S_K^κ its associated logic on Fm_κ . Then for every regular cardinal $\lambda \leq \max(|\tau|, \kappa)^+$, if S_K^κ is λ -compact, the class of all the matrices that are models of S_K^κ is*

$$\mathbf{H}^{-1}\mathbf{HSP}_{\mathbf{R}_\lambda}(K'),$$

where K' is K plus some trivial matrix. Moreover, if λ is strongly compact or is ω it is

$$\mathbf{H}^{-1}\mathbf{HSPP}_{\mathbf{U}_\lambda}(K').$$

PROOF. By Corollary 8 and Theorem 9 for the strict case, and the previous observations. \dashv

Now we prove the analogous theorem for reduced matrices.

THEOREM 30. *Let K be a class of matrices of type τ and let S_K^κ its associated logic on Fm_κ . Then for every regular cardinal $\lambda \leq \max(|\tau|, \kappa)^+$, if S_K^κ is λ -compact, the class of all the reduced matrices that are models of S_K^κ is*

$$\mathbf{S}^*\mathbf{P}_{\mathbf{R}_\lambda}^*(K'),$$

where K' is K plus some trivial matrix. Moreover, if λ is strongly compact or is ω it is

$$\mathbf{S}^*\mathbf{P}^*\mathbf{P}_{\mathbf{U}_\lambda}^*(K').$$

PROOF. By the analogous results to Theorems 16 and 18 but for the strict case, and the previous observations. \dashv

We will now explain a result of W. J. Blok and D. Pigozzi that is related to this theorem. Given a class K of matrices we can consider the set of sequents of finite cardinality on the algebra Fm_ω that hold in every matrix in the class. This set of sequents determines a finitary logic, the finitary logic (on Fm_ω) determined by the class K .

W. J. Blok and D. Pigozzi in [2] proved a characterization result for the class of reduced matrix models of the finitary logic on Fm_ω determined by a given class of matrices. In fact they proved it for what they call k -deductive systems. A k -deductive system is like a finitary logic but its “formulas” (k -formulas), instead of being elements of Fm_ω are finite sequences of length k of elements of Fm_ω . For these k -deductive systems we have the corresponding notion of sequent, and each finite sequent translates into a strict universal Horn formula without identity of the first-order language whose similarity type is the algebraic type of the deductive system augmented with a k -ary relation symbol. The matrices for these k -deductive

systems consist of an algebra and a k -ary relation on its domain. For the sake of simplicity we state the theorem for finitary logics, the 1-deductive systems.

THEOREM 31 (Blok, Pigozzi). *Let K be a class of reduced matrices of type τ and let S_K be its associated finitary logic on Fm_ω . Then the class of reduced matrices that are models of S_K is $\mathbf{S}^*\mathbf{P}^*\mathbf{P}_\cup^*(K')$, where K' is K plus the trivial structures.*

PROOF. By Corollary 19 but for the strict case, taking into account that what corresponds to the finitary deductive system associated to K is the first-order strict universal Horn theory without equality of K . \dashv

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